OSU PHYSICS DEPARTMENT
COMPREHENSIVE EXAMINATION #106

March 30 and 31, 2009

Spring 2009 Comprehensive Examination

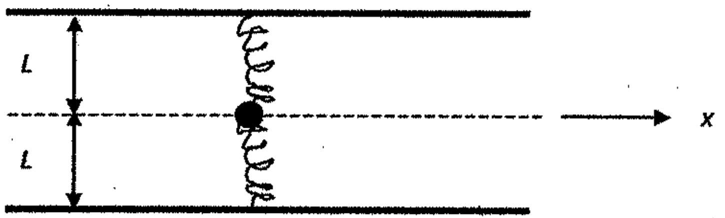
PART 1, Monday, March 30, 9:00 am

General Instructions

This Spring 2009 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, March 30, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third-and fourth parts will be administered on Tuesday, March 31, at 9:00 am and 1:00 pm., respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit - especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination. If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials' books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required – calculators will be then provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Comprehensive Examination, Spring 2009 – Problem 1

Consider the following arrangement of identical springs with spring constant *k* connected to a mass *m*. In equilibrium, as shown, the springs are stretched a distance *d* beyond their unstretched length.



When the mass is displaced in the direction x and released, it oscillates in the ± x direction.

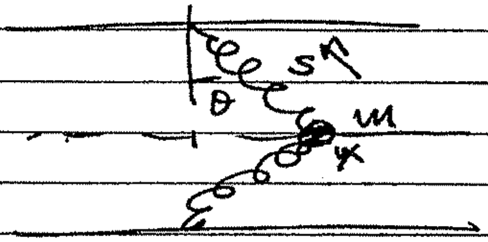
(1) Write the Hamiltonian of this system in a form valid for small displacements (x/L << 1). Retain terms up to $(x/L)^2$.

(2) Show that this system is non-linear for *any* amplitude if $d = 0$, i.e., if the springs are not stretched in equilibrium.

(3) Assume that the maximum displacement $(x/L)_{max} = 0.1$ and estimate quantitatively the range of (d/L) in which the non-linear properties of this oscillator will dominate. Pick a reasonable criterion and do a "back-of-the-envelope calculation." **Do not use a calculator!**

(4) Write the equation of motion for the variable *x* for the case $(d/L) \neq 0$ and the oscillator is driven in the ± x direction with frequency ω. Use successive approximations (perturbation theory) to solve for *x(t)* in terms of two harmonic frequencies.

#1



① $F_s = k\left(s-(L-d)\right) = k\left(s-L+d\right)$

$F_x = -2K\left[s-L+d\right]\sin\theta = -2k\left[s-L+d\right]\frac{x}{s}$

$s=\sqrt{L^2+x^2}$

$= -2k\frac{\left[\sqrt{}-L+d\right]x}{\sqrt{}} = -2k\left[1-\frac{(L-d)}{\sqrt{L^2+x^2}}\right]$

$= -2k\left[1-\frac{1-(d/L)}{\sqrt{1+(x/L)^2}}\right]x$

$\epsilon \equiv d/L$

$F_x = -2k\left[1-\frac{1-\epsilon}{\sqrt{1+(x/L)^2}}\right]x$

expand:

$= -2k\left[1-(1-\epsilon)\left(1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right)\right]x$

$= -2kx\left[1-1+\frac{1}{2}\left(\frac{x}{L}\right)^2\right] -2kx\left[\epsilon-\epsilon\frac{1}{2}\left(\frac{x}{L}\right)^2\right]$

$$F_x = -\frac{kx^3}{L^2} - 2k\epsilon x\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$

$$= -\frac{kx^3}{L^2}\left[1 - \epsilon\,\text{[illegible]}\right] - 2k\epsilon x$$

$$F_x = -\frac{k(1-\epsilon)}{L^2}x^3 - 2k\epsilon x$$

$$U = \frac{k(1-\epsilon)x^4}{4L^2} + k\epsilon x^2$$

$$\therefore\ \boxed{H = \frac{p^2}{2m} + \frac{k(1-\epsilon)}{4L^2}x^4 + k\epsilon x^2}$$

$$\boxed{H = \frac{p^2}{2m} + \frac{1}{2}kx^2\left[\frac{(1-\epsilon)}{2L^2}x^2 + 2\epsilon\right]}$$

(2) Any amplitude:

$$F_x = -2k\left[1 - \frac{(1-\epsilon)}{\sqrt{1+(x/L)^2}}\right]x$$

@ $\epsilon=0$ $F_x = -2k\left[1 - \frac{1}{\sqrt{1+(x/L)^2}}\right]$ not $\propto x$:

@ $\frac{x}{L}$ small: $F_x = -\frac{k(1-\epsilon)}{L^2}x^3 - 2k\epsilon x$

@ $\epsilon=0$

$$F_x = -\frac{kx^3}{L^2} \sim \underline{\underline{x^3}}$$

(3) $\left(\frac{x}{L}\right)_{max} = 0.1$

Look @ $U = \frac{1}{2}kx^2\left[\frac{1-\epsilon}{2L^2}x^2 + 2\epsilon\right]$

@ what ϵ $\frac{(1-\epsilon)x^2}{2L^2} = 2\epsilon$ $\frac{x}{L} = 0.1$

$$\left(\frac{1-\epsilon}{2}\right)\cdot 10^{-2} = 2\epsilon$$

$$\left(\frac{1}{2}-\frac{\epsilon}{2}\right)10^{-2} = 2\epsilon$$

$$\boxed{\epsilon = \frac{d}{L} \simeq \frac{1}{400}}$$

$$\frac{1}{2}\cdot 10^{-2} - \frac{1}{2}\epsilon\cdot 10^{-2} = 2\epsilon$$

$$\frac{1}{2}10^{-2} = \epsilon\left[2 + \frac{1}{2}\cdot 10^{-2}\right] \sim 2\epsilon \quad \epsilon = \frac{1}{4}10^{-2} = \frac{1}{400}$$

⑤

(4) $F_x = -kx\left[\frac{(1-\epsilon)x^2}{L^2} + 2\epsilon\right]$

Newton: $m\ddot{x} = \underbrace{-2k\epsilon}_{a/m} x - \underbrace{k\frac{(1-\epsilon)}{L^2}}_{b/m} x^3 + A_0 \cos\omega t$

$$\ddot{x} = -ax - bx^3 + (A_0/m)\cos\omega t$$

1st approximation - assume linear response:

$$x_1 = A_1 \cos\omega t$$

Substitute:

$$\ddot{x}_2 = (-aA_1 + A_0/m)\cos\omega t - bA_1^3 \cos^3\omega t$$

$$\cos^3\omega t = \frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t$$

$$\ddot{x}_2 = (-aA_1 + A_0/m - 3bA_1^3/4)\cos\omega t - \frac{bA_1^3}{4}\cos 3\omega t$$

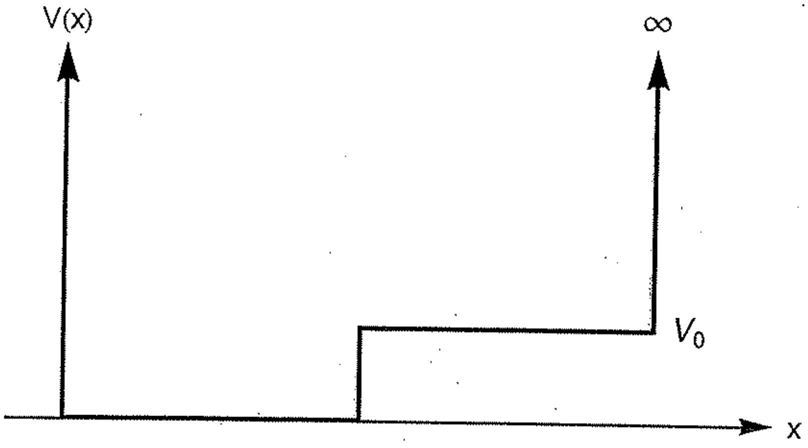
integrate twice:

$$x_2(t) = \frac{1}{\omega^2}(aA_1 - A_0/m + 3bA_1^3/4)\cos\omega t$$
$$+ \frac{1}{9\omega^2}\frac{bA_1^3}{4}\cos 3\omega t$$

$\uparrow$ get third harmonic response

Comprehensive Examination, Spring 2009 – Problem 2

Consider a particle of mass m bound in an infinite square well with an added "shelf" at the bottom of the well, as shown in the figure below. The potential energy is 0 in the left half of the well and V_0 in the right half of the well. The well has total width L.



(a) Find the transcendental equation that determines the energy eigenvalues of the system for energies that lie above the shelf energy V_0.
(b) Show that the transcendental equation in (a) reproduces the energy eigenvalues of the standard infinite square well (*i.e.* with a flat bottom).
(c) Use perturbation theory to find the energies of the system for the case that the shelf height V_0 is small compared to the ground state energy.
(d) Show that the exact result in (a) and the perturbation result in (c) agree to lowest order in the perturbation for the ground state shift.

(a) The infinite potential outside the well implies that the energy eigenstates are zero outside the well. Inside the well there are different energy eigenvalue equations in the left and right halves:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+0\right)\varphi_E(x)=E\varphi_E(x) \quad \textit{left half}$$
$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V_0\right)\varphi_E(x)=E\varphi_E(x) \quad \textit{right half}$$

For the case that the energy E is greater than the potential V_0, the solutions in each half of the well are sinusoidal, with different wave vectors in each half:

$$k_1=\sqrt{\frac{2mE}{\hbar^2}}$$
$$k_2=\sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

The general solution is

$$\varphi_E(x)=\begin{cases} A\sin k_1x+B\cos k_1x, & -L/2\le x\le 0 \\ C\sin k_2x+D\cos k_2x, & 0\le x\le L/2 \end{cases}$$

Apply the boundary condition on the wave function continuity at the middle and sides of the well, and the boundary condition on the continuity of the first derivative of the wave function at the middle of the well (recall that the infinite potential on the sides means that the derivative condition is not applicable there). The four boundary conditions are

$$\begin{aligned} \varphi_E(-L/2)&:-A\sin(k_1L/2)+B\cos(k_1L/2)=0 \\ \varphi_E(L/2)&:C\sin(k_2L/2)+D\cos(k_2L/2)=0 \\ \varphi_E(0)&:B=D \\ \left.\frac{d\varphi_E(x)}{dx}\right|_{x=0}&:k_1A=k_2C \end{aligned}$$

These four equations contain five unknowns: the amplitudes A, B, C, and D, and the energy E through the wave vectors k_1 and k_2. The normalization condition supplies the fifth equation required to solve for all unknowns. Eliminate the amplitude coefficients from the four boundary condition equations:

$$k_1 A = k_2 C \to C = A\frac{k_1}{k_2}$$

$$-A\sin(k_1 L/2) + B\cos(k_1 L/2) = 0 \to B = D = A\frac{\sin(k_1 L/2)}{\cos(k_1 L/2)}$$

$$A\frac{k_1}{k_2}\sin(k_2 L/2) + A\frac{\sin(k_1 L/2)}{\cos(k_1 L/2)}\cos(k_2 L/2) = 0$$

to arrive at the transcendental equation for the energy eigenvalues (HW):

$$\boxed{k_1\cos(k_1 L/2)\sin(k_2 L/2) + k_2\sin(k_1 L/2)\cos(k_2 L/2) = 0}$$

(b) If $V_0 = 0$, then the two wave vectors are equal and the transcendental equation becomes:

$$\begin{aligned}
&k_1\cos(k_1 L/2)\sin(k_1 L/2) + k_1\sin(k_1 L/2)\cos(k_1 L/2) = 0 \\
&k_1\sin\left[(k_1 L/2) + (k_1 L/2)\right] = 0 \\
&k_1\sin k_1 L = 0
\end{aligned}$$

This yields

$$\begin{aligned}
&\sin k_1 L = 0 \\
&k_1 L = n\pi \\
&k_1 = \frac{n\pi}{L} \\
&E = \frac{\hbar^2 k_1^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2mL^2}
\end{aligned}$$

which are the energy eigenvalues of the infinite square well.

(c) Perturbation theory dictates that the shift of the energy states is given by the expectation value of the perturbation in the zero-order states:

$$E_n^{(1)} = \left\langle n^{(0)}\middle|H'\middle|n^{(0)}\right\rangle = \int_{-\infty}^{\infty}\psi_n^{(0)*}(x)H'\psi_n^{(0)}\,dx$$

We can do this integral without knowing the explicit form of the unperturbed eigenstates, using their symmetry about the well origin.

$$\begin{aligned}
E_n^{(1)} &= \int_{-\infty}^{\infty}\psi_n^{(0)*}(x)H'\psi_n^{(0)}\,dx \\
&= \int_{-L/2}^{0}\psi_n^{(0)*}(x)0\psi_n^{(0)}\,dx + \int_0^{L/2}\psi_n^{(0)*}(x)V_0\psi_n^{(0)}\,dx \\
&= V_0\int_0^{L/2}\left|\psi_n^{(0)}\right|^2 dx \\
&= \frac{V_0}{2}
\end{aligned}$$

(d) Expand the transcendental equation when the perturbation is small. For the ground state, the unperturbed wave vector is π/L, so write the perturbed left side wave vector as

$$k_1 L = \pi(1+\varepsilon)$$

and the perturbed the right side wave vector as

$$k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} = k_1(1-V_0/E)^{1/2} \cong k_1(1-V_0/2E) \equiv k_1(1-\eta)$$

$$k_2 L \cong \pi(1+\varepsilon)(1-\eta) \cong \pi(1+\varepsilon-\eta)$$

The new parameter is $\eta = V_0/2E$. The transcendental equation has *sin* and *cos* functions of the half angles $k_i L/2$, which are nearly $\pi/2$. Hence, the *sin* terms are unity and the *cos* terms are $\pi/2$ minus the half angles:

$$k_1 \cos(k_1 L/2)\sin(k_2 L/2) + k_2 \cos(k_2 L/2)\sin(k_1 L/2) = 0$$

$$\frac{\pi}{L}(1+\varepsilon)\left(-\frac{\varepsilon}{2}\right) + \frac{\pi}{L}(1+\varepsilon-\eta)\left(-\frac{\varepsilon}{2}+\frac{\eta}{2}\right) \cong 0$$

$$\left(-\frac{\varepsilon}{2}\right) + \left(-\frac{\varepsilon}{2}+\frac{\eta}{2}\right) \cong 0$$

$$\varepsilon \cong \eta/2$$

The perturbed energy is

$$E = \frac{\hbar^2 k_1^2}{2m} = \frac{\hbar^2\pi^2}{2mL^2}(1+\varepsilon)^2 \cong \frac{\hbar^2\pi^2}{2mL^2}(1+2\varepsilon) = \frac{\hbar^2\pi^2}{2mL^2}(1+\eta) = \frac{\hbar^2\pi^2}{2mL^2}\left(1+\frac{V_0}{2E}\right)$$

$$E \cong \frac{\hbar^2\pi^2}{2mL^2} + \frac{V_0}{2}$$

$$\Delta E = E - E^{(0)} = \frac{V_0}{2}$$

Thus the perturbation result and the exact result agree to this order.

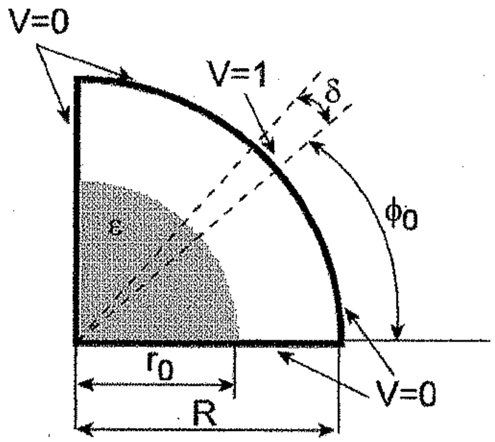
Comprehensive Examination, Spring 2009 – Problem 3

This is a two-dimensional problem. Consider a wedge (radius R, wedge angle $\pi/2$) shown in Fig.1. The wedge is partially filled with material with relative permittivity ε up to $r \leq r_0$ and contains air for $r_0 < r < R$. The radial sides of the wedge are kept at zero electric potential. The electric potential at $r = R$ is given by

$$V(R,\phi) = \begin{cases} 1, \phi_0 < \phi < \phi_0 + \delta \\ 0, \text{otherwise} \end{cases}$$

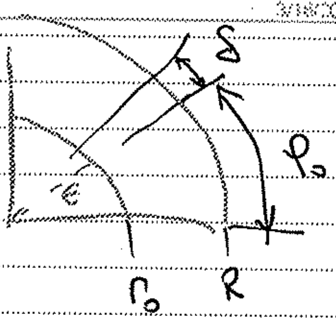
with $\delta \ll 1$.

Find the electric potential everywhere inside the wedge.



#3

Laplace eq in cylindrical coord:



(1) $\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} = 0$

Use separation of variables:

(2) $V = R(r)\,\Phi(\varphi)$

(3) $\Phi\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2}R\cdot\Phi'' = 0$

To satisfy b conditions on the sides of the wedge, choose:

(4) $\frac{\Phi''}{\Phi} = -k^2 \Rightarrow \Phi(\varphi) = \sin m\varphi\pi \Rightarrow k_n^2 = (2m)^2$, $m = 1, \ldots$

(4) → (3) (assuming that $k \neq 0$) ⇒

(5) $r\frac{dR}{dr} + r^2\frac{d^2R}{dr^2} - k^2 R = 0$

looking for a solution in the form:

(6) $R(r) = r^\alpha \Rightarrow$

$\alpha \cancel{r^\alpha} + \alpha(\alpha-1)\cancel{r^\alpha} - k^2\cancel{r^\alpha} = 0$

$\alpha = \pm k$

(7) $\Rightarrow \boxed{V(r) = \sum_n C_n^+ r^{k_n}\sin k_n\varphi + C_n^- r^{-k_n}\sin k_n\varphi} + a_0 \ln r$

Assuming that $V(r \to 0)$ is finite, $a_0 = 0$

The potential must be finite everywhere $\Rightarrow$

(8) $$V(r) = \begin{cases} \sum_m a_m^+ r^{k_m} \sin k_m\varphi \, , & r < r_0 \\ \sum_m b_m^+ r^{k_m} \sin k_m\varphi + b_m^- r^{-k_m} \sin k_m \varphi \, , & r_0 < r < R \end{cases}$$

Boundary conditions:

(9) $$\begin{cases} V(R) = V_0(\varphi) = \sum_m \left(b_m^+ R^{k_m} + b_m^- \cdot R^{-k_m}\right) \sin k_m\varphi \\ \epsilon \frac{\partial V}{\partial n}\Big|_{r=r_0} = \text{const} \Rightarrow \sum_m \left(b_m^+ r_0^{k_m - 1} - b_m^- r_0^{-k_m - 1}\right) k_m \sin k_m\varphi = \\ \qquad = \epsilon_r \sum k_m a_m^+ r_0^{k_m - 1} \sin k_m \varphi \\ V\Big|_{r_0} = \text{const} \quad \sum \left(b_m^+ r_0^{k_m} + b_m^- r_0^{-k_m}\right) \sin k_m\varphi \end{cases}$$

Note that:

$$\int_0^{\pi/2} \sin k_m\varphi \, \sin k_n\varphi \, d\varphi = \int_0^{\pi/2} \sin 2m\varphi \cdot \sin 2n\varphi \, d\varphi =$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(\cos 2(m-n)\varphi - \cos 2(m+n)\varphi \right) d\varphi = \begin{cases} \frac{\pi}{4} \, , & m = n \\ 0 \, , & m \neq n \end{cases}$$

Thus, B.C. reduce to:

$$(9.1) \rightarrow \frac{4}{\pi} \int_0^{\pi/2} V_0(\varphi) \cdot \sin \frac{\pi n \varphi}{\varphi_0} \, d\varphi = V_m \frac{4}{\pi} = b_m^+ R^{k_m} + b_m^- R^{-k_m}$$

$$(9.2) \rightarrow \left(b_m^+ r_0^{k_m} - b_m^- r_0^{-k_m}\right) = \epsilon \, a_m^+ r_0^{k_m}$$

(a.3) $\Rightarrow$ $\left(b_n^+ r_0^{k_n} + b_n^- r_0^{-k_n}\right) = a_n^+ r_0^{k_n}$

The last two eqs give:

$$\left(b_n^+ r_0^{k_n} - b_n^- r_0^{-k_n}\right) = \epsilon\left(b_n^+ r_0^{k_n} + b_n^- r_0^{-k_n}\right)$$

$$\begin{cases} b_n^- = \dfrac{1-\epsilon}{1+\epsilon}\, r_0^{2k_n} \cdot b_n^+ \\ a_n^+ = b_n^+ \cdot \dfrac{2}{1+\epsilon} \\ b_n^+ = \dfrac{4}{\pi}\left[R^{k_n} + \dfrac{1-\epsilon}{1+\epsilon}\, r_0^{2k_n} R^{-k_n}\right] \displaystyle\int_0^{\pi/2} V(\varphi) \sin k_n\varphi \, d\varphi \\ \quad = \dfrac{4}{\pi}\left[R^{k_n} + \dfrac{1-\epsilon}{1+\epsilon}\, r_0^{2k_n} R^{-k_n}\right] a \cdot \sin k_n\varphi_0 . \end{cases}$$

Comprehensive Examination, Spring 2009 - Problem 4

Consider a "semi-infinite" crystal with a surface at $z = 0$, and with dimensions spanning to $+\infty$ in the z direction, and to $\pm\infty$ in the x and y directions.

The crystal consists of close-packed atomic layers (all atoms identical) perpendicular to the z axis, and stacked upon one another in the z direction (see Fig. 1 (a)). In such structure, atoms in the $i = 2,\ 3,\ \ldots$ layers are in direct contact with 12 nearest-neighbor atoms, and atoms in the surface layer (i.e., that with $i = 1$) are in direct contact with only 9 nearest-neighbor atoms (six located in the same layer, and three in the $i = 2$ layer).

The nearest-neighbor atoms are coupled by interatomic binding forces, and the energy of coupling between any two such atoms[1] is $-\epsilon$ (where ϵ is a positive number). We assume that there are no interactions of any kind between atoms that are **not** nearest-neighbors.

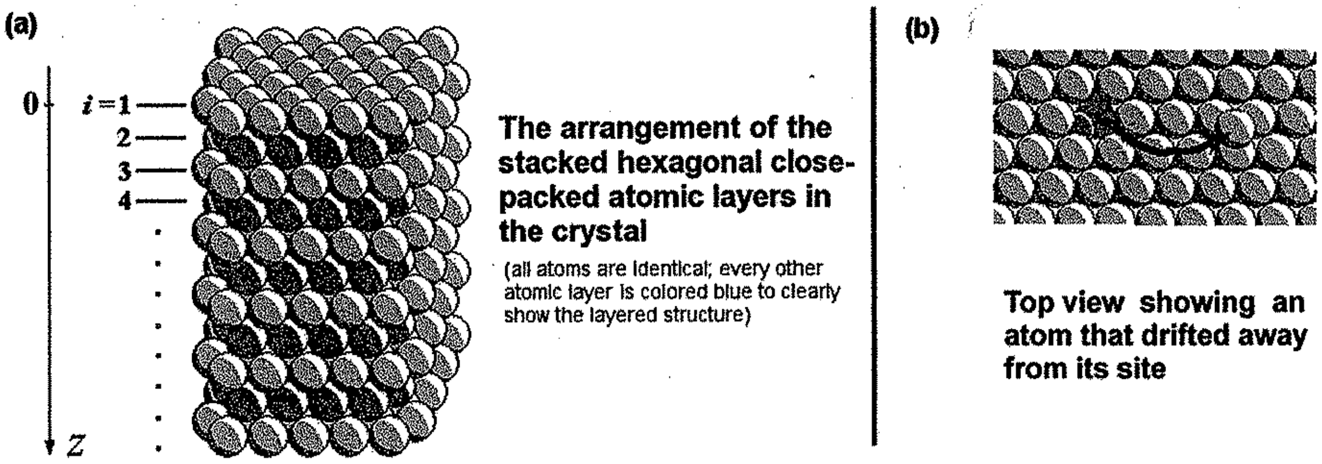
At $T = 0$, all atomic sites in the surface layer are occupied (in other words, the surface is "perfectly smooth"). However, at finite temperatures some surface atoms are pushed out of their sites by thermal fluctuations. Such atoms drift away, leaving vacancies, or "holes" in the surface, and they settle in nearby "dips" formed by three atoms, as illustrated in Fig. 1 (b).

Your task in this problem is to find what fraction of atoms from the surface layer migrate to the "dips" at temperature T (we assume that only atoms from the surface layer are able to migrate, while those from the $i = 2,\ 3, \ldots$ layers do not take part in the process).

(i) Start with assuming that the total number of atoms in the surface layer, N, is a large yet *finite* number. Let n be the number of atoms relocated from layer sites to "dips" at certain temperature T. Find the Helmholtz free energy F_{lay} of the surface layer with n vacancies, the free energy F_{migr} of the system of n migrant atoms, and the total free energy $F_{\text{lay+migr}}$ of the combined system of the surface layer plus the migrant atoms. *Hint:* For large N values, $\ln N! \approx N \ln N - N$.

(ii) Based on the results of (i), find the equilibrium average value $\langle n \rangle$ of the nuber of migrant atoms at temperature T.

(iii) Express the value of the $\langle n \rangle / N$ fraction as a function of T. Does the result change if $N \to \infty$?



The arrangement of the stacked hexagonal close-packed atomic layers in the crystal

(all atoms are identical; every other atomic layer is colored blue to clearly show the layered structure)

Top view showing an atom that drifted away from its site

Fig. 1 (a) shows the "side view" of the layer, with every other constituent layer colored blue for clarity; (b) a migrant atom in its new position, and the "hole" (vacancy) it left behind – in this plot, one can recognize all nine nearest-neighbor atoms the migrant atom was bound to before leaving its site.

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[1] We assume that the atom-atom interactions are perfectly isotropic, i.e., the energy of interaction of an atom, call it A, with another atom – call it B – which is a nearest-neighbor of A, is always the same, $-\epsilon$, no matter whether atom B resides in the same layer as A, in an adjacent layer, or in a surface "dip".

Problem 4, solution, p. 1

(i) There are N−n atoms in the depleted surface layer and N sites. N−n atoms and n "holes" can be distributed over N sites in $g(N,n) = \frac{N!}{(N-n)!\,n!}$ ways.

The energy of a fully occupied surface layer is:

$N \times (-3\varepsilon)$ – each atom is coupled with three in the layer underneath;

plus

$\frac{1}{2} \times N \times (-6\varepsilon) = N \times (-3\varepsilon)$ – each atom is coupled with 6 other surface layer atoms, but each such bond is "shared" by two atoms, therefore $\frac{1}{2}$

So, the energy of the top layer $-6N\varepsilon$ (that much energy would be needed for totally "peeling off" the surface layer from the film).

The energy needed to extract a single surface layer atom from its site is $+9\varepsilon$ (that extraction cuts off 9 atom-atom bonds) – so the total energy of a surface layer with n vacancies is $-6N\varepsilon + 9n\varepsilon$.

The free energy is $F_{lay} = U - TS$, and $S = k \ln g(N,n)$,

hence:

$$F_{lay} = -6N\varepsilon + 9n\varepsilon - Tk \ln \frac{N!}{(N-n)!\,n!}$$

The energy of the system of n migrant atoms is $n\cdot(-3\varepsilon)$. Each atom in the surface layer is sorounded by 6 "dips". If the atom migrates, all 6 dips disappear. So, n migrant atoms can be distributed over $N-6n$ "dips", and the number of ways they can be distributed is:

$$g(N-6n, n) = \frac{(N-6n)!}{[(N-6n)-n]!\,n!} = \frac{N-6n!}{(N-7n)!\,n!}$$

So, the free energy of the migrant atom system is

$$F_{migr} = -3n\varepsilon - kT\ln\frac{(N-6n)!}{(N-7n)!\,n!}$$

The free energy is additive, so the total free energy of the layer with n vacancies, and n atoms resting in the "dips" is: $F_{lay+migr} = -6N\varepsilon + 6n\varepsilon - kT\ln\frac{(N-6n)!}{(N-7n)!(n!)^2}$

$$- kT\ln\frac{(N-6n)!}{(N-7n)!\,n!} = -6N\varepsilon + 6n\varepsilon - kT\ln\frac{N!\,(N-6n)!}{(N-n)!\,(N-7n)!\,(n!)^2}$$

(ii)

In the state of equlibrium the free energy has a maximum, so that:

$$\frac{\partial}{\partial n}\left(F_{lay+migr}\right) = 0$$

Problem 4, sol. p. 3

For calculating the derivative, on can use the Stirling approximation:

$$\ln(N-kn)! \cong \left[(N-kn)\ln(N-kn)\right] - (N-kn) \quad \text{where } k=0,1,2,\ldots$$

Then:

$$\frac{\partial}{\partial n}\ln(N-kn)! \cong N\frac{-k}{N-kn} - kn\frac{-k}{N-kn} - k\ln(N-kn) + k$$

$$= -k\frac{N}{N-kn} + k\frac{nk}{N-kn} + k - k\ln(N-kn)$$

$$= -k\frac{N-kn}{N-kn} + k - k\ln(N-kn) = -k\ln(N-kn)$$

So:

$$\frac{\partial}{\partial n}\left[\ln\frac{N!(N-6n)!}{(N-n)!(N-7n)!(n!)^2}\right]$$

$$\cong -6\ln(N-6n) + \ln(N-n) + 7\ln(N-7n) - 2\ln(n)$$

$$\boxed{\begin{array}{l}\text{One can use another approximation:}\\ \ln(1\pm x) \approx \pm x, \text{ for } x \ll 1\\ \ln(N-kn) = \ln\left[N\left(1-k\frac{n}{N}\right)\right] = \ln N + \ln\left(1-k\frac{n}{N}\right) \overset{\text{when } n \ll N}{\cong} \ln N - k\frac{n}{N}\end{array}}$$

$$\cong -6\ln N + 36\frac{n}{N} + \ln N - \frac{n}{N} + 7\ln N - 49\frac{n}{N} - 2\ln(n)$$

$$= 2\ln N - 2\ln n - 14\frac{n}{N} = 2\left[\ln\frac{N}{n} - 7\frac{n}{N}\right]$$

$$\boxed{-7\frac{n}{N} \cong \ln\left[1-7\frac{n}{N}\right]}$$

$$\cong 2\left[\ln\frac{N}{n} + \ln\left(1-7\frac{n}{N}\right)\right] = 2\ln\left(\frac{N}{n} - 7\right)$$

So, we get:

$$\frac{\partial}{\partial n} F_{lay+migr} \cong 6\varepsilon - kT\cdot 2\ln\left(\frac{N}{n}-7\right) = 0$$

$$\ln\left(\frac{N}{n}-7\right) = \frac{3\varepsilon}{kT}$$

$$\frac{N}{n}-7 = e^{3\varepsilon/kT}$$

$$\boxed{n = N\frac{1}{e^{3\varepsilon/kT}+7}}$$

The low-T range here would be the one where $e^{3\varepsilon/kT} \gg 7 \Rightarrow T \ll \frac{3\varepsilon}{k\cdot\ln 7} \cong \frac{3\varepsilon}{k\times 1.946} \approx \frac{3}{2}\frac{\varepsilon}{k}$

In this region,

$$n = N\cdot e^{-\frac{3\varepsilon}{kT}}$$

The result suggest that when $T\to\infty$, $n \approx \frac{N}{7}$

but this is a dubious result – the $n \ll N$ condition we used is no longer fulfilled, and the approximations we used in the procedure were good only for $n \ll N$.

(iii) $\frac{n}{N} = e^{-\frac{3\varepsilon}{kT}}$, does not change when $N \to \infty$

Comprehensive Examination, Spring 2009 – Problem 5

In this problem, you will calculate quantum mechanical probabilities and show that they violate the Bell inequalities for local hidden variable theories.

In the spin version of the Einstein-Podolsky-Rosen *gedanken* experiment, a spin 0 source decays into two spin ½ particles, which by conservation of angular momentum must have opposite spin projections and by conservation of linear momentum must head in opposite directions. Consider observers *A* and *B* on opposite sides of the source, as shown below. Each observer has a Stern-Gerlach apparatus to measure the spin projection of the particle headed in his direction. The quantum state of the two-particle system is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right]$$

Each observer can orient his Stern-Gerlach apparatus along one of three directions $\hat{\mathbf{a}},\hat{\mathbf{b}},\hat{\mathbf{c}}$ in a plane, each 120° from any of the other two. Each observer makes measurements of the spin projection along one of these three directions. The two observers do not communicate with each other during the experiment, and they choose the orientations of their respective Stern-Gerlach devices independently and randomly. The entanglement of the quantum state vector leads to correlations in the measured results of the two observers. John Bell showed that the quantum mechanical results for these correlations exclude the possibility of local hidden variable theories.

In Bell's argument, we classify the measurement results into two categories. Either the two observers record the "same" spin projections (++ or −−), or they record "opposite" spin projections (+− or −+). For example if observer *A* measures particle 1 to have *spin up* along direction $\hat{\mathbf{a}}$ and observer *B* measures particle 2 to have *spin up* along direction $\hat{\mathbf{b}}$, then that result is classified as "same." Bell showed that any local hidden variable theory would produce results that obey these inequalities:

$$\mathcal{P}_{same} \leq \frac{4}{9} \quad , \quad \mathcal{P}_{opp} \geq \frac{5}{9}$$

when averaged over all possible relative orientations of the two Stern-Gerlach devices, given the three fixed directions $\hat{\mathbf{a}},\hat{\mathbf{b}},\hat{\mathbf{c}}$.

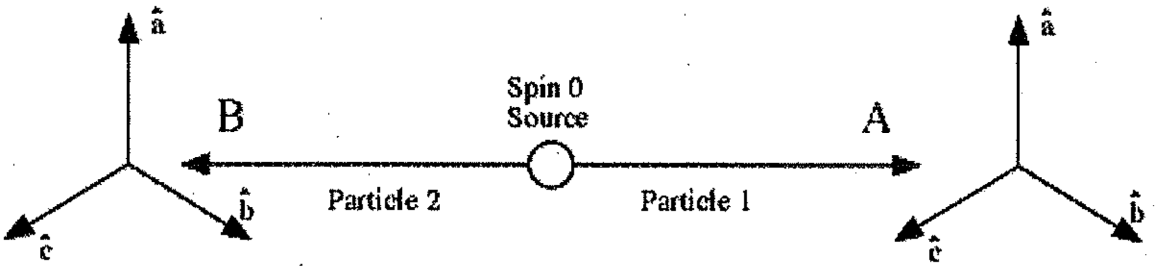
By explicitly calculating the two averaged probabilities $\mathcal{P}_{same}$ and $\mathcal{P}_{opp}$, show that the quantum mechanical predictions of this experiment violate the Bell inequalities.

The single-particle spin ½ projection eigenstates for an arbitrary direction $\hat{\mathbf{n}}$ are

$$|+\hat{\mathbf{n}}\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$$

$$|-\hat{\mathbf{n}}\rangle = \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}e^{i\phi}|-\rangle$$

where $|+\rangle,|-\rangle$ are the projection eigenstates along the *z*-axis.



(51)

Calculate the probability that observer A records a "+" along the direction $\mathbf{\hat{a}}$ and that observer B records a "+" along the direction $\mathbf{\hat{n}}$ oriented at an angle θ with respect to the z-axis. Choose the direction $\mathbf{\hat{a}}$ as the z-axis. We get

$$\mathcal{P}_{+\mathbf{\hat{a}},+\mathbf{\hat{n}}} = \left|\left[{}_{1:\mathbf{\hat{a}}}\langle+|\,{}_{2:\mathbf{\hat{n}}}\langle+|\right]|\psi\rangle\right|^2 = \left|\left[{}_1\langle+|\,{}_{2:\mathbf{\hat{n}}}\langle+|\right]\frac{1}{\sqrt{2}}\left[|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right]\right|^2$$

$$= \frac{1}{2}\left|\left\{\cos\frac{\theta}{2}\,{}_2\langle+| + e^{-i\phi}\sin\frac{\theta}{2}\,{}_2\langle-|\right\}|-\rangle_2\right|^2 = \frac{1}{2}\sin^2\frac{\theta}{2}$$

Now calculate the probability that observer A records a "+" along the direction $\mathbf{\hat{a}}$ and that observer B records a "-" along the direction $\mathbf{\hat{n}}$ oriented at the angle θ with respect to the z-axis. We get

$$\mathcal{P}_{+\mathbf{\hat{a}},-\mathbf{\hat{n}}} = \left|\left[{}_{1:\mathbf{\hat{a}}}\langle+|\,{}_{2:\mathbf{\hat{n}}}\langle-|\right]|\psi\rangle\right|^2 = \left|\left[{}_1\langle+|\,{}_{2:\mathbf{\hat{n}}}\langle-|\right]\frac{1}{\sqrt{2}}\left[|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right]\right|^2$$

$$= \frac{1}{2}\left|\left\{\sin\frac{\theta}{2}\,{}_2\langle+| - e^{-i\phi}\cos\frac{\theta}{2}\,{}_2\langle-|\right\}|-\rangle_2\right|^2 = \frac{1}{2}\cos^2\frac{\theta}{2}$$

Similarly, we get

$$\mathcal{P}_{-\mathbf{\hat{a}},-\mathbf{\hat{n}}} = \left|\left[{}_{1:\mathbf{\hat{a}}}\langle-|\,{}_{2:\mathbf{\hat{n}}}\langle-|\right]|\psi\rangle\right|^2 = \left|\left[{}_1\langle-|\,{}_{2:\mathbf{\hat{n}}}\langle-|\right]\frac{1}{\sqrt{2}}\left[|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right]\right|^2$$

$$= \frac{1}{2}\left|\left\{\sin\frac{\theta}{2}\,{}_2\langle+| - e^{-i\phi}\cos\frac{\theta}{2}\,{}_2\langle-|\right\}|+\rangle_2\right|^2 = \frac{1}{2}\sin^2\frac{\theta}{2}$$

and

$$\mathcal{P}_{-\mathbf{\hat{a}},+\mathbf{\hat{n}}} = \left|\left[{}_{1:\mathbf{\hat{a}}}\langle-|\,{}_{2:\mathbf{\hat{n}}}\langle+|\right]|\psi\rangle\right|^2 = \left|\left[{}_1\langle-|\,{}_{2:\mathbf{\hat{n}}}\langle+|\right]\frac{1}{\sqrt{2}}\left[|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\right]\right|^2$$

$$= \frac{1}{2}\left|\left\{\cos\frac{\theta}{2}\,{}_2\langle+| + e^{-i\phi}\sin\frac{\theta}{2}\,{}_2\langle-|\right\}|+\rangle_2\right|^2 = \frac{1}{2}\cos^2\frac{\theta}{2}$$

We would obtain similar results for the other possible cases. To summarize

$$\mathcal{P}_{++} = \mathcal{P}_{--} = \frac{1}{2}\sin^2\frac{\theta}{2}$$

$$\mathcal{P}_{+-} = \mathcal{P}_{-+} = \frac{1}{2}\cos^2\frac{\theta}{2}$$

Now use these results to calculate the average probability that the results are the same $(\mathcal{P}_{same} = \langle\mathcal{P}_{++} + \mathcal{P}_{--}\rangle)$ and the probability that the results are opposite $(\mathcal{P}_{opp} = \langle\mathcal{P}_{+-} + \mathcal{P}_{+-}\rangle)$, considering all possible measurements. There are 9 different combinations of measurement directions for the pair of observers.

The angle θ is 0° in 1/3 of the measurements ($\hat{\mathbf{a}}\hat{\mathbf{a}}, \hat{\mathbf{b}}\hat{\mathbf{b}}, \hat{\mathbf{c}}\hat{\mathbf{c}}$) and 120° in 2/3 of the measurements ($\hat{\mathbf{a}}\hat{\mathbf{b}}, \hat{\mathbf{a}}\hat{\mathbf{c}}, \hat{\mathbf{b}}\hat{\mathbf{a}}, \hat{\mathbf{b}}\hat{\mathbf{c}}, \hat{\mathbf{c}}\hat{\mathbf{a}}, \hat{\mathbf{c}}\hat{\mathbf{b}}$), so the average probabilities are

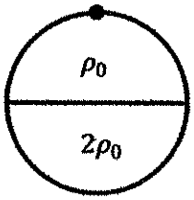
$$\mathcal{P}_{same} = \frac{1}{3}\cdot\sin^2\frac{0^\circ}{2} + \frac{2}{3}\cdot\sin^2\frac{120^\circ}{2} = \frac{1}{3}\cdot 0 + \frac{2}{3}\cdot\frac{3}{4} = \frac{1}{2}$$

$$\mathcal{P}_{opp} = \frac{1}{3}\cdot\cos^2\frac{0^\circ}{2} + \frac{2}{3}\cdot\cos^2\frac{120^\circ}{2} = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot\frac{1}{4} = \frac{1}{2}$$

These predictions of quantum mechanics are inconsistent with the inequalities derived from hidden variable theories. These probabilities have been experimentally measured to test whether local hidden variable theories are possible. The results agree with quantum mechanics and hence exclude the possibility of local hidden variable theories.

Comprehensive Examination, Spring 2009 – Problem 6

The two halves of a thin circular disk of radius R are composed of different materials whose densities are ρ_0 and $2\rho_0$. The disk is suspended symmetrically from a pivot on the rim in such a way that the diameter separating the two halves is horizontal at equilibrium:



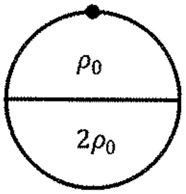
The disk is free to swing back and forth in the plane of the disk.

(a) Find the location of the center of mass of the disk.

(b) Determine the moment of inertia for rotation about the pivot point.

(c) Calculate the frequency of small angle oscillations of the disk.

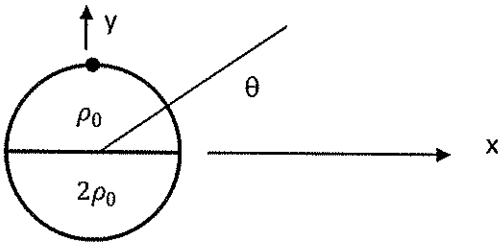
The two halves of a thin circular disk of radius *R* are composed of different materials whose densities are ρ_0 and $2\rho_0$. The disk is suspended symmetrically from a pivot on the rim in such a way that the diameter separating the two halves is horizontal at equilibrium:



The disk is free to swing back and forth in the plane of the disk.

(a) Find the location of the center of mass of the disk.

Set up a coordinate system with the origin at the center of the disk:



By symmetry, the CM will on a vertical diameter, i.e. $X = 0$. Then,

$$Y = \frac{1}{M}\int y\, dm = \frac{1}{M}\int y\,\rho\, t\, dA$$

where *t* is the thickness of the disk and *dA* is the differential area element.

$dA = r\, dr\, d\theta$ and $y = r\sin\theta$.

Therefore, $Y = \frac{\rho_0 t}{M}\left[2\int_0^R r^2 dr \int_\pi^{2\pi} \sin\theta\, d\theta \ + \int_0^R r^2 dr \int_0^\pi \sin\theta\, d\theta\right]$

$$Y = \frac{\rho_0 t}{M}\int_0^R r^2 dr\ \left[2\int_\pi^{2\pi} \sin\theta\, d\theta + \int_0^\pi \sin\theta\, d\theta\right] = \frac{\rho_0 t}{M}\int_0^R r^2\, dr \int_\pi^{2\pi} \sin\theta\, d\theta = -\frac{2}{3}\frac{\rho_0 t}{M}R^3$$

Now, note that $M = \frac{3}{2}\pi R^2 \rho_0 t \quad \rightarrow \quad \frac{\rho_0\, t}{M} = \frac{2}{3\pi R^2}$ and $Y = -\frac{4}{9\pi}R,\ X = 0.$

(b) Determine the moment of inertia for rotation about the pivot point.

First, find the moment of inertial through the center of the disk:

$$I_C = \int_{lower} r^2\ 2\rho_0 t\, dA\ +\ \int_{upper} r^2\ \rho_0 t\, dA$$

$$= \rho_0 t\left[2\int_0^R r^3\, dr \int_\pi^{2\pi} d\theta\ +\ \int_0^R r^3\, dr \int_0^\pi d\theta\ \right] = \rho_0 t\left[2\frac{R^4}{4}\pi\ +\ \frac{R^4}{4}\pi\right] = \frac{3\pi\rho_0 t}{4}R^4$$

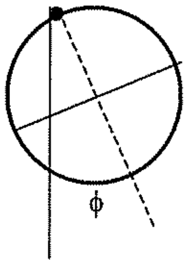
and since $\frac{\rho_0\, t}{M} = \frac{2}{3\pi R^2}$ we have $I_C = \frac{1}{2}MR^2$

Now, use the parallel axis theorem: $I_c = I_{CM} + Mh^2$ where $h = \frac{4}{9\pi}R$ is the distance from the center-of-mass to the center of the disk (from part (a)). And using the theorem again, the moment of inertial at the pivot will be

$$I_P = I_{CM} + M(R+h)^2 = I_C - Mh^2 + M(R+h)^2 = MR^2\left(\frac{3}{2} - \frac{8}{9\pi}\right).$$

(c) Calculate the frequency of small angle oscillations of the disk.

There are various ways to do this. A straightforward approach is to set up the equation of motion for the angular variable ϕ:



The height of the center of mass above the minimum (equilibrium) point is

$$y = (R+h)(1-\cos\phi)$$

so that the potential energy is

$$U = -MgR\left(1+\frac{4}{9\pi}\right)\cos\phi$$

Where we have dropped the constant term.

With the kinetic energy $T = \frac{1}{2}I_P\dot{\phi}^2$ the Lagrangian is $L = \frac{1}{2}I_P\dot{\phi}^2 \ + \ MgR\left(1+\frac{4}{9\pi}\right)\cos\phi$

The Lagrange equation is $\frac{\partial L}{\partial\phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = 0$ gives

$$I_P\ddot{\phi} + \quad MgR\left(1+\frac{4}{9\pi}\right)\sin\phi = 0$$

and for small amplitude oscillations, we get a harmonic oscillator equation

$$I_P\ddot{\phi} + \quad MgR\left(1+\frac{4}{9\pi}\right)\phi = 0$$

Rewriting,

$$\ddot{\phi} + \quad \frac{MgR}{I_p}\left(1+\frac{4}{9\pi}\right)\phi = \ddot{\phi} + \quad \frac{MgR}{MR^2}\frac{\left(1+\frac{4}{9\pi}\right)}{\left(\frac{3}{2}-\frac{8}{9\pi}\right)}\phi = \ddot{\phi} + \omega^2\phi = 0$$

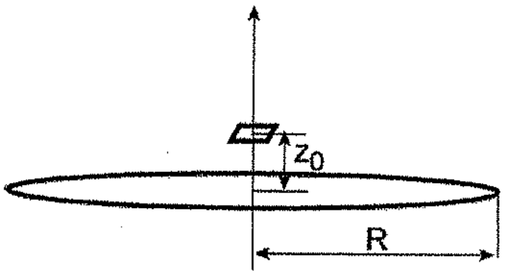
where $\omega^2 = \frac{g}{R}\left(\frac{1+\frac{4}{9\pi}}{\frac{3}{2}-\frac{8}{9\pi}}\right) = \frac{2g}{R}\left(\frac{9\pi+4}{27\pi-16}\right)$.

Comprehensive Examination, Spring 2009 – Problem 7

Magnetic dipole in inhomogeneous magnetic field experiences the force:

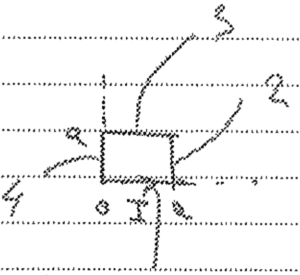
$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$$

1. Derive the above expression starting from the general expression for the Lorentz force by assuming that magnetic dipole is a square loop (side a) carrying current I_1
2. Calculate the force on the magnetic dipole positioned at the point $z = z_0$ along the axis of a relatively large circular loop (radius R) carrying current I_2. Consider two cases when (i) $\vec{m}$ is parallel to $+\hat{z}$ and when (ii) $\vec{m}$ is anti-parallel to $+\hat{z}$
3. Describe the motion of the magnetic dipole released at $z_0 \neq 0, z_0 \ll R$ in both cases



Solution to UG problem

① For simplicity, assume that the current lies in xy plane as shown in fig. 1



$$F_m = \int \bar{I}_1 \times \bar{B}\, dl = \left(\int_{(1)} + \int_{(2)} + \int_{(3)} + \int_{(4)} \right) I_1 \times B\, dl =$$

$$= \left[\int_{(1)} + \int_{(3)} \right] I_1 \times B\, dl + \left[\int_{(2)} + \int_{(3)} \right] I_1 \times \bar{B}\, dl =$$

$$= \int_{(1)} I_1 \times \left[B(x,0) - B(x,a) \right] dx + \int_{(2)} I \times \left[B(a,y) - B(0,y) \right] dy =$$

$$\left(B(x,y) \approx \bar{B}(0,0) + \frac{\partial \bar{B}}{\partial x}\Big|_{0,0} x + \frac{\partial \bar{B}}{\partial y}\Big|_{0,0} y \right.$$

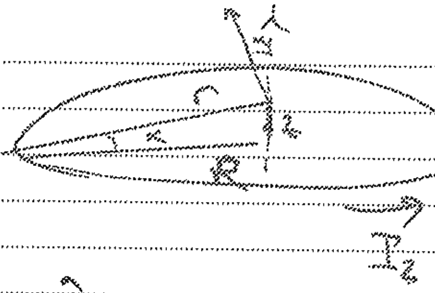
$$= -\int_{(1)} I_1 \times \left[\frac{\partial \vec{B}}{\partial y} \cdot a \right] dx + \int_{(2)} I_1 \times \frac{\partial \bar{B}}{\partial x}\, a\, dy = I_1 a^2 \cdot \left[+ \hat{y} \cdot \frac{\partial B_z}{\partial y} - \right.$$

$$\left. - \hat{z} \cdot \frac{\partial B_y}{\partial y} + \hat{x} \cdot \frac{\partial B_z}{\partial x} - \hat{z} \frac{\partial B_x}{\partial x} \right] = m_z \cdot \left[\hat{x} \frac{\partial B_z}{\partial x} + \hat{y} \frac{\partial B_z}{\partial y} + \right.$$

$$\left. + \hat{z} \underbrace{\left(-\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} \right)}_{\frac{\partial B_z}{\partial z} \quad \text{since } \operatorname{div} \bar{B} = 0} \right] = m_z \left[\hat{x} \frac{\partial B_z}{\partial x} + \hat{y} \frac{\partial B_z}{\partial y} + \hat{z} \frac{\partial B_z}{\partial z} \right] =$$

$$= \vec{\nabla} \left(\bar{m} \cdot \bar{B} \right)$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int\frac{I\times\hat{r}}{r^2} = \frac{I_2}{2}\frac{\mu_0\cos^3\alpha}{R^2}\,R$$



$$= \mu_0\frac{I}{2}\;\frac{R^3}{R\left(\sqrt{R^2+z_0^2}\right)^3} = \mu_0\frac{I_2}{2}\frac{R^2}{(R^2+z_0^2)^{3/2}}\hat{z}$$

$$F_{m_2} = \vec{\nabla}\left(\vec{m}\cdot B\right) = -3\,\mu_0\frac{I_2}{2}m_2\cdot\frac{R^2\,z_0}{(R^2+z_0^2)^{5/2}}$$

③ when $m_2 \parallel +\hat{z}$, $(I_1 \parallel I_2)$, the two loops attract, so the small loop oscillates along $\hat{z}$ axis

when $m_2 \parallel -\hat{z}$, $(I_1 \uparrow\downarrow I_2)$, the loops repelse. the smaller loop will fly away from the bigger one

Comprehensive Examination, Spring 2009 – Problem 8

It is a well-known phenomenon that even on sunny days with clear skies "cap clouds" form over mountain tops – especially, if the mountain is located not far from a large body of water, and there is an inland-blowing wind. On the attached page, one picture shows such a "cap cloud" over Mount Rainier, and the other shows a large cloud of this kind over the plateau of the famous Table Mountain, a prominent landmark overlooking the city of Cape Town, South Africa.

Due to the atmospheric pressure decrease with increasing altitude, moist air driven uphill by the wind expands in volume. The expansion is approximately adiabatic, so that the temperature drops. When the dew point for the water vapor carried by the air is reached, the vapor starts condensing into mist, and a cloud forms.

Your task in this problem is to quantitatively analyze the physics of the cloud formation process at Table Mountain.

(a) Assume that the barometric pressure just above the sea water ($h = 0$) is 100 kPa, and the air temperature is 293 K. Find the barometric pressure at the altitude of $h = 1100$ m above the sea level, assuming that the air temperature over the sea is approximately constant up to this altitude. Assume that atmospheric air obeys the ideal gas law. The molar mass of air is 2.9×10^{-2} kg/mole, and the "gas constant" $R = 8.314$ J/K·mole.

(b) The air blown uphill from the sea level up to the Table Mountain plateau, which is at 1100 m altitude, undergoes an adiabatic expansion. Find the temperature of the air when it reaches the plateau. Assume that the barometric pressure at the plateau is the same as that over the sea at $h = 1100$ m.

(c) Assume that humidity of the air blown inland from the sea is 100% (i.e., the H_2O vapor it carries is a *saturated vapor*. Knowing that at $T = 293$ K the pressure of saturated H_2O vapor is 2.339 kPa, that the latent heat of water evaporation at $T = 293$ K is $l = 4.417 \times 10^4$ J/mole, and using the result from (b), determine the saturated vapor pressure in the cooled air blowing over the Table Mountain plateau. Next, find what fraction of the vapor originally carried by the air over the sea changes into a mist.

Hints: These formulae may be helpful:

$$\mu = k_B T \ln\left(\frac{n}{n_Q}\right)$$

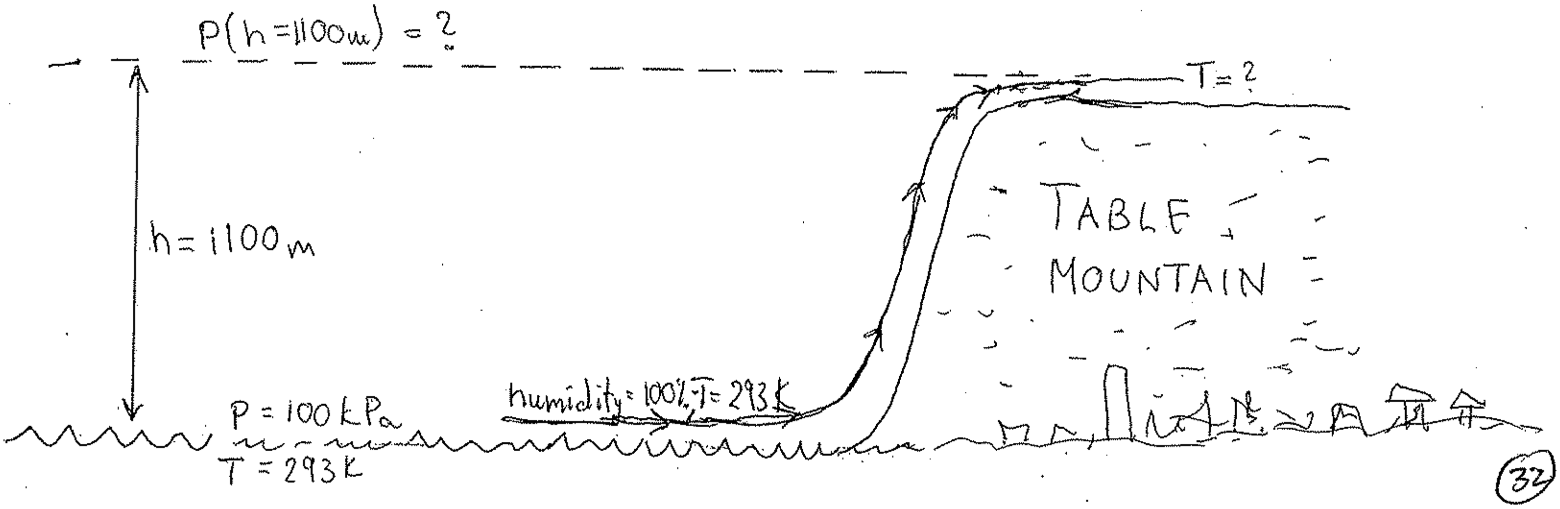
For atmospheric air one can use:

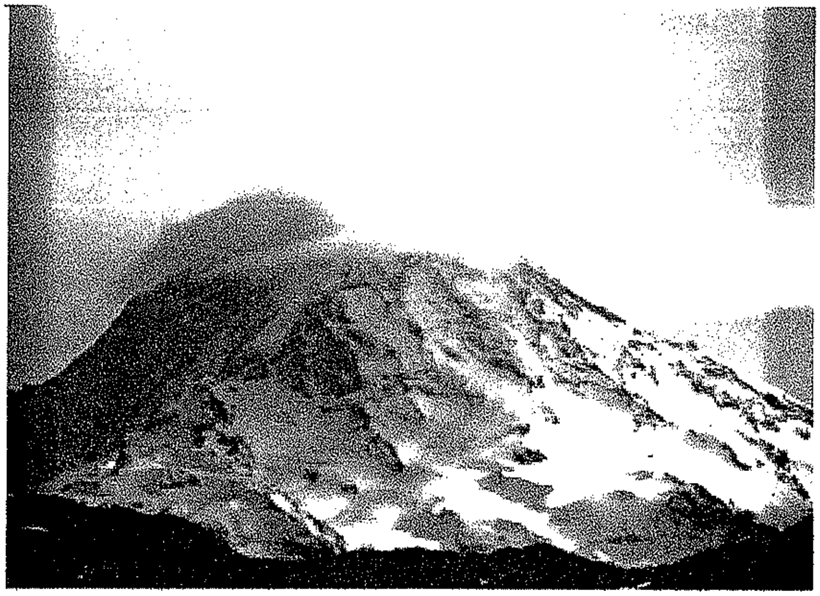
$$s = s_0(u_0, v_0) + R\ln(u^{5/2} \cdot v) \quad \text{and} \quad u = \frac{5}{2}RT$$

The Clapeyron Equation:

$$\frac{dp}{dT} = \frac{l}{T\Delta v}$$

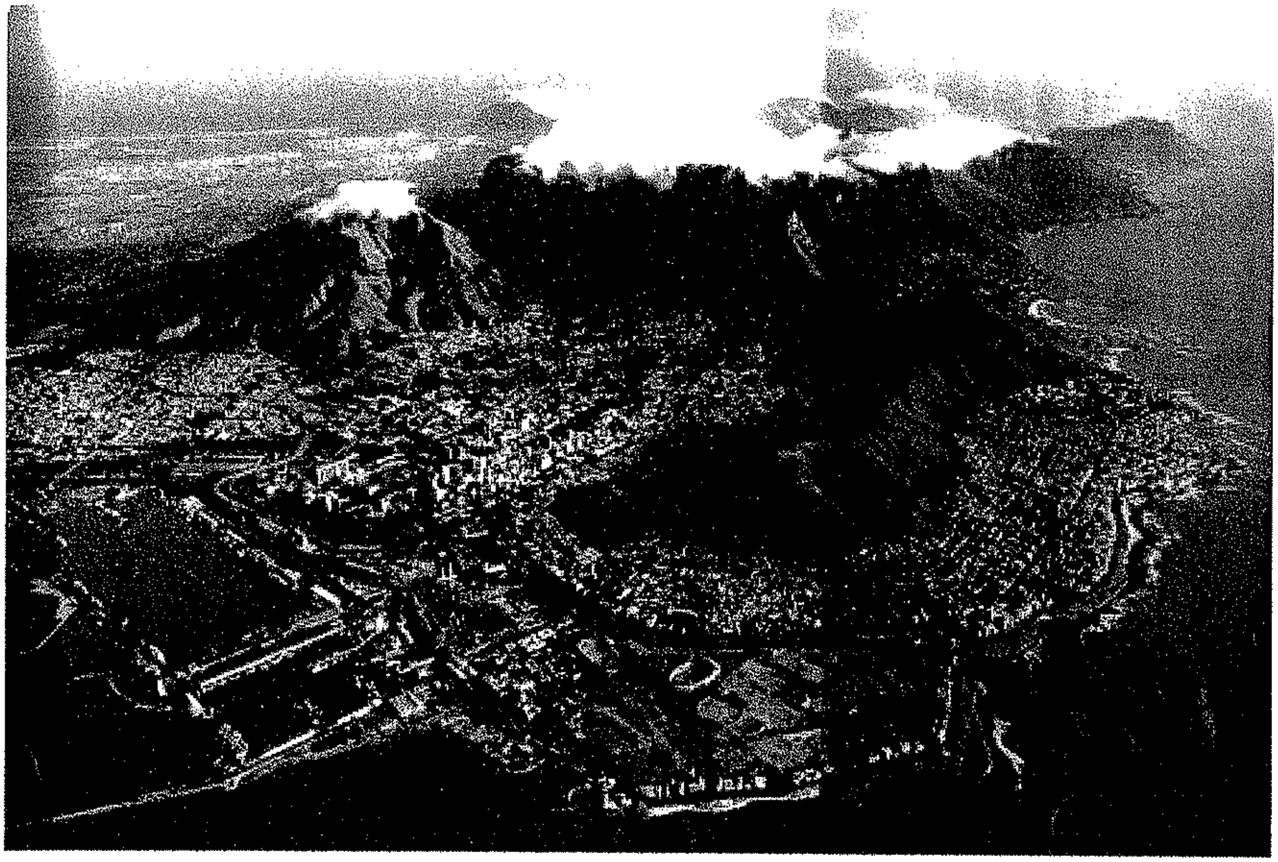
Since at $T = 293$ K the molar volume of water vapor is about 60 000 times larger, than that of liquid water, it is fair to use $\Delta v \approx v_{vap.}$; and it is a reasonable assumption that water vapor at $T = 293$ K obeys the ideal gas law.





Left: A cap cloud over Mt. Rainier

Below: an aerial view of a large cap cloud (called a "tablecloth cloud" by the locals) over the Table Mountain at Cape Town, South Africa.



Left: Table Mountain from another perspective

Solution, problem 8

(a) The effective chemical potential is the sum a "thermal part" $kT\ln n$ and the particle potential energy, and in an atmospere in equlibrium it is constant, so that

$$kT\ln\left(\frac{n}{n_Q}\right) + mgh = \text{const.}$$

Therefore $kT\ln\left[\frac{n(h=0)}{n_Q}\right] = kT\ln\left[\frac{n(h)}{n_Q}\right] + mgh$

$$\ln\left[\frac{n(h)}{n(0)}\right] = -\frac{mgh}{kT} \Rightarrow n(h) = n(0)\,e^{-\frac{mgh}{kT}}$$

From the ideal gas law, $pV = NkT \Rightarrow p = \frac{N}{V}kT = nkT$

(V – total volume, N – # of particles, so that $N/V = n$, concentr.)

Therefoe, for constant T, $p \propto$, so that $p(h) = p(0)\,e^{-\frac{mgh}{kT}}$

But particle mass = (molar mass)/(Avogadro number)

and $k \cdot$ (Avogadro #) $= R$, which leads to the final formula: $p(h) = p(0) \cdot \exp\left[-\frac{(\text{molar mass}) \cdot g \cdot h}{RT}\right]$

Plugging in the numbers, one obtains:

$$p(1100\,\text{m}) = 10^2\,\text{kB} \cdot \exp\left[-\frac{0.029\,\text{kg/mole} \cdot 10\,\text{m/s}^2 \cdot 1100\,\text{m}}{8.314\,\text{J/K}\cdot\text{mole} \cdot 293\,\text{K}}\right]$$

$$= 10^2\,\text{kB} \cdot e^{-0.131} = 87.7\,\text{kB}$$

(b)

Since in adiabatic expansion the entropy does not change, so $S = S_0(u_0, v_0) + R\ln(u^c \cdot v) = \text{const}$ which means that $u^c \cdot v = \text{const}$.

Using the gas energy, $u = cRT$, and the gas law, $pv = RT$, we get $(cRT)^c \cdot \frac{RT}{p} = \text{const}$

or: $\frac{T^{c+1}}{p} = \text{const}'$, or $T \cdot p^{\frac{-1}{c+1}} = \text{const}''$

Hence: $T(1100\,m) = T(0\,m) \cdot \left[\frac{p(0)}{p(1100\,m)}\right]^{-\frac{1}{c+1}}$

$= 293\,K \cdot \left[\frac{100\,kPa}{87.7\,kPa}\right]^{-\frac{2}{7}} = 282.2\,K$

$$\boxed{\begin{aligned} c &= 2.5 = \tfrac{5}{2} \\ c+1 &= \tfrac{7}{2} \\ -\tfrac{1}{c+1} &= -\tfrac{2}{7} \end{aligned}}$$

The temperature will drop by $-10.8\,K$.

(c). The Clapeyrone Equation: $\frac{dp}{dT} = \frac{l}{T\Delta v}$. Here p is the pressure at which two phases (e.g., water and water vapor) are in equlibrium at temperature T. l is the latent heat of the phase transition, and Δv is the change in the molar volume. Taking $pv = RT \Rightarrow v = \frac{RT}{p}$ we find that the molar volume of saturated water vapor at 293K is: $v = \frac{8.314\,J/K\text{-}mole \cdot 293\,K}{2,339} = 1.04\,m^3/mole$

while $\overset{(for)}{}$ liquid water it is $\sim 18\,cm^3 = 1.8 \times 10^{-5}\,m^3/mole$

Therefore, we can take $\Delta v \approx v_{vapor}$

From the ideal gas law, $v_{vapor} = RT/p_{vapor}$, and we get:

$$\frac{dp_{VAP}}{dT} \simeq \frac{p_{vap} \cdot l}{T^2 R} \quad \text{or} \quad \frac{dp_{VAP}}{p_{vap}} = \frac{l}{R}\frac{dT}{T^2}$$

Then:

$$\int_{T_i}^{T_f} \frac{dp_{VAP}}{p_{vap}} = \frac{l}{R}\int_{T_i}^{T_f} \frac{dT}{T^2} \Rightarrow \ln\frac{p_{vap}(T_f)}{p_{vap}(T_i)} = \frac{l}{R}\left(\frac{1}{T_i} - \frac{1}{T_f}\right)$$

$$\Rightarrow p_{vap}(T_f) = p_{vap}(T_i) \cdot \exp\left[\frac{l}{R}\left(\frac{1}{T_i} - \frac{1}{T_f}\right)\right]$$

$$p_{vap}(282.2K) = 2.339\,kPa \cdot \exp\left[\frac{4.417\times10^4}{8.314\,J/K\text{-}mole}\left(\frac{1}{293K} - \frac{1}{282.2K}\right)\right]$$

$$= 1.169\ kPa$$

It is almost exactly 50% of the initial value, so that one can conclude that approximately 50% of the water (vapor) condensates into mist.

This is an approximation because, in fact, the air that carries the vapor expands. The volume and the temperature of the vapor changes by the same rate as the volume and T of the air. So, since the air pressure drops to 87.7% of its initial value, so does the vapor pressure – if it did not condensate, i.e., it stayed in a "supersaturated" state, its pressure would be 87.7% of 2.339 kPa, which is 2.052 kPa. It is this number that should be compared with the saturated vapor pressure at 282.2K: 1.169 kPa/2.052 kPa = 57% would remain in the vapor state, and 43% would condensate. So, the 50% we obtained before is not a bad estimate.