

OSU PHYSICS DEPARTMENT  
COMPREHENSIVE EXAMINATION #103

September 24 and 25, 2007

Comprehensive examination for Fall 2007

PART 1, Monday September 24, 9:00 am

General Instructions

This Comprehensive Examination for Fall 2007 consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, September 24, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, September 25, at 9:00 am and 1:30 pm.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

OSU Physics Department Comprehensive Exam Fall 2007

Problem 1

Consider a 3-state quantum mechanical system whose Hamiltonian  $H_0$  has eigenstates labeled  $|1\rangle$ ,  $|2\rangle$ , and  $|3\rangle$ , with corresponding energies  $E_1 = 3E_0$ ,  $E_2 = 3E_0$ , and  $E_3 = E_0$ . A perturbation  $W$  is applied to the system and is characterized by the following equations:

$$W|1\rangle = a|1\rangle + b|2\rangle + c|3\rangle$$

$$W|2\rangle = b|1\rangle + a|2\rangle + d|3\rangle$$

$$W|3\rangle = c|1\rangle + d|2\rangle$$

where  $a$ ,  $b$ ,  $c$ , and  $d$  are positive, real constants with dimensions of an energy.

- a) Write down the matrix representation of the full Hamiltonian  $H = H_0 + W$  in the basis defined by the states  $|1\rangle$ ,  $|2\rangle$ , and  $|3\rangle$ .
  
- (b) Find the first non-zero correction to each energy level caused by the perturbation.

a) 3 state system

$$|1\rangle \rightarrow E_1^{(0)} = 3E_0$$

$$|2\rangle \rightarrow E_2^{(0)} = 3E_0$$

$$|3\rangle \rightarrow E_3^{(0)} = E_0$$

unperturbed energies

$$\Rightarrow H_0 = \begin{pmatrix} 3E_0 & 0 & 0 \\ 0 & 3E_0 & 0 \\ 0 & 0 & E_0 \end{pmatrix}$$

Perturbation  $W$ :

$$W|1\rangle = a|1\rangle + b|2\rangle + c|3\rangle$$

$$W|2\rangle = b|1\rangle + a|2\rangle + d|3\rangle$$

$$W|3\rangle = c|1\rangle + d|2\rangle$$

$$\Rightarrow \langle 1|W|1\rangle = a$$

$$\langle 1|W|2\rangle = b$$

$$\langle 1|W|3\rangle = c$$

etc

$$\Rightarrow W = \begin{pmatrix} a & b & c \\ b & a & d \\ c & d & d \end{pmatrix}$$

$$\Rightarrow H = H_0 + W = \begin{pmatrix} 3E_0 + a & b & c \\ b & 3E_0 + a & d \\ c & d & E_0 \end{pmatrix}$$

note that it is Hermitian as expected

b) levels  $|1\rangle, |2\rangle$  are degenerate  
 level  $|3\rangle$  is non-degenerate  
 so we need to use appropriate  
 perturbation theory.

Do level  $|3\rangle$  first:

$$E_n^{(1)} = \langle n | W | n \rangle$$

$$\Rightarrow E_3^{(1)} = \langle 3 | W | 3 \rangle = 0$$

$\Rightarrow$  no 1<sup>st</sup> order correction  
 so go to 2<sup>nd</sup> order

$$E_n^{(2)} = \sum_{k \neq n} \frac{|\langle k | W | n \rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$\Rightarrow E_3^{(2)} = \frac{|\langle 1 | W | 3 \rangle|^2}{E_3^{(0)} - E_1^{(0)}} + \frac{|\langle 2 | W | 3 \rangle|^2}{E_3^{(0)} - E_2^{(0)}}$$

$$= \frac{c^2}{E_0 - 3E_0} + \frac{d^2}{E_0 - 3E_0}$$

$$E_3^{(2)} = -\frac{c^2 + d^2}{2E_0}$$

level pushed down  
 (repelled from  $|1\rangle, |2\rangle$ )  
 as expected.

For levels  $|1\rangle, |2\rangle$  we deg. pert. th.,  
 which means diagonalize pert. in  
 degenerate subspace spanned by  $|1\rangle, |2\rangle$   
 call this  $W'$ :

$$W' = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)^2 - b^2 = 0$$

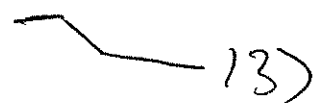
$$a-\lambda = \pm b$$

$$\Rightarrow \lambda = a \pm b$$

$$\Rightarrow \begin{cases} E_1^{(1)} = a + b \\ E_2^{(1)} = a - b \end{cases}$$

So perturbed energies are

$$\begin{cases} \tilde{E}_1 = 3E_0 + a + b \\ \tilde{E}_2 = 3E_0 + a - b \\ \tilde{E}_3 = E_0 - \frac{c^2 + d^2}{2E_0} \end{cases}$$



Problem Th-1 (“undergraduate level”)

Any object of finite temperature is the source of thermal radiation. The total flux of radiative energy  $\phi$  emitted by a unit surface area of an ideal blackbody is given by the known Stefan-Boltzmann Law:  $\phi = \sigma T^4$ , where  $\sigma$  is the Stefan-Boltzmann constant.

In vacuum-insulated cryogenic vessels (Dewars) and in modern refrigerators used in low-temperature physics studies, thermal radiation through the vacuum jacket is the major source of heat transferred to the inner low-temperature container. A technique of reducing this heat transfer is to place “heat shields” in the vacuum space between the inner and the outer containers.

Idealize this situation by considering two infinite plates (labelled as L and H, see Figs. 1a - 1c), made of perfect blackbody material, and separated by a vacuum space. The temperatures of the plates are, respectively,  $T_L$  and  $T_H$  ( $T_L < T_H$ ).

- Calculate the energy flux (at equilibrium) between the plates when there is no extra shield between them (Fig. 1a).
- Next, suppose that a single infinite sheet (a heat shield), also made of a perfect blackbody material, is placed between the two plates (the sheet labelled as “1” in Fig. 1b). This shield is not in direct thermal contact with any other body – energy is transferred to it and out of it only through blackbody absorption/emission. Find the equilibrium temperature  $T_1$  of the shield, and the effective energy flux from plate H to plate L when the shield is in place.
- Then, suppose that  $N$  such sheets are placed between the L and H plates (Fig. 1c). Calculate the effective equilibrium energy flux between the H and L plates when such a “multi-layered shield” is applied. Find the temperature of the first sheet ( $T_1$ ) and of the last sheet ( $T_N$ ).

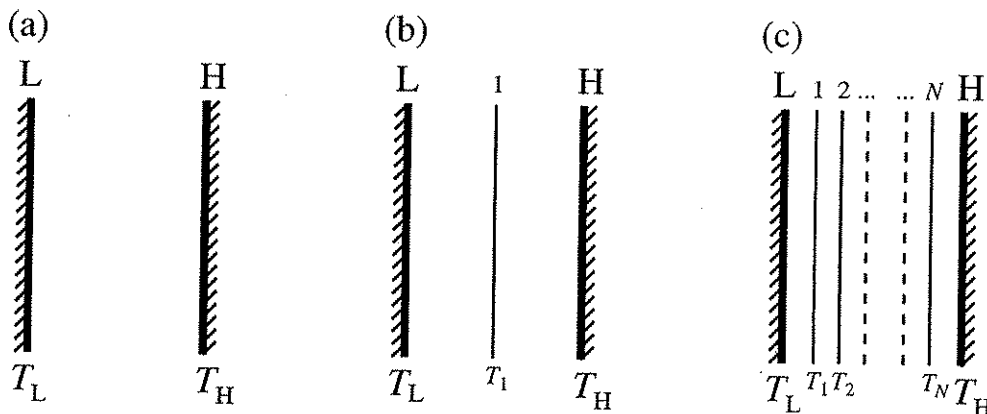


Figure 1: (a) Two infinite blackbody plates separated by a vacuum space; (b) the two plates with “heat shield” consisting of single blackbody sheet in between; (c) the two plates with a “heat shield” consisting of  $N$  parallel blackbody sheets.

**Problem Th-1, solutions:**

**Task (a):** The energy flux radiated by the H plate, and absorbed by the L plate is  $\phi_{H \rightarrow L} = \sigma T_H^4$ , where  $\sigma$  is the Stefan-Boltzmann constant. The flux radiated by the L plate and absorbed by the H plate is  $\phi_{L \rightarrow H} = \sigma T_L^4$ . Thus, the effective flux between the plates with no shield in place is

$$\phi_0^{\text{eff}} = \phi_{H \rightarrow L} - \phi_{L \rightarrow H} = \sigma(T_H^4 - T_L^4).$$

where the subscript "0" in the  $\phi_0^{\text{eff}}$  is supposed to mean "no (zero) sheets between the plates".

**Task (b):** In a system with a single sheet, we can identify four fluxes:

$$\phi_{H \rightarrow 1} = \sigma T_H^4 \quad - \quad \text{flux emitted by plate H and absorbed by sheet 1;}$$

$$\phi_{L \rightarrow 1} = \sigma T_L^4 \quad - \quad \text{flux emitted by plate L and absorbed by sheet 1;}$$

$$\phi_{1 \rightarrow H} = \sigma T_1^4 \quad - \quad \text{flux emitted by sheet 1 and absorbed by plate H;}$$

$$\phi_{1 \rightarrow L} = \sigma T_1^4 \quad - \quad \text{flux emitted by sheet 1 and absorbed by plate L.}$$

Because the system is in equilibrium, the sum of the fluxes absorbed by sheet 1 must be equal to the sum of the fluxes emitted by it:

$$\phi_{H \rightarrow 1} + \phi_{L \rightarrow 1} = \phi_{1 \rightarrow H} + \phi_{1 \rightarrow L}$$

meaning that:

$$\sigma T_H^4 + \sigma T_L^4 = \sigma T_1^4 + \sigma T_1^4$$

from which one obtains:

$$T_1^4 = \frac{1}{2}(T_H^4 + T_L^4)$$

or:

$$T_1 = \left[ \frac{1}{2}(T_H^4 + T_L^4) \right]^{1/4}.$$

Now, the effective flux transferred from plate H to sheet 1 can be readily found:

$$\phi_{H \rightarrow 1}^{\text{eff}} = \sigma T_H^4 - \sigma T_1^4 = \sigma \left[ T_H^4 - \frac{1}{2}(T_H^4 + T_L^4) \right] = \frac{1}{2}\sigma(T_H^4 - T_L^4) = \frac{1}{2}\phi_0^{\text{eff}}$$

And the effective flux transferred from sheet 1 to plate L is:

$$\phi_{1 \rightarrow L}^{\text{eff}} = \sigma T_1^4 - \sigma T_L^4 = \sigma \left[ \frac{1}{2}(T_H^4 + T_L^4) - T_L^4 \right] = \frac{1}{2}\sigma(T_H^4 - T_L^4) = \frac{1}{2}\phi_0^{\text{eff}}.$$

As expected, both these fluxes are equal because of the equilibrium – and they are also equivalent with the effective flux transfer from plate H to plate L. So, when a single plate is used as a "heat shield", the effective energy transfer  $\phi_1^{\text{eff}}$  between the plates is one-half of that when no shield is used:

$$\phi_1^{\text{eff}} = \frac{1}{2}\phi_0^{\text{eff}}$$

Or, in other words, a "heat shield" consisting of a single sheet reduces the radiative energy transfer by 50%.

**Task (c):** When the “heat shield” consists of  $N$  sheets and the system is in equilibrium, one can write the following equations for the effective fluxes transferred between plate H and sheet  $N$ , sheet  $N$  and sheet  $N - 1$ , ... , sheet 2 and sheet 1, and sheet 1 and plate L:

$$\begin{aligned}\phi_{\text{H} \rightarrow \text{N}}^{\text{eff}} &= \sigma T_{\text{H}}^4 - \sigma T_{\text{N}}^4 \\ \phi_{\text{N} \rightarrow \text{N}-1}^{\text{eff}} &= \sigma T_{\text{N}}^4 - \sigma T_{\text{N}-1}^4 \\ \phi_{\text{N}-1 \rightarrow \text{N}-2}^{\text{eff}} &= \sigma T_{\text{N}-1}^4 - \sigma T_{\text{N}-2}^4 \\ &\dots = \dots \\ &\dots = \dots \\ \phi_{2 \rightarrow 1}^{\text{eff}} &= \sigma T_2^4 - \sigma T_1^4 \\ \phi_{1 \rightarrow \text{L}}^{\text{eff}} &= \sigma T_1^4 - \sigma T_{\text{L}}^4\end{aligned}$$

Because of the equilibrium, all these fluxes are the same, and are equal to the effective energy flux between the plate H and the plate L, which we now denote as  $\phi_{\text{N}}^{\text{eff}}$ . If we now add all these equations, all right-side terms cancel out with the exception of two, and we obtain:

$$(N + 1)\phi_{\text{N}}^{\text{eff}} = \sigma(T_{\text{H}}^4 - T_{\text{L}}^4) = \phi_0^{\text{eff}}$$

or:

$$\phi_{\text{N}}^{\text{eff}} = \frac{1}{N + 1}\phi_0^{\text{eff}}$$

So, a multi-layered “heat shield” consisting of  $N$  sheets reduces the heat transfer  $(N + 1)$  times compared to the “no-shield” situation.

One can readily obtain expressions for the temperatures of the first and last sheet by solving the equations for the effective fluxes  $\phi_{\text{H} \rightarrow \text{N}}^{\text{eff}}$  and  $\phi_{1 \rightarrow \text{L}}^{\text{eff}}$ :

$$\phi_{\text{H} \rightarrow \text{N}}^{\text{eff}} = \sigma(T_{\text{H}}^4 - T_{\text{N}}^4) = \frac{\sigma}{N + 1}(T_{\text{H}}^4 - T_{\text{L}}^4)$$

and

$$\phi_{1 \rightarrow \text{L}}^{\text{eff}} = \sigma(T_1^4 - T_{\text{H}}^4) = \frac{\sigma}{N + 1}(T_{\text{H}}^4 - T_{\text{L}}^4)$$

from which one obtains:

$$T_{\text{N}} = \left[ T_{\text{H}}^4 - \frac{T_{\text{H}}^4 - T_{\text{L}}^4}{N + 1} \right]^{1/4}$$

and

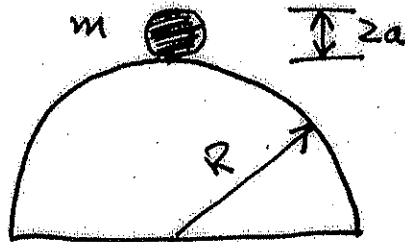
$$T_1 = \left[ T_{\text{L}}^4 + \frac{T_{\text{H}}^4 - T_{\text{L}}^4}{N + 1} \right]^{1/4}$$

*Comment:* Having calculated  $T_1$  and  $T_{\text{N}}$ , one can in a similar way obtain expressions for  $T_2$  and  $T_{\text{N}-1}$ , then for  $T_3$  and  $T_{\text{N}-2}$ , and so on. It can be shown that the general expression for the temperature of the  $i^{\text{th}}$  sheet is:

$$T_i = \left[ T_{\text{L}}^4 + i \frac{T_{\text{H}}^4 - T_{\text{L}}^4}{N + 1} \right]^{1/4}$$

However, showing this requires additional calculations that are straightforward, but rather time-consuming, and therefore finding the general expression for  $T_i$  is not on the task list in the problem. Nevertheless, we show this expression here, because deriving it may be a good exercise for students preparing for the future comprehensive exams.

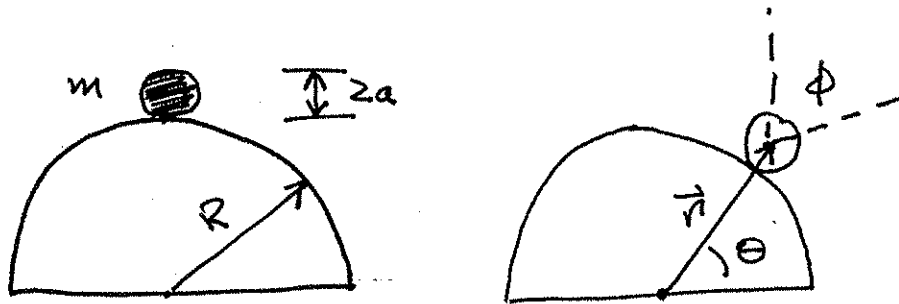
A disk of mass  $m$  and radius  $a$  rolls without slipping from rest at the top of a hemisphere of radius  $R$ .



- (i) Obtain the equations of motion in terms of appropriate generalized coordinates.
- (ii) Derive an expression for the point at which the disk leaves the surface of the hemisphere.
- (iii) What will be the angular velocity of the disk when it leaves the hemisphere?

Problem 3.

A disk of mass  $M$  and radius  $a$  rolls without slipping from rest at the top of a hemisphere of radius  $R$ .



- (i) Obtain the equations of motion in terms of appropriate generalized coordinates.

Let the coordinates be:

- $r$   $\equiv$  unconstrained radial position of the center of mass of the disk
- $\theta$   $\equiv$  angular coordinate of the center of mass of the disk
- $\phi$   $\equiv$  angular orientation of the rolling disk.

Then the Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr\dot{\theta}^2 + \frac{1}{2}I\dot{\phi}^2 - mgr \cos \theta$$

Now, rolling without slipping implies  $R\theta = a\phi$ . The moment of inertia of a disk rotating about an axis perpendicular to its plane is  $I = \frac{1}{2}ma^2$ . Then the Lagrangian becomes

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\left(mr^2 + \frac{1}{2}mR^2\right)\dot{\theta}^2 - mgr \cos \theta.$$

The constraint when the disk is rolling on the hemisphere is described by the function  $f(r, \theta) = r - a - R = 0$ .

Now, write the Lagrange equations with an undetermined multiplier ( $\lambda$ ), which will represent the force of constraint:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) + \lambda \frac{\partial f}{\partial q_i} = 0 \quad \text{where} \quad q_i = r, \theta$$

For r:  $mr\dot{\theta}^2 - mg \cos \theta - m\ddot{r} + \lambda \cdot 1 = 0$ . But  $r = a + R$  and  $\dot{r} = 0$  so that

$$m(a+R)\dot{\theta}^2 - mg \cos \theta + \lambda = 0 \quad (1)$$

For  $\theta$ :  $mgr \sin \theta - \left( mr^2 + \frac{1}{2}mR^2 \right) \ddot{\theta} - 2mr\dot{\theta} + \lambda \cdot 0 = 0$

$$r = a + R \quad \dot{r} = 0$$

$$g(a+R)\sin \theta - \left[ (a+R)^2 + \frac{1}{2}R^2 \right] \ddot{\theta} = 0 \quad (2)$$

Equations (1) and (2) are the equations of motion in terms of the generalized coordinates  $r$  and  $\theta$ .

- (ii) Derive an expression for the point at which the disk leaves the surface of the hemisphere.

We need to find the force of constraint, lamda, and then determine the point at which it vanishes. First, solve (2) for  $\ddot{\theta}^2$ , then substitute in (1) and set  $\lambda = 0$ .

Eq. (2) gives:

$$\ddot{\theta} = \dot{\theta} \frac{d\dot{\theta}}{d\theta} = c_1 \sin \theta \quad \text{where} \quad c_1 \equiv \frac{g(a+R)}{\left[ (a+R)^2 + \frac{1}{2}R^2 \right]}$$

$$\dot{\theta} d\dot{\theta} = c_1 \sin \theta d\theta$$

Integrate:  $\frac{1}{2}\dot{\theta}^2 = -c_1 \cos \theta + c_2$

Disk starts from rest ( $\dot{\theta} = 0$  @  $\theta = 0$ ) so that  $c_2 = c_1$  and  $\dot{\theta}^2 = 2c_1(1 - \cos \theta)$ .

Then, from Eq. (1),

$$\begin{aligned}
\lambda &= mg \cos \theta - m(a + R)\dot{\theta}^2 \\
&= mg \cos \theta - \frac{2mg(a + R)^2}{\left[(a + R)^2 + \frac{1}{2}R^2\right]}(1 - \cos \theta) \\
&= mg \left\{ \frac{3(a + R)^2 + \frac{1}{2}R^2}{\left[(a + R)^2 + \frac{1}{2}R^2\right]} \right\} \cos \theta - \frac{2mg(a + R)^2}{\left[(a + R)^2 + \frac{1}{2}R^2\right]}
\end{aligned}$$

For simplicity, let  $x \equiv \frac{R}{a+R}$ . Then  $\lambda = mg \left[ \frac{3 + x^2/2}{1 + x^2/2} \right] \cos \theta - \frac{2mg}{1 + x^2/2}$ .

The disk will leave the surface when  $\lambda = 0$ , i.e. when

$$\cos \theta = \frac{2}{3 + x^2/2} \quad \text{where } x = \frac{R}{a + R}.$$

(iii) What will be the angular velocity of the disk when it leaves the hemisphere?

$$\text{Recall } \dot{\theta}^2 = \left\{ \frac{2g(a + R)}{\left[(a + R)^2 + R^2/2\right]} \right\} (1 - \cos \theta) = \frac{2g}{a + R} \cdot \frac{(1 - \cos \theta)}{1 + x^2/2}.$$

At the point where  $\lambda = 0$ ,

$$\dot{\theta}^2 = \frac{2g}{a + R} \cdot \frac{1}{1 + x^2/2} \cdot \left( 1 - \frac{2}{3 + x^2/2} \right) = \frac{2g}{a + R} \cdot \frac{1}{3 + x^2/2}.$$

Angular velocity:

$$\dot{\phi} = \left( \frac{R}{a} \right) \dot{\theta} = \left( \frac{R}{a} \right) \sqrt{\frac{2g}{a + R} \cdot \frac{1}{3 + x^2/2}}$$

OSU Physics Comprehensive Exam, Fall 2007, Problem 4

Find the capacitance of the two parallel cylindrical conductors with radii  $R_1$  and  $R_2$  separated by the distance  $d$  (see Fig.1).

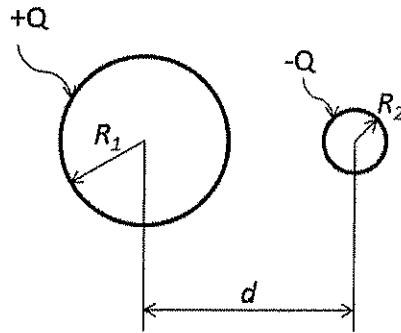
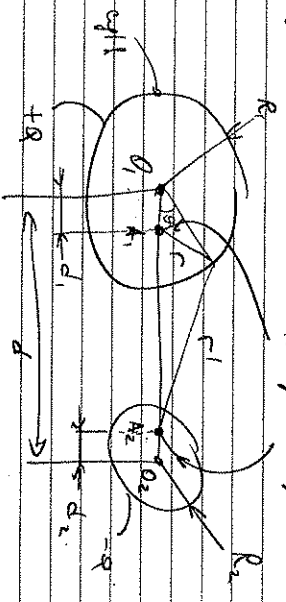


Figure 1

Hint: you may use method of images

According to the method of images original two cylinder system should be replaced by a set of live charges yielding the constant potential @ the cylinder surfaces

Two image charges are located on the line connecting the centers of the cylinders. The charges of image charges are equal to those of cylinders



The location of image charges can be found from the condition:

$$d_1 \cdot (d - d_2) = R_1^2 ; d_2 \cdot (d - d_1) = R_2^2$$

Leading to:

$$\frac{C}{\epsilon_0} = \frac{R_1}{QA_2} = \frac{R_1}{d - d_2} = \frac{d_1}{R_1}$$

From the potential @ left cylinder:

$$V = 2Q \ln \frac{R_1}{R_1 + (d - d_2)} = 2Q \ln \frac{R_1}{R_1 + (d - d_2)}$$

$$= Q \ln \frac{R_1^2 + d_1^2 - 2R_1 d_1 \cos \theta}{R_2^2 + \frac{R_1^2}{d_1^2} - 2 \frac{R_1^2}{d_1} \cos \theta} = Q \ln \frac{d_1^2}{R_1^2} = \text{const}$$

and similar for the second cylinder.

Potentials @ the cylinders:

$$V_1 = 2Q \ln \frac{R_1}{R_1} = 2Q \ln \frac{d_1}{R_1}$$

$$V_2 = -2Q \ln \frac{d_2}{R_2}$$

$$C = \frac{Q}{V_1 - V_2} = \frac{Q}{2Q \ln \left( \frac{d_1 R_2}{d_2 R_1} \right)}$$

$$\ln \frac{d_1 R_2}{d_2 R_1} = y ; \frac{d_1}{d_2} \frac{R_2}{R_1} = e^y ; \frac{d_2}{d_1} \frac{R_1}{R_2} = e^{-y}$$

$$\frac{d_1}{d_2} \frac{R_2}{R_1} + \frac{d_2}{d_1} \frac{R_1}{R_2} = 2 \cosh y$$

$$\frac{d_1}{d_2} \frac{R_2}{R_1} + \frac{d_2}{d_1} \frac{R_1}{R_2} = \frac{d_1^2 R_2^2 + d_2^2 R_1^2}{d_1 d_2 R_1 R_2} =$$

$$= \frac{1}{R_1 R_2} \left( d_1^2 \frac{R_2^2}{d_2} + d_2 \frac{R_1^2}{d_1} \right) = \frac{1}{R_1 R_2} \left( R_2^2 \frac{d_1}{d_2} + R_1^2 \frac{d_2}{d_1} \right) =$$

$$= \frac{1}{R_1 R_2} (d_2^2 \frac{R_2^2}{R_2} - R_2^2)$$

$$\Rightarrow y = R_2 \frac{d}{dt} \frac{R_1}{R_1} = \cosh^{-1} \left[ \frac{d^2 - R_1^2 - R_2^2}{2R_1 R_2} \right]$$

$$\Rightarrow C = \frac{1}{2 \cosh^{-1} \left[ \frac{d^2 - R_1^2 - R_2^2}{2R_1 R_2} \right]}$$

Electricity and Magnetism

Theory of relativity predicts that electric and magnetic fields transform into each other in moving inertial reference frames.

1. Find the field invariants (combinations of electric and magnetic fields that do not depend on the reference frame).
2. Given the electric and magnetic fields in a stationary reference frame  $K$ , find the relative velocity  $\vec{v}_0$  of an inertial reference frame  $K'$  where the electric field  $\vec{E}$  and magnetic field  $\vec{H}$  are parallel to each other and are both perpendicular to  $\vec{v}_0$ . Express  $\vec{v}_0$  in terms of invariants from p.1
3. Prove that  $\vec{E} \parallel \vec{H}$  in any inertial reference frame  $K''$  moving along the common direction of  $\vec{E}$  and  $\vec{H}$  with respect to  $K'$ .

You may use the fact that E&M fields form a 4-tensor:

$$F^{ik} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix}$$

1) The two invariants are given by:

$$F_{ik} F^{ik} = iuv$$

$$e^{iklm} F_{ik} F_{lm} = iuv$$

leading to:

$$\begin{cases} H^2 - E^2 = iuv \\ \vec{E} \cdot \vec{H} = iuv \end{cases}$$

2) Rotating the x axis along  $\vec{S}_0$  we see:

$$E'_x = H'_x = 0;$$

moreover,  $E'_x H'_x = 0 \Rightarrow$

$$E'_y H'_y - E'_z H'_z = 0$$

Using the transformation rules

$$E'_x = E_x, \quad E'_y = \gamma(E_y - \frac{v}{c} H_z), \quad E'_z = \gamma(E_z + \frac{v}{c} H_y)$$

$$H'_x = H_x, \quad H'_y = \gamma(H_y + \frac{v}{c} E_z), \quad H'_z = \gamma(H_z - \frac{v}{c} E_y)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

also:

$$E_y H'_y - \frac{v^2}{c^2} (H_y^2 + E_z^2) + \frac{v^2}{c^2} H_z E_y -$$

$$- E_z H'_z - \frac{v^2}{c^2} (H_z^2 + E_y^2) - \frac{v^2}{c^2} H_y E_z = 0$$

$$(E_y H_z - E_z H_y) \left(1 + \frac{v^2}{c^2}\right) - \frac{v^2}{c^2} (E^2 + H^2) = 0$$

$$\frac{\vec{S}_0 \cdot \vec{E} \times \vec{H}}{1 + v^2/c^2} = \frac{[\vec{E} \times \vec{H}]}{E^2 + H^2}$$

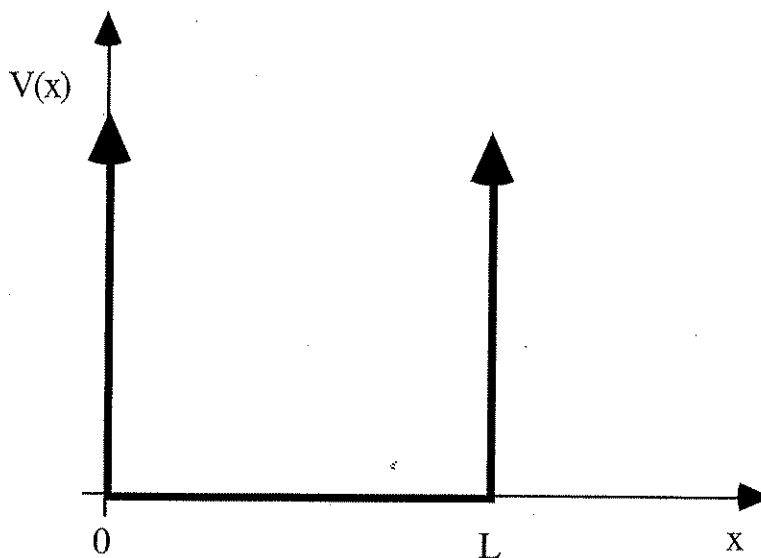
3) According to transformation rules for the fields, since

$$\vec{S}_0 \parallel \vec{E}' \parallel \vec{H}'$$

$$E'' = E', \text{ and } H'' = H'$$

$$\text{Thus } E'' \times H'' = E' \times H' = 0$$

Consider a particle of mass  $m$  confined to a one-dimensional infinite square well potential as shown below.



At time  $t = 0$ , the system is in a state for which the probability that a measurement of the energy would yield the ground state energy is  $1/2$ , the probability that a measurement of the energy would yield the first excited state energy is  $1/2$ , and the expectation value of the momentum  $p$  is  $8\hbar/3L$ . This information completely specifies the initial state of the system.

- Determine the initial quantum state of the system ( $|\psi(0)\rangle$ ).
- Find the minimum value of the probability that the particle is measured to be in the left half of the well and the time when this minimum first occurs (after  $t = 0$ ).

The following information may be useful:

$$\int \sin mx \sin nx dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, \quad m^2 \neq n^2$$

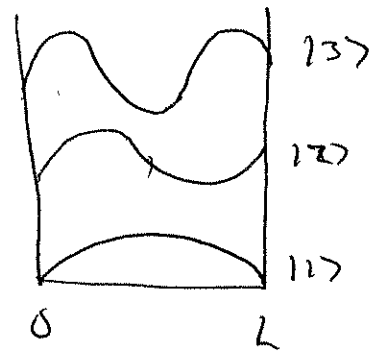
$$\int \sin mx \cos nx dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}, \quad m^2 \neq n^2$$

# Square Well

$$L = n \cdot \frac{\lambda_n}{2} \Rightarrow \lambda_n = \frac{2L}{n}$$

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}$$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2}{2mL^2} n^2$$



$$\phi_n(x) = \langle x | n \rangle = \sqrt{\frac{2}{L}} \sin k_n x = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

$$a) \mathcal{P}(E_1) = \frac{1}{2} = |\langle 1 | \psi(0) \rangle|^2$$

$$\mathcal{P}(E_2) = \frac{1}{2} = |\langle 2 | \psi(0) \rangle|^2$$

$$\text{let } |\psi(0)\rangle = \sum_n c_n |n\rangle$$

$$\Rightarrow \mathcal{P}(E_1) = |\langle 1 | \psi(0) \rangle|^2 = |c_1|^2 = \frac{1}{2}$$

$$\mathcal{P}(E_2) = |c_2|^2 = \frac{1}{2}$$

$$\Rightarrow |\psi(0)\rangle = \frac{1}{\sqrt{2}} |1\rangle + \frac{1}{\sqrt{2}} e^{i\phi} |2\rangle$$

since overall phase not measurable.

find  $\phi$  from  $\langle p \rangle(0)$  info.

$$\langle p \rangle (0) = \langle \psi(0) | p | \psi(0) \rangle$$

$$= \frac{1}{\sqrt{2}} \left[ \langle 1 | + e^{-i\phi} \langle 2 | \right] p \frac{1}{\sqrt{2}} \left[ |1\rangle + e^{i\phi} |2\rangle \right]$$

$$= \frac{1}{2} \left[ \langle 1 | p | 1 \rangle + \langle 2 | p | 2 \rangle + e^{-i\phi} \langle 2 | p | 1 \rangle + e^{i\phi} \langle 1 | p | 2 \rangle \right]$$

$$\langle m | p | n \rangle = \int_0^L \phi_m^*(x) \left( \frac{\hbar}{i} \frac{d}{dx} \right) \phi_n(x) dx$$

$$= \frac{2}{L} \int_0^L \sin k_m x \left( \frac{\hbar}{i} \frac{d}{dx} \right) \sin k_n x dx$$

$$= \frac{2\hbar}{iL} k_n \int_0^L \sin k_m x \cos k_n x dx$$

$m=n$  case:

$$\langle m | p | m \rangle = \frac{2\hbar}{iL} k_m \int_0^L \sin k_m x \cos k_m x dx$$

$$= \frac{2\hbar}{iL} k_m \left[ \frac{1}{2k_m} \sin^2 k_m x \right]_0^L$$

$$= \frac{\hbar}{iL} \left[ \sin^2 k_m L \right]$$

$$= \frac{\hbar}{iL} \sin^2 m\pi = 0$$

$m \neq n$  case

$$\langle m | p | n \rangle = \frac{2\hbar}{iL} k_m \int_0^L \sin k_m x \cos k_n x dx$$

$$= \frac{2\hbar k_m}{iL} \left[ -\frac{\cos(k_m - k_n)x}{2(k_m - k_n)} - \frac{\cos(k_m + k_n)x}{2(k_m + k_n)} \right]_0^L$$

$$= \frac{\hbar k_m}{iL} \left[ \frac{2k_m}{(k_m^2 - k_n^2)} - \frac{(-1)^{m-n}}{(k_m - k_n)} - \frac{(-1)^{m+n}}{(k_m + k_n)} \right]$$

$$\langle 2 | p | 1 \rangle = \frac{8\hbar}{i3L}$$

$$\Rightarrow \langle p \rangle(0) = \frac{1}{2} \left[ e^{-i\phi} \frac{8\hbar}{i3L} + e^{i\phi} \frac{8\hbar}{(-i3L)} \right]$$

$$= \frac{4\hbar}{3L} (-2 \sin \phi)$$

$$= -\frac{8\hbar}{3L} \sin \phi$$

since  $\langle p \rangle(0) = +\frac{8\hbar}{3L} \rightarrow \sin \phi = -1$

$$\Rightarrow \boxed{\phi = \frac{3\pi}{2}}$$

$$\Rightarrow |\psi(0)\rangle = \frac{1}{\sqrt{2}} \left[ |1\rangle + e^{i\frac{3\pi}{2}} |2\rangle \right]$$

$$b) |\psi(x,t)| = \frac{1}{\sqrt{2}} \left[ e^{-iE_1 t/\hbar} |1\rangle + e^{i\frac{3\pi}{2}} e^{-i\frac{E_2 t}{\hbar}} |2\rangle \right]$$

$$P(x \leq \frac{L}{2}) = \int_0^{\frac{L}{2}} |\psi(x,t)|^2 dx$$

$$= \int_0^{\frac{L}{2}} \frac{1}{\sqrt{2}} \left[ e^{+i\frac{E_1 t}{\hbar}} \phi_1(x) + e^{-i\frac{3\pi}{2} + i\frac{E_2 t}{\hbar}} \phi_2(x) \right]^*$$

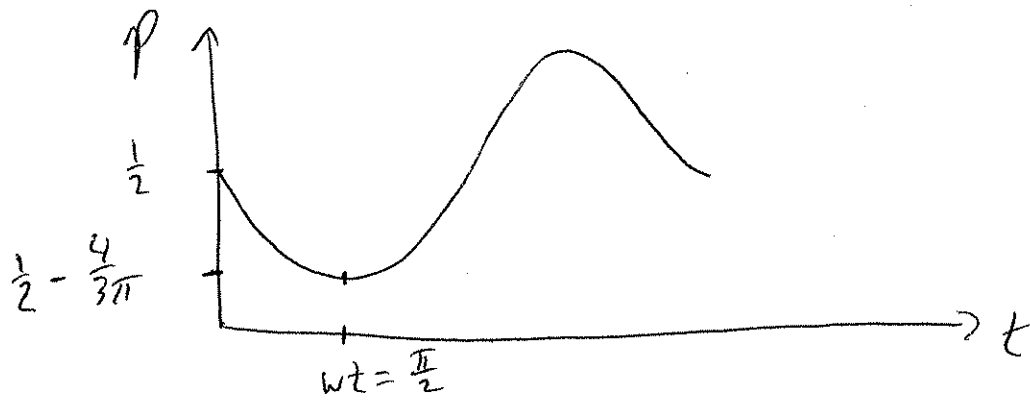
$$\cdot \frac{1}{\sqrt{2}} \left[ e^{-iE_1 t/\hbar} \phi_1(x) + e^{i\frac{3\pi}{2} - i\frac{E_2 t}{\hbar}} \phi_2(x) \right] dx$$

$$= \frac{1}{2} \cdot \frac{2}{L} \int_0^{\frac{L}{2}} \left[ \sin^2 k_1 x + \sin^2 k_2 x + \sin k_1 x \sin k_2 x \left\{ e^{i\frac{(E_2-E_1)}{\hbar} t} - e^{-i\frac{(E_2-E_1)}{\hbar} t} \right\} \right] dx$$

$$= \frac{1}{L} \left[ \frac{1}{2} \cdot \frac{L}{2} + \frac{1}{2} \frac{L}{2} - 2 \sin\left(\frac{E_2-E_1}{\hbar} t\right) \left[ \frac{\sin(k_2-k_1)x}{2(k_2-k_1)} - \frac{\sin(k_2+k_1)x}{2(k_2+k_1)} \right]_0^{\frac{L}{2}} \right]$$

$$P(x \leq \frac{L}{2}) = \frac{1}{2} - \frac{4}{3\pi} \sin\left(\frac{E_2-E_1}{\hbar} t\right)$$

$$P(x \leq \frac{L}{2}) = \frac{1}{2} - \frac{4}{3\pi} \sin\left(\frac{3\hbar\pi^2}{2mL^2} t\right)$$



$$P_{\min} = \frac{1}{2} - \frac{4}{3\pi}$$

$$t_{\min} = \frac{\pi}{2} \cdot \frac{1}{\frac{3\pi^2}{2mL^2}} = \frac{mL^2}{3\pi h}$$

$$P_{\min} = \frac{1}{2} - \frac{4}{3\pi}$$

$$t_{1st \min} = \frac{mL^2}{3\pi h}$$

Problem Th-2 (“graduate level”)

The free-electron model of conduction electrons seems naive but is successful in describing a range of physical phenomena seen in metals. Among other things, it gives a reasonably good account of the compressibility for certain metals.

Consider a gas of  $N$  non-interacting electrons enclosed in a three-dimensional cubic “box” of size  $L \times L \times L$ .

- (a) Calculate the Fermi energy  $\epsilon_F$  of the gas;
- (b) Find the density of states function  $DS(\epsilon)$  for this system;
- (c) Calculate the total internal energy  $U$  of the gas at  $T = 0$ , and express the result in terms of  $\epsilon_F$ ;
- (d) Find the Helmholtz free energy  $F$  of the gas at  $T = 0$ ;
- (e) Based on the results of (c) and (d), find the pressure of the electron gas at  $T = 0$ . Express the result as a function of  $U$ . Show that this result agrees with the result given by the theory of classical monoatomic gas.
- (d) Calculate the isothermal compressibility  $\kappa_T$  of the electron gas and show that this result *does not* agree with the classical gas theory.

## Problem Th-2, solutions:

**Task (a):** The electrons obey the Fermi-Dirac (F-D) statistics. According to the F-D statistics, at temperature  $T$  the occupancy  $f(\epsilon)$  of a state (“orbital<sup>1</sup>”) with energy  $\epsilon$  is:

$$f_{\text{F-D}}(\epsilon, T) = \frac{1}{\exp\left(\frac{\epsilon - \mu}{kT}\right) + 1}$$

where  $\mu$  is the chemical potential. The value of the chemical potential at  $T = 0$  is called the “Fermi energy” and is denoted as  $\epsilon_{\text{F}}$ . At  $T = 0$  the  $f_{\text{F-D}}(\epsilon, T)$  takes the value of 1 for all energies  $\epsilon < \epsilon_{\text{F}}$ , and is zero for all energies  $\epsilon > \epsilon_{\text{F}}$ . In other words, all orbitals with energy lower than the Fermi energy are fully occupied, and all orbitals with energies higher than  $\epsilon_{\text{F}}$  are empty (in the case of electrons, “fully occupied” means *two* electrons per each orbital,

The wavevectors of allowed quantum states of a particle in a 3-dimensional  $L \times L \times L$  “box” are:

$$\vec{k} = \left(\frac{\pi}{L}\right) (n_x, n_y, n_z)$$

where  $n_x, n_y, n_z$  are *positive* integers ( $n_x = 1, 2, 3, \dots$ ;  $n_y = 1, 2, 3, \dots$ ;  $n_z = 1, 2, 3, \dots$ ). The energy corresponding to a state with a given  $\vec{k}$  is:

$$\epsilon(n_x, n_y, n_z) = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m} \left(\frac{\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2).$$

where  $m$  is the particle mass.

At  $T = 0$ , the maximum energy of the electrons in the box is  $\epsilon_{\text{F}}$ . We can introduce a number  $n_{\text{max}}$  satisfying:

$$\epsilon_{\text{F}} = \frac{\hbar^2}{2m} \left(\frac{\pi}{L}\right)^2 n_{\text{max}}^2 \quad \Rightarrow \quad n_{\text{max}} = \left(\frac{2m\epsilon_{\text{F}}}{\hbar^2}\right)^{1/2} \left(\frac{L}{\pi}\right) \quad (1)$$

The “point vectors”  $(n_x, n_y, n_z)$  corresponding to *occupied* states – i.e., those with energies  $\epsilon < \epsilon_{\text{F}}$  – must all satisfy:

$$n_x^2 + n_y^2 + n_z^2 < n_{\text{max}}^2$$

which means that all these points are located within a *sphere* of radius  $n_{\text{max}}$  – or, more precisely stating, within the *first octant* of such a sphere (because  $n_x, n_y, n_z$  are all positive). So, the *total number* of all occupied states is simply equal to 1/8 of the volume of a sphere of radius  $n_{\text{max}}$ , and, since there are two electrons per each such orbital, the total number  $N$  of electrons in the system is:

$$N = 2 \times \frac{1}{8} \times \frac{4}{3} \pi n_{\text{max}}^3$$

This result can be combined with with Eq. 1:

$$N = \frac{\pi}{3} \left(\frac{2m\epsilon_{\text{F}}}{\hbar^2}\right)^{3/2} \left(\frac{L}{\pi}\right)^3 \quad (2)$$

From which, solving for  $\epsilon_{\text{F}}$ , we obtain:

$$\epsilon_{\text{F}} = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V}\right)^{2/3} \quad (3)$$

---

<sup>1</sup>In statistical mechanical “lingo”, “orbital” means the same as “quantum state”. In the present text, we use these two terms interchangeably.

where  $V = L^3$  is the system (box) volume.

**Task (b):** Now, let  $\epsilon$  be an arbitrary energy value. What is the number  $\mathcal{N}$  of particles that can be accommodated by all states with energies lower than  $\epsilon$ ? We can apply exactly the same reasoning scheme as was used above, and, without repeating any calculations, we can use Eq. 2, putting in it  $\epsilon$  instead of  $\epsilon_F$ :

$$\mathcal{N} = \frac{\pi}{3} \left( \frac{2m\epsilon}{\hbar^2} \right)^{3/2} \left( \frac{L}{\pi} \right)^3$$

If we now increase the energy by  $d\epsilon$ , the total number of electrons that can be accommodated by states with energies lower than  $\epsilon + d\epsilon$  is:

$$\mathcal{N} + d\mathcal{N} = \mathcal{N} + \left( \frac{d\mathcal{N}}{d\epsilon} \right) d\epsilon = \mathcal{N} + \frac{\pi}{3} \left( \frac{2m}{\hbar^2} \right)^{3/2} \left( \frac{L}{\pi} \right)^3 \times \frac{3}{2} \epsilon^{1/2} d\epsilon = \mathcal{N} + \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \epsilon^{1/2} d\epsilon$$

In this equation, the last right-hand term simply expresses the number of particles that can be accommodated by states in the  $(\epsilon, \epsilon + d\epsilon)$  range – and it is called the density of the states function  $D(\epsilon)$ :

$$D(\epsilon)d\epsilon \equiv \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \epsilon^{1/2} d\epsilon \quad (4)$$

**Task (c):** Using the  $D(\epsilon)$  function, the expression for the internal energy  $U$  of the electron gas can be written in a simple integral form:

$$U(T) = \int_0^\infty \epsilon D(\epsilon) f_{F-D}(\epsilon, T) d\epsilon$$

which is valid for any temperature. Because at  $T = 0$  the  $f_{F-D}$  function is 1 for  $\epsilon < \epsilon_F$ , and zero for higher energies, the expression simplifies to:

$$U(T=0) = \int_0^{\epsilon_F} \epsilon D(\epsilon) d\epsilon$$

Substituting Eq. 4 for  $D(\epsilon)$ , we get:

$$U(T=0) = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\epsilon_F} \epsilon^{3/2} d\epsilon = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \times \frac{2}{5} \times \epsilon_F^{5/2}$$

By splitting  $\epsilon_F^{5/2} = \epsilon_F^{3/2} \epsilon_F$  and invoking Eq. 3, we readily obtain the energy  $U$  in terms of Fermi energy:

$$U(T=0) = \frac{V}{5\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \times \left( \frac{\hbar^2}{2m} \right)^{3/2} \left( \frac{3\pi^2 N}{V} \right) \epsilon_F = \frac{3}{5} N \epsilon_F \quad (5)$$

**Task (d):** The Helmholtz free energy is  $F = U - TS$ , where  $S$  is the entropy. At  $T = 0$ , the free energy is simply equal to the internal energy:  $F = U$ .

**Task (e):** The pressure can be readily obtained from the free energy. Recall some basic relations involving the  $F$  function: taking the  $F$  differential  $dF = dU - TdS - SdT$ , and substituting  $dU = TdS - pdV$  (the differential form of the First Law of Thermodynamics), one obtains:

$$dF = -SdT - pdV$$

On the other hand, if  $F = F(T, S)$ , as implied by the above differential form, one can write a “generic” absolute differential:

$$dF = \left( \frac{\partial F}{\partial T} \right)_V dT + \left( \frac{\partial F}{\partial S} \right)_T dS$$

By comparing the two forms of  $dF$ , one obtains the known relations:

$$S = - \left( \frac{\partial F}{\partial T} \right)_V \quad \text{and} \quad p = - \left( \frac{\partial F}{\partial V} \right)_T$$

Now, since we already know  $F$  of the electron gas at  $T = 0$ , we can use the latter relation for calculating the pressure:

$$\begin{aligned} p &= - \left[ \frac{\partial}{\partial V} \left( \frac{3}{5} N \epsilon_F \right) \right]_T = - \frac{3}{5} N \left( \frac{\hbar^2}{2m} \right) (3\pi^2 N)^{2/3} \frac{\partial}{\partial V} V^{-2/3} = - \frac{3}{5} N \left( \frac{\hbar^2}{2m} \right) (3\pi^2 N)^{2/3} \left( -\frac{2}{3} \right) V^{-5/3} \\ &= \frac{2}{5} N \left( \frac{\hbar^2}{2m} \right) (3\pi^2 N)^{2/3} V^{-5/3} \end{aligned} \quad (6)$$

By invoking Eq. 3, this result can be written in a compact form:

$$p = \frac{2}{5} N \left( \frac{\hbar^2}{2m} \right) \left( \frac{3\pi^2 N}{V} \right)^{2/3} \times \frac{1}{V} = \frac{2N\epsilon_F}{5V} \quad (7)$$

Combining this result with Eq. 8, we obtain the relation between  $p$ ,  $U$  and  $V$ :

$$p = \frac{2U}{3V} \quad (8)$$

The equations describing classical monoatomic gas are  $pV = NkT$  and  $U = \frac{3}{2}NkT$ . Solving for pressure, one obtains  $p = \frac{2U}{3V}$ . So, at  $T = 0$  the relation between  $p$ ,  $U$  and  $V$  for the electron gas indeed agrees with the result given by the classical monoatomic gas theory.

**Task (f):**

$$\kappa_T = - \frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T = - \frac{1}{V} \left[ \left( \frac{\partial p}{\partial V} \right)_T \right]^{-1} \quad (9)$$

For calculating the  $\partial p/\partial V$  derivative, we can use the result for  $p$  in Eq. 6:

$$\begin{aligned} \frac{\partial p}{\partial V} &= \frac{\partial}{\partial V} \left[ \frac{2}{5} N \left( \frac{\hbar^2}{2m} \right) (3\pi^2 N)^{2/3} V^{-5/3} \right] = \frac{2}{5} N \left( \frac{\hbar^2}{2m} \right) (3\pi^2 N)^{2/3} \left( -\frac{5}{3} \right) V^{-8/3} \\ &= - \frac{2}{3} N \left( \frac{\hbar^2}{2m} \right) \left( \frac{3\pi^2 N}{V} \right)^{2/3} V^{-2} = - \frac{2}{3} N \epsilon_F V^{-2} \end{aligned}$$

Inserting this result into Eq. 9 yields

$$\kappa_T = \frac{3}{2} \frac{V}{N \epsilon_F} \quad (10)$$

If we now invoke Eq. 7, we obtain:

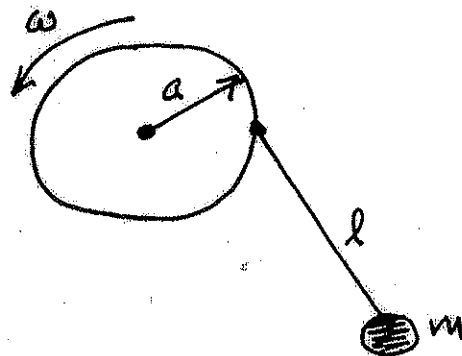
$$\kappa_T = \frac{3}{5} \times \frac{1}{p} \quad (11)$$

For a classical gas (any classical gas – not necessarily monoatomic),  $pV = NkT$ , so that:

$$\kappa_T = - \frac{1}{V} \frac{\partial NkT}{\partial p} \frac{1}{p} = \frac{1}{V} \frac{NkT}{p^2} = \frac{1}{V} \frac{V}{p} = \frac{1}{p}$$

So, the compressibility of electron gas at  $T = 0$  does not agree with the result of the classical gas theory.

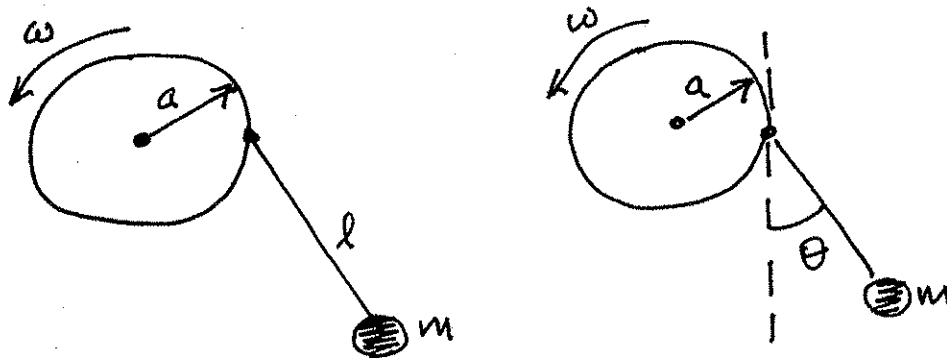
A plane pendulum of mass  $m$  and length  $\ell$  is suspended from a point on the rim of a disk of radius  $a$ . The disk is rotating at a constant angular velocity  $\omega$ . The pendulum can swing freely as the disk rotates, i.e. the string does not wrap around the edge of the disk.



- (i) Define an appropriate set of canonical variables for this problem and express the Hamiltonian in terms of these variables.
- (ii) Obtain Hamilton's equations of motion.
- (iii) Compare the Hamiltonian to the total energy and comment on this comparison.
- (iv) Consider the case  $\sqrt{\frac{g}{\ell}} \cong 10\omega$ . Suppose that the pendulum is swinging with small amplitude at some initial time. Using your intuition, rather than the detailed mathematics, sketch a phase-space diagram for the subsequent motion of the system for the first few rotations of the disk.

Problem 8

A plane pendulum of mass  $m$  and length  $\ell$  is suspended from a point on the rim of a disk of radius  $a$ . The disk is rotating at a constant angular velocity  $\omega$ . The pendulum can swing freely as the disk rotates, i.e. the string does not wrap around the edge of the disk.



- (i) Define an appropriate set of canonical variables for this problem and express the Hamiltonian in terms of these variables.

First, set up the Lagrangian:

$$x = a \cos \omega t + \ell \sin \theta$$

$$y = a \sin \omega t - \ell \cos \theta$$

$$T = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) = \frac{m}{2} \left[ (-a\omega \sin \omega t + \ell \cos \theta \dot{\theta})^2 + (a\omega \cos \omega t + \ell \sin \theta \dot{\theta})^2 \right]$$

$$= \frac{m}{2} \left[ (a\omega)^2 \sin^2 \omega t - 2a\omega \ell \sin \omega t \cos \theta \dot{\theta} + \ell^2 \cos^2 \theta \dot{\theta}^2 \right. \\ \left. + (a\omega)^2 \cos^2 \omega t + 2a\omega \ell \cos \omega t \sin \theta \dot{\theta} + \ell^2 \sin^2 \theta \dot{\theta}^2 \right]$$

$$= \frac{m}{2} \left[ (a\omega)^2 + 2a\omega \ell \sin(\theta - \omega t) \dot{\theta} + \ell^2 \dot{\theta}^2 \right]$$

$$V = mgy = mg(a \sin \omega t - \ell \cos \theta)$$

$$L = T - V = \frac{m}{2} \left[ (a\omega)^2 + 2a\omega \ell \sin(\theta - \omega t) \dot{\theta} + \ell^2 \dot{\theta}^2 \right] - mg(a \sin \omega t - \ell \cos \theta)$$

The canonical variables are the angle  $\theta$  and the canonical momentum

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m a \omega \ell \sin(\theta - \omega t) + m \ell^2 \dot{\theta}$$

The Hamiltonian is

$$\begin{aligned} H &= p_\theta \dot{\theta} - L \\ &= m a \omega \ell \sin(\theta - \omega t) \dot{\theta} + m \ell^2 \dot{\theta}^2 - \frac{m}{2} (a \omega)^2 - m a \omega \ell \sin(\theta - \omega t) \dot{\theta} \\ &\quad - \frac{m}{2} \ell^2 \dot{\theta}^2 + mg(a \sin \omega t - \ell \cos \theta) \\ &= \frac{m}{2} \ell^2 \dot{\theta}^2 - \frac{m}{2} (a \omega)^2 + mg(a \sin \omega t - \ell \cos \theta) \end{aligned}$$

Now, eliminate  $\dot{\theta}$  to get  $H(\theta, p_\theta)$ :  $\dot{\theta} = \frac{p_\theta}{m \ell^2} - \frac{a \omega}{\ell} \sin(\theta - \omega t)$

$$H = \frac{p_\theta^2}{2 m \ell^2} - \frac{a \omega}{\ell} p_\theta \sin(\theta - \omega t) - \frac{m}{2} (a \omega)^2 \cos^2(\theta - \omega t) + mg(a \sin \omega t - \ell \cos \theta)$$

(ii) Obtain Hamilton's equations of motion.

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m \ell^2} - \frac{a \omega}{\ell} \sin(\theta - \omega t)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{a \omega}{\ell} p_\theta \cos(\theta - \omega t) - m (a \omega)^2 \cos(\theta - \omega t) \sin(\theta - \omega t) - mg \ell \sin \theta$$

Note:  $\frac{\partial H}{\partial t} \neq 0 \Rightarrow H$  is not a constant of the motion.

(iii) Compare the Hamiltonian to the total energy and comment on this comparison.

$$E = T + V = L + 2V = \frac{m}{2} \left[ (a\omega)^2 + 2a\omega\ell \sin(\theta - \omega t)\dot{\theta} + \ell^2\dot{\theta}^2 \right] + mg(a \sin \omega t - \ell \cos \theta)$$

$$H = \frac{m}{2} \ell^2 \dot{\theta}^2 - \frac{m}{2} (a\omega)^2 + mg(a \sin \omega t - \ell \cos \theta)$$

$$E - H = m(a\omega)^2 + ma\omega\ell \sin(\theta - \omega t)\dot{\theta}$$

$$\dot{\theta} = \frac{p_{\theta}}{m\ell^2} - \frac{a\omega}{\ell} \sin(\theta - \omega t)$$

$$E - H = m(a\omega)^2 \cos^2(\theta - \omega t) + \frac{a\omega}{\ell} \sin(\theta - \omega t)p_{\theta} \neq 0$$

Neither the energy nor the Hamiltonian are constants of the motion for this system. The kinetic energy is not a homogeneous quadratic function of the velocity ( $\dot{\theta}$ ) and the potential energy is time dependent. The oscillator (pendulum) is being driven periodically and the energy increases over time.

- (iv) Consider the case  $\sqrt{\frac{g}{\ell}} \cong 10\omega$ . Suppose that the pendulum is swinging with small amplitude at some initial time. Using your intuition, rather than the detailed mathematics, sketch a phase-space diagram for the subsequent motion of the system for the first few rotations of the disk.

Note: If  $\omega = 0$ ,  $E = H = \frac{m}{2} \ell^2 \dot{\theta}^2 - mg\ell \cos \theta = \frac{p_\theta^2}{2m\ell^2} - mg\ell \cos \theta$ .

For small amplitudes,  $\cos \theta \cong 1 - \frac{\theta^2}{2}$  and the path in  $p_\theta - \theta$  phase-space for  $\omega = 0$  is an ellipse described by  $\frac{p_\theta^2}{2m\ell^2} + \frac{mg\ell}{2} \theta^2 = E + mg\ell = \text{constant}$ . (When the pendulum is at rest,  $E = -mg\ell$ .)

If  $\omega \neq 0$  and  $\omega \ll \sqrt{\frac{g}{\ell}}$ ,  $E$  gradually increases (but not at a constant rate) while the pendulum completes many cycles of its motion, i.e. the phase path spirals outward.

