OSU PHYSICS DEPARTMENT
COMPREHENSIVE EXAMINATION #102

April 2 and 3, 2007

Comprehensive examination for Spring 2007

PART 1, Monday April 2, 9:00 am

General Instructions

This Comprehensive Examination for Spring 2007 consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, April 2, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, April 3, at 9:00 am and 1:30 pm.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Problem 1

This problem concerns the spin configurations of two particles with spins $s_1 = ½$ and $s_2 = 1$, respectively.

(1) Show that if $\vec{J} = \vec{S}_1 + \vec{S}_2$ is the operator for the total spin, then J^2 commutes with the z-component of the total spin $J_z = S_{1z} + S_{2z}$.

(2) Find the matrix representation of J^2 in the basis of states $\left|s_1, m_{1z}; s_2, m_{2z}\right\rangle = \left|\frac{1}{2}, m_{1z}; 1, m_{2z}\right\rangle \equiv \left|m_1, m_2\right\rangle$.

Hint: It will be convenient to define the rows and columns with the states in the order:

$$\left|\frac{1}{2},1\right\rangle, \left|-\frac{1}{2},1\right\rangle, \left|\frac{1}{2},0\right\rangle, \left|-\frac{1}{2},0\right\rangle, \left|\frac{1}{2},-1\right\rangle, \left|-\frac{1}{2},1\right\rangle$$

(3) Calculate the eigenvalues of J^2.

(4) Find the eigenstates of J^2, $\left|J,M\right\rangle$, in terms of the basis states $\left|m_1, m_2\right\rangle$ and label them using the quantum numbers J and M for the total spin and J_z, respective.

(5) Define the Clebsch-Gordon coefficients and given an explicit example of one from this problem.

Solution to Problem 1

This problem concerns the spin configurations of two particles with spins s_1 = ½ and s_2 = 1, respectively.

(1) Show that if $\vec{J} = \vec{S}_1 + \vec{S}_2$ is the operator for the total spin, then J^2 commutes with the z-component of the total spin $J_z = S_{1z} + S_{2z}$.

Recall the commutation rules for the components of a single spin: $\vec{S} \times \vec{S} = i\hbar\,\vec{S}$. Thus,

$$[S_x, S_y] = i\hbar\, S_z \qquad [S_y, S_z] = i\hbar\, S_x \qquad [S_z, S_x] = i\hbar\, S_y$$

$$[J^2, J_z] = [J^2, S_{1z}] + [J^2, S_{2z}]$$

$$J^2 = (\vec{S}_1 + \vec{S}_2)\cdot(\vec{S}_1 + \vec{S}_2) = S_2^2 + S_1^2 + 2S_{1x}S_{2x} + 2S_{1y}S_{2y} + 2S_{1z}S_{2z}$$

$$\begin{aligned}[J^2, S_{1z}] &= 2[S_{1x}, S_{1z}]S_{2x} + 2[S_{1y}, S_{1z}]S_{2y} + 2[S_{1z}, S_{1z}]S_{2z} \\ &= -2(i\hbar)S_{1y}S_{2x} + 2(i\hbar)S_{1x}S_{2y} + 0\end{aligned}$$

Similarly,

$$\begin{aligned}[J^2, S_{2z}] &= 2S_{1x}[S_{2x}, S_{2z}] + 2S_{1y}[S_{2y}, S_{2z}] + 2S_{2z}[S_{2z}, S_{2z}] \\ &= -2(i\hbar)S_{1x}S_{2y} + 2(i\hbar)S_{1y}S_{2x} + 0 = -[J^2, S_{1z}]\end{aligned}$$

Thus, $[J^2, J_z] = [J^2, S_{1z}] + [J^2, S_{2z}] = 0$.

(2) Find the matrix representation of J^2 in the basis of states $\left|s_1, m_{1z}; s_2, m_{2z}\right\rangle = \left|\frac{1}{2}, m_{1z}; 1, m_{2z}\right\rangle \equiv \left|m_1, m_2\right\rangle$.

Hint: It will be convenient to define the rows and columns with the states in the order:

$$\left|\frac{1}{2},1\right\rangle, \left|-\frac{1}{2},1\right\rangle, \left|\frac{1}{2},0\right\rangle, \left|-\frac{1}{2},0\right\rangle, \left|\frac{1}{2},-1\right\rangle, \left|-\frac{1}{2},-1\right\rangle$$

Use $\vec{S}_1 \cdot \vec{S}_2 = S_{1z}S_{2z} + \frac{1}{2}\left(S_{1+}S_{2-} + S_{1-}S_{2+}\right)$ so that

$$J^2 = S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

Now consider the action of the successive terms of J^2 on $\left|s_1, m_{1z}; s_1, m_{2z}\right\rangle \equiv \left|m_1, m_2\right\rangle$:

$$S_1^2\left|m_1, m_2\right\rangle = s_1(s_1+1)\hbar^2\left|m_1, m_2\right\rangle \qquad S_2^2\left|m_1, m_2\right\rangle = s_2(s_2+1)\hbar^2\left|m_1, m_2\right\rangle$$

$$S_{1z}S_{2z}\left|m_1, m_2\right\rangle = m_1 m_2 \hbar^2\left|m_1, m_2\right\rangle$$

$$\begin{aligned} S_{1+}S_{2-}\left|m_1, m_2\right\rangle &= S_{1+}\cdot\hbar\sqrt{s_2(s_2+1) - m_2(m_2-1)}\left|m_1, m_2-1\right\rangle \\ &= \hbar^2\sqrt{s_1(s_1+1) - m_1(m_1+1)}\sqrt{s_2(s_2+1) - m_2(m_2-1)}\left|m_1+1, m_2-1\right\rangle \end{aligned}$$

$$S_{1-}S_{2+}\left|m_1, m_2\right\rangle = \hbar^2\sqrt{s_1(s_1+1) - m_1(m_1-1)}\sqrt{s_2(s_2+1) - m_2(m_2+1)}\left|m_1-1, m_2+1\right\rangle$$

Now, with $s_1 = ½$ and $s_2 = 1$, we have

$$J^2\left|m_1, m_2\right\rangle = \hbar^2 \times \left[\begin{array}{l} \left(\frac{3}{4} + 2 + 2m_1 m_2\right)\left|m_1, m_2\right\rangle \\ + \sqrt{\frac{3}{4} - m_1(m_1+1)}\sqrt{2 - m_2(m_2-1)}\left|m_1+1, m_2-1\right\rangle \\ + \sqrt{\frac{3}{4} - m_1(m_1-1)}\sqrt{2 - m_2(m_2+1)}\left|m_1-1, m_2+1\right\rangle \end{array}\right]$$

Evaluate diagonal elements:

$$\left\langle\frac{1}{2},1\middle|J^2\middle|\frac{1}{2},1\right\rangle = \left\langle-\frac{1}{2},-1\middle|J^2\middle|-\frac{1}{2},-1\right\rangle = \hbar^2\left(\frac{11}{4} + 2\cdot\left(\pm\frac{1}{2}\right)\cdot(\pm 1)\right) = \frac{15}{4}\hbar^2$$

$$\left\langle \frac{1}{2},0 \right| J^2 \left| \frac{1}{2},0 \right\rangle = \left\langle -\frac{1}{2},0 \right| J^2 \left| -\frac{1}{2},0 \right\rangle = \hbar^2\left(\frac{11}{4}+2\cdot\left(\pm\frac{1}{2}\right)\cdot 0\right) = \frac{11}{4}\hbar^2$$

$$\left\langle -\frac{1}{2},1 \right| J^2 \left| -\frac{1}{2},1 \right\rangle = \left\langle \frac{1}{2},-1 \right| J^2 \left| \frac{1}{2},-1 \right\rangle = \hbar^2\left(\frac{11}{4}+2\cdot\left(\mp\frac{1}{2}\right)\cdot(\pm 1)\right) = \frac{7}{4}\hbar^2$$

The non-zero off-diagonal elements are of the form
$\langle m_1-1,\, m_2+1 | J^2 | m_1, m_2\rangle$ or $\langle m_1+1,\, m_2-1 | J^2 | m_1, m_2\rangle$

$$\left\langle -\frac{1}{2},1 \right| J^2 \left| \frac{1}{2},0 \right\rangle = \hbar^2\sqrt{\frac{3}{4}-\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}\cdot\sqrt{2-0} = \sqrt{2}\,\hbar^2$$

$$\left\langle -\frac{1}{2},0 \right| J^2 \left| \frac{1}{2},-1 \right\rangle = \hbar^2\sqrt{\frac{3}{4}-\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}\cdot\sqrt{2-(-1)(-1+1)} = \sqrt{2}\,\hbar^2$$

and similarly $\left\langle \frac{1}{2},-1 \right| J^2 \left| -\frac{1}{2},0 \right\rangle = \left\langle \frac{1}{2},0 \right| J^2 \left| -\frac{1}{2},1 \right\rangle = \sqrt{2}\,\hbar^2$

Thus

$$J^2 = \begin{array}{c} \\ \left|\frac{1}{2},1\right\rangle \\ \left|-\frac{1}{2},1\right\rangle \\ \left|\frac{1}{2},0\right\rangle \\ \left|-\frac{1}{2},0\right\rangle \\ \left|\frac{1}{2},-1\right\rangle \\ \left|-\frac{1}{2},-1\right\rangle \end{array} \begin{array}{c} \begin{array}{cccccc} \left|\frac{1}{2},1\right\rangle & \left|-\frac{1}{2},1\right\rangle & \left|\frac{1}{2},0\right\rangle & \left|-\frac{1}{2},0\right\rangle & \left|\frac{1}{2},-1\right\rangle & \left|-\frac{1}{2},-1\right\rangle \end{array} \\ \begin{pmatrix} \frac{15}{4}\hbar^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{4}\hbar^2 & \sqrt{2}\,\hbar^2 & 0 & 0 & 0 \\ 0 & \sqrt{2}\,\hbar^2 & \frac{11}{4}\hbar^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{11}{4}\hbar^2 & \sqrt{2}\,\hbar^2 & 0 \\ 0 & 0 & 0 & \sqrt{2}\,\hbar^2 & \frac{7}{4}\hbar^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{4}\hbar^2 \end{pmatrix} \end{array}$$

(3) Calculate the eigenvalues of J^2.

We can obtain two eigenvalues by inspection: $J(J+1)\hbar^2 = \frac{15}{4}\hbar^2 \Rightarrow J = \frac{3}{2}$ for the states $\left|\frac{1}{2},1\right\rangle \equiv |\psi_1\rangle$ and $\left|-\frac{1}{2},-1\right\rangle \equiv |\psi_6\rangle$.

To continue, we have to diagonalize the 2 x 2 manifolds of the J^2 matrix:

$$\begin{vmatrix} \frac{7}{4}\hbar^2-\lambda & \sqrt{2}\,\hbar^2 \\ \sqrt{2}\,\hbar^2 & \frac{11}{4}\hbar^2-\lambda \end{vmatrix} = \left(\frac{7}{4}\hbar^2-\lambda\right)\left(\frac{11}{4}\hbar^2-\lambda\right)-2\hbar^4 = \lambda^2-\frac{9}{2}\hbar^2\lambda + \frac{45}{16}\hbar^4 = 0$$

where $\lambda = J(J+1)\hbar^2$ is the eigenvalue. Solving the quadratic, gives $\lambda = \frac{\hbar^2}{4}(9 \pm 6)$.
Thus, $J(J+1)\hbar^2 = \frac{15}{4}\hbar^2 \left(J=\frac{3}{2}\right)$ or $\frac{3}{4}\hbar^2 \left(J=\frac{1}{2}\right)$.

The other 2 x 2, $\begin{vmatrix} \frac{11}{4}\hbar^2-\lambda & \sqrt{2}\,\hbar^2 \\ \sqrt{2}\,\hbar^2 & \frac{7}{4}\hbar^2-\lambda \end{vmatrix} = 0$ gives the same results so that all together we have a <u>four-fold</u> degenerate eigenvalue $J(J+1)\hbar^2 = \frac{15}{4}\hbar^2 \left(J=\frac{3}{2}\right)$ and a <u>two-fold</u> degenerate eigenvalue $J(J+1)\hbar^2 = \frac{3}{4}\hbar^2 \left(J=\frac{1}{2}\right)$.

(4) Find the eigenstates of J^2, $|J,M\rangle$, in terms of the basis states $|m_1,m_2\rangle$ and label them using the quantum numbers J and M for the total spin and J_z, respective.

$$\begin{pmatrix} \frac{15}{4}\hbar^2-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{4}\hbar^2-\lambda & \sqrt{2}\,\hbar^2 & 0 & 0 & 0 \\ 0 & \sqrt{2}\,\hbar^2 & \frac{11}{4}\hbar^2-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{11}{4}\hbar^2-\lambda & \sqrt{2}\,\hbar^2 & 0 \\ 0 & 0 & 0 & \sqrt{2}\,\hbar^2 & \frac{7}{4}\hbar^2-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{4}\hbar^2-\lambda \end{pmatrix} \begin{pmatrix} c_{\frac{1}{2},1} \\ c_{-\frac{1}{2},1} \\ c_{\frac{1}{2},0} \\ c_{-\frac{1}{2},0} \\ c_{\frac{1}{2},-1} \\ c_{-\frac{1}{2},-1} \end{pmatrix} = 0$$

The second row of the J^2 matrix gives the equation $\left(\frac{7}{4}\hbar^2 - \lambda\right)c_{-\frac{1}{2},1} + \sqrt{2}\,\hbar^2 c_{\frac{1}{2},0} = 0$.

With $\lambda = \frac{15}{4}\hbar^2$, we have $-2\,c_{-\frac{1}{2},1} + \sqrt{2}\,c_{\frac{1}{2},0} = 0 \quad \Rightarrow \quad c_{\frac{1}{2},0} = \sqrt{2}c_{-\frac{1}{2},1}$ and

$$|\psi_2\rangle = c_{-\frac{1}{2},1}\left|-\frac{1}{2},1\right\rangle + \sqrt{2}\,c_{-\frac{1}{2},1}\left|\frac{1}{2},0\right\rangle$$

Normalize: $\left|c_{-\frac{1}{2},1}\right|^2 + 2\left|c_{-\frac{1}{2},1}\right|^2 = 3\left|c_{-\frac{1}{2},1}\right|^2 = 1 \quad \Rightarrow \quad c_{-\frac{1}{2},1} = \frac{1}{\sqrt{3}}$

$$|\psi_2\rangle = \frac{1}{\sqrt{3}}\left|-\frac{1}{2},1\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2},0\right\rangle$$

With $\lambda = \frac{3}{4}\hbar^2$, we have $c_{-\frac{1}{2},1} + \sqrt{2}\,c_{\frac{1}{2},0} = 0 \Rightarrow \quad c_{-\frac{1}{2},1} = -\sqrt{2}c_{\frac{1}{2},0}$ and

$$|\psi_3\rangle = \sqrt{\frac{2}{3}}\left|-\frac{1}{2},1\right\rangle - \frac{1}{\sqrt{3}}\left|\frac{1}{2},0\right\rangle$$

The fourth row of the J^2 matrix gives the equation $\left(\frac{11}{4}\hbar^2 - \lambda\right)c_{-\frac{1}{2},0} + \sqrt{2}\hbar^2 c_{\frac{1}{2},-1} = 0$.

With $\lambda = \frac{15}{4}\hbar^2$, we have $-c_{-\frac{1}{2},0} + \sqrt{2}c_{\frac{1}{2},-1} = 0 \quad \Rightarrow \quad c_{-\frac{1}{2},0} = \sqrt{2}c_{\frac{1}{2},-1}$ and

$$|\psi_4\rangle = \sqrt{\frac{2}{3}}\left|-\frac{1}{2},0\right\rangle + \frac{1}{\sqrt{3}}\left|\frac{1}{2},-1\right\rangle$$

With $\lambda = \frac{3}{4}\hbar^2$, we have $2c_{-\frac{1}{2},0} + \sqrt{2}\,c_{\frac{1}{2},-1} = 0 \quad \Rightarrow \quad c_{\frac{1}{2},-1} = -\sqrt{2}c_{-\frac{1}{2},0}$ and

$$|\psi_5\rangle = \frac{1}{\sqrt{3}}\left|-\frac{1}{2},0\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},-1\right\rangle$$

Thus, making use of $M = m_1 + m_2$, the six eigenstates and their J, M values are:

$$|\psi_1\rangle = \left|\frac{1}{2},1\right\rangle \qquad J = \frac{3}{2},\ M = \frac{3}{2}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{3}}\left|-\frac{1}{2},1\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2},0\right\rangle \qquad J = \frac{3}{2},\ M = \frac{1}{2}$$

$$|\psi_3\rangle = \sqrt{\frac{2}{3}}\left|-\frac{1}{2},1\right\rangle - \frac{1}{\sqrt{3}}\left|\frac{1}{2},0\right\rangle \qquad J = \frac{1}{2},\ M = \frac{1}{2}$$

$$|\psi_4\rangle = \sqrt{\frac{2}{3}}\left|-\frac{1}{2},0\right\rangle + \frac{1}{\sqrt{3}}\left|\frac{1}{2},-1\right\rangle \quad J = \frac{3}{2},\ M = -\frac{1}{2}$$

$$|\psi_5\rangle = \frac{1}{\sqrt{3}}\left|-\frac{1}{2},0\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},-1\right\rangle \quad J = \frac{1}{2},\ M = -\frac{1}{2}$$

$$|\psi_6\rangle = \left|-\frac{1}{2},-1\right\rangle \qquad J = \frac{3}{2},\ M = -\frac{3}{2}$$

(5) Define the Clebsch-Gordon coefficients and given an explicit example of one from this problem.

The general definition of a Clebsch-Gordon coefficient in our notation is $C^{s_1 s_2 J}_{m_1 m_2 M}$ where $|J,M\rangle = \sum_{M=m_1+m_2} C^{s_1 s_2 J}_{m_1 m_2 M}\ |s_1,m_1;s_2,m_2\rangle$.

Particular examples from the state $\left|\frac{1}{2},-\frac{1}{2}\right\rangle = |\psi_5\rangle = \frac{1}{\sqrt{3}}\left|-\frac{1}{2},0\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2},-1\right\rangle$ would be $C^{\frac{1}{2},1,\frac{1}{2}}_{-\frac{1}{2},0,-\frac{1}{2}} = \frac{1}{\sqrt{3}}$, $\quad C^{\frac{1}{2},1,\frac{1}{2}}_{\frac{1}{2},-1,-\frac{1}{2}} = -\sqrt{\frac{2}{3}}$, etc.

Problem 2

A particle with mass M and initial velocity $\vec{v}_{01}$ elastically hits another particle with mass m which is initially at rest $(\vec{v}_{02} = 0)$.

1. Find a maximum angle $\theta_{\max}$ between velocities of the particles after the collision ($\vec{v}_{11}$ and $\vec{v}_{12}$) as a function of M / m.

2. Find $\vec{v}_{11}$ and $\vec{v}_{12}$ when $\theta = \theta_{\max}$. Consider the limits when $M / m \to 0$ and $M / m \to \infty$.

Comp. exam 2007 - solution to cm. undergrad. problem

Note Title 3/29/20

(i) Energy and (ii) momentum are conserved under elastic collisions.

To find the directions of particle motion after collision, use momentum conservation

Denote:

(1)
$$\begin{cases} \vec{p}_{01} = M\vec{v}_{01} = \text{momentum of first particle before the collision} \\ \vec{p}_{02} = 0 = \text{momentum of second particle before the collision} \\ \vec{p}_{11} = M\vec{v}_{11} = \text{momentum of first particle after collision} \\ \vec{p}_{12} = m\vec{v}_{12} = \text{momentum of second particle after collision} \end{cases}$$

(2) $\vec{p}_{01} + \cancelto{0}{\vec{p}_{02}} = \vec{p}_{11} + \vec{p}_{12}$ - momentum conservation

$$M\vec{v}_{01} = M\vec{v}_{11} + m\vec{v}_{12}$$

(3)
$$\boxed{\vec{v}_{01} = \vec{v}_{11} + \frac{m}{M}\vec{v}_{12}}$$

multiplying this equation by itself, obtain:

(4)
$$|v_{01}|^2 = |v_{11}|^2 + \frac{m^2}{M^2}|v_{12}|^2 + 2\frac{m}{M}|v_{11}||v_{12}|\cdot\cos\theta$$

Adding energy conservation condition:

(5) $$|v_{01}|^2 = |v_{11}|^2 + \frac{m}{M}|v_{12}|^2$$

obtain:

$$|v_{01}|^2 = \cancel{|v_{11}|^2} + \frac{m^2}{M^2}|v_{12}|^2 + 2\frac{m}{M}|v_{11}||v_{12}|\cdot\cos\theta = \cancel{|v_{11}|^2} + \frac{m}{M}|v_{12}|^2$$

$$\cos\theta = \frac{1}{2}\frac{M}{m}\left(\frac{m}{M} - \frac{m^2}{M^2}\right)\frac{|v_{12}|}{|v_{11}|} = \frac{1}{2}\left(1 - \frac{m}{M}\right)\left|\frac{v_{12}}{v_{11}}\right|$$

→ (a) $m < M$

$$\cos\theta = \frac{1}{2}\left(1 - \frac{m}{M}\right)\left|\frac{v_{12}}{v_{11}}\right| > 0 \rightarrow 0 < \theta < \frac{\pi}{2}$$

looking for minimum of $\frac{1}{2}\left(1 - \frac{m}{2}\right)\left|\frac{v_{12}}{v_{11}}\right| \Rightarrow$

minimum of v_{12}, which is achieved when

$$\boxed{\begin{array}{l} v_{11} \rightarrow v_{10};\ v_{12} \rightarrow 0 \\ \cos\theta_{max} = 0 \rightarrow \theta_{max} = \frac{\pi}{2}. \end{array}}$$

(b) $m > M$

$$\cos\theta = \frac{1}{2}\left(1 - \frac{m}{M}\right)\left|\frac{v_{12}}{v_{11}}\right| < 0 \rightarrow \frac{\pi}{2} < \theta < \pi$$

$\cos\theta_{max} = -1$

$$\boxed{\begin{array}{l} \theta_{max} = \pi, \text{ realized @:} \\ |v_{11}| = \frac{1}{2}\left(\frac{m-M}{M}\right)|v_{12}| \end{array}}$$

$$|v_{01}|^2 = \frac{1}{2}\left(\frac{m-M}{M}\right)^2 |v_{12}|^2 + \frac{m}{M}|v_{12}|^2 = |v_{12}|^2 \frac{m^2+M^2}{2M^2}$$

$$\boxed{v_{12} = v_{01} \cdot \sqrt{\frac{2M^2}{m^2+M^2}}}$$

when $m \ll M$, $v_{11} \to v_{01}$; $v_{12} \to 0$

when $m \gg M$; $v_{12} \to 0$; $v_{11} = -v_{01}$

Problem 3

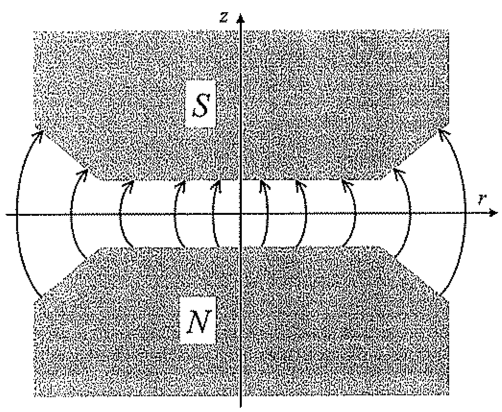
A cyclotron is a particle accelerator consisting of two magnets, as shown in the figure below. The magnets produce a perpendicular magnetic field causing the particles to go almost in a circle. The magnetic field is expressed as

$$\mathbf{B} = B_r(r,z)\,\hat{r} + B_z(r,z)\,\hat{z}\,,\ \left(B_r \ll B_z\right)$$

where $r = \sqrt{x^2 + y^2}$ is the distance from the axis of the polê faces. The vertical component of the field near the median plane ($z = 0$) is given as

$$B_z(r,z) \cong B_0 \exp(-ar^2 + bz^2)\,,$$

where a and b are positive constants.

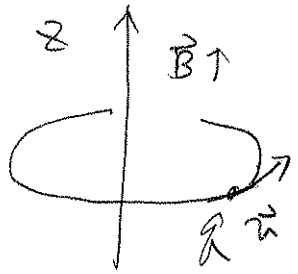


(a) An electron in the magnetic field goes in a circular motion in the median plane. Assume the motion is nonrelativistic. What is the total energy of the particle when the radius of the circle is r_0? Show that the frequency of the circular motion is independent of the radius if the magnetic field is uniform.

(b) Show that the lines of the magnetic intensity bow outward, as shown in the figure. It may be useful to use the equation for curl of **B**. Also use the fact that $B_r = 0$ on the median plane.

(c) Show that accelerated particles that drift away from the median plane experience a force tending to restore them to the median plane, regardless of whether they are positively or negatively charged.

P3. (a) Lorentz force balances the centri-fugal force.



$$\vec{F}_B = q\,\vec{v}\times\vec{B} \qquad (B_r = 0 \text{ at } z=0)$$

$$\Rightarrow F_B = q v B_z(r_0, 0)$$

$$= \frac{mv^2}{r_0^{\,2}}$$

$$\Rightarrow v = \frac{q r_0 B_z(r_0,0)}{m}$$

$$\therefore E = \frac{1}{2}mv^2 = \boxed{\frac{q^2 r_0^2 B_z^2(r_0,0)}{2m}}$$

The frequency of the circular motion is

$$f = \frac{v}{2\pi r_0} = \frac{q \cancel{r_0} B_z(r_0,0)}{2\pi \cancel{r_0}\, m} = \frac{qB_z}{2\pi m}$$

If the magnetic field is uniform, B_z is constant.

Thus, f is independent of radius.

(b)
$$0 = \nabla\times\vec{B} = \left\{\frac{1}{r}\cancelto{0}{\frac{\partial B_z}{\partial\phi}} - \cancelto{0}{\frac{\partial B_\phi}{\partial z}}\right\}\hat{r}$$

$$+ \left\{\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right\}\hat{\phi}$$

$$+ \frac{1}{r}\left\{\frac{\partial}{\partial r}\cancelto{0}{(rB_\phi)} - \cancelto{0}{\frac{\partial B_r}{\partial\phi}}\right\}\hat{z}$$

$$\Rightarrow \boxed{\frac{\partial B_r}{\partial z} = \frac{\partial B_z}{\partial r}}$$

Since $\left.\dfrac{\partial B_z}{\partial r}\right|_{z=0} = -2ar\, B_0\, e^{-ar^2} < 0$,

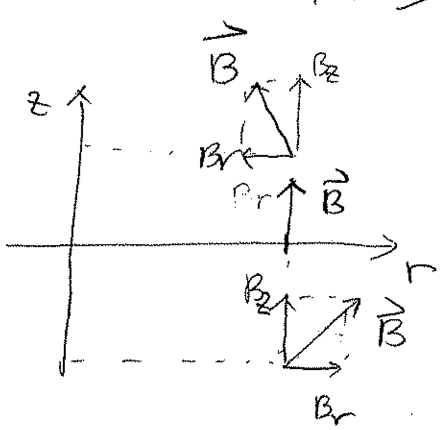
$$\left.\frac{\partial B_r}{\partial z}\right|_{z=0} = \left.\frac{\partial B_z}{\partial r}\right|_{z=0} < 0 .$$

Near the median plane,

$$B_r(r,z) = \cancelto{0}{B_r(r, z=0)} + \overbrace{\left.\frac{\partial B_r}{\partial z}\right|_{z=0}}^{-\alpha\ (>0)} z$$

$$= -\alpha z \quad (\alpha > 0)$$

$$\Rightarrow \begin{cases} B_r < 0 & \text{for } z > 0 \\ B_r > 0 & \text{for } z < 0 \end{cases}$$



The direction of the $\vec{B}$ field is the tangent of the field line. Because the sign of B_r chges from + to − when z varies from − to +, the field lines bow outward.

(c) The velocity of the charged particle is

$$\vec{v} = -\frac{q r_0 B_z}{m} \hat{\phi}$$

The z-component of the Lorentz force is

$$F_z = [q \vec{v} \times \vec{B}]_z = [q v_\phi B_r \underbrace{(\hat{\phi} \times \hat{r})}_{-\hat{z}}]_z$$

$$= -q v_\phi B_r$$

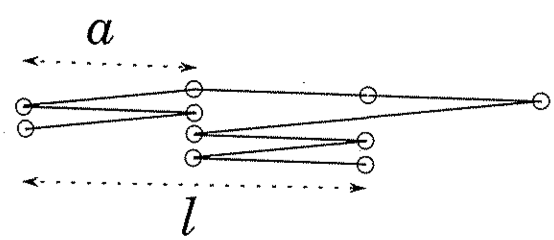
$$= \frac{q^2 r_0 B_z}{m} B_r = \beta B_r \quad (\beta > 0)$$

Therefore,
$$\begin{cases} F_z < 0, & \text{for } z > 0 \\ F_z > 0, & \text{for } z < 0 \end{cases}$$

F_z is a restoring force near the median plane.

Problem 4

Rubber can be qualitatively modelled by taking into account only entropic effects. In this model, a rubber molecule is viewed as a linear chain of rigid segments, as illustrated below. In this model, the molecule is composed of N segments of length a, and the distance between start and end is l. We will treat the distance l as a macroscopic control parameter.



For simplicity, we restrict ourselves to one dimension, so each link will have only two possible states, that is, each segment is either pointing to the right, or to the left, and these states will have an equal energy.

(a) What is the entropy of a rubber molecule with a distance from start to end l? You may assume that $l \ll Na$ and $N \gg 1$?

(b) What is the free energy at temperature T of a rubber molecule of length l? *Note:* If you have not yet done part (a), you may wish to express your answer in terms of the entropy.

(c) What is the force needed in order to hold the molecule at length l?

(d) If a rubber molecule held in contact with a heat bath at temperature T as it is isothermally stretched from l_i to length l_f ($l_f > l_i$), how much heat flows to or from the heat bath? Where does this heat go?

Possibly helpful formulae:

- The binomial coefficient, often read as "n choose k" is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

  and gives the number of ways that k objects can be chosen from n objects, regardless of order.

- Stirling's approximation:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

4(a) $S = k \log \Omega$

where Ω is the number of microstates. In this case, the number of microstates is determined by the number of segments pointing to the right (N_R) and the number pointing to the left (N_L)

$$\Omega = \binom{N}{N_R} = \binom{N}{N_L} = \frac{N!}{N_R!\,N_L!}$$

$$(N_R - N_L)\,a = l$$

$$N_R + N_L = N$$

$$N_R = \frac{l}{a} + N_L = \frac{l}{a} + (N - N_R)$$

$$N_R = \frac{N + \frac{l}{a}}{2} \qquad N_L = \frac{N - \frac{l}{a}}{2}$$

$$S = k \log \Omega = k \log \left(\frac{N!}{\left(\frac{N+\frac{l}{a}}{2}\right)!\left(\frac{N-\frac{l}{a}}{2}\right)!} \right)$$

$$\frac{S}{k} = \log N! - \log \left(\frac{N+\frac{l}{a}}{2}\right)! - \log \left(\frac{N-\frac{l}{a}}{2}\right)!$$

using Stirling's approximation

$$\log N! \approx N \log N - N$$

$$\frac{S}{k} \approx N \log N - N - \left(\frac{N+\frac{l}{a}}{2}\right) \log\left(\frac{N+\frac{l}{a}}{2}\right) + \frac{N+\frac{l}{a}}{2}$$
$$- \left(\frac{N-\frac{l}{a}}{2}\right) \log\left(\frac{N-\frac{l}{a}}{2}\right) + \frac{N-\frac{l}{a}}{2}$$

$$= N\log N - N - \frac{N+\frac{\ell}{a}}{2}\log\left(N+\frac{\ell}{a}\right) - \frac{N-\frac{\ell}{a}}{2}\log\left(N-\frac{\ell}{a}\right)$$

$$+ \frac{N+\frac{\ell}{a}+N-\frac{\ell}{a}}{2}\log 2 \ \cancel{+N+\ell} + N$$

$$= N\log 2 + N\log N - \frac{N+\frac{\ell}{a}}{2}\log\left(\cancel{N+\frac{\ell}{a}}\ N\cdot\left(1+\frac{\ell}{Na}\right)\right) - \frac{N-\frac{\ell}{a}}{2}\log\left(N\cdot\left(1-\frac{\ell}{Na}\right)\right)$$

$$= N\log 2 + N\log N - \frac{N+\frac{\ell}{a}}{2}\left(\log N + \log\left(1+\frac{\ell}{Na}\right)\right)$$

$$- \frac{N-\frac{\ell}{a}}{2}\left(\log N + \log\left(1-\frac{\ell}{Na}\right)\right)$$

$$= N\log 2 + \frac{\ell}{2a}\log\left(1-\frac{\ell}{Na}\right) - \frac{\ell}{2a}\log\left(1+\frac{\ell}{Na}\right)$$

$$\approx N\log 2 - \frac{\ell^2}{2Na^2} - \frac{\ell^2}{2Na^2}$$

$$= N\log 2 - \frac{\ell^2}{Na^2}$$

so $\boxed{S \approx k\left(N\log 2 - \frac{\ell^2}{Na^2}\right)}$

(b)

$$A = U - TS$$

internal energy is zero in this model!

$$\boxed{A = -kT\left(N\log 2 - \frac{\ell^2}{Na^2}\right)}$$

(c)

$$F = -\frac{dA}{d\ell} = -2\frac{kT}{Na^2}\ell$$

where the negative sign indicates that the rubber molecule wants to have $\ell = 0$.

(d) The work done ~~is~~ by the ~~rubber mole~~ person stretching the rubber is

$$W = \int_{\ell_i}^{\ell_f} -F\,d\ell = \int_{\ell_i}^{\ell_f} +\frac{2kT}{Na^2}\ell\,d\ell = \frac{kT}{Na^2}\left(\ell_f^2 - \ell_i^2\right)$$

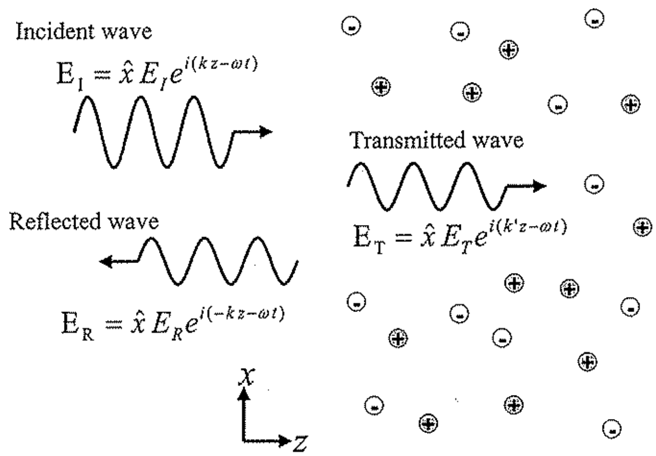
the $-$ sign comes because the person stretching it is exerting an equal and opposite force to that exerted by the rubber molecule.

By the first law of thermodynamics, the difference between the work done on a system and the heat flowing out of it is equal to its change in internal energy. Since our molecule always has zero internal energy, the work done on the molecule must equal the heat flowing out of it—or into the heat bath. So the heat flowing into the heat bath is

$$\boxed{Q = \frac{kT}{Na^2}\left(\ell_f^2 - \ell_i^2\right)}$$

Problem 5

A simple model to describe an n-doped semiconductor is collisionless electron gas embedded in a uniform positive charge balancing the negative charge of electrons. Let N, m, and $-e$ be the number density (number of electrons per unit volume), mass and charge of the free electrons. A plane electro-magnetic wave of angular frequency ω is incident on the semiconductor with normal angle. Assuming the motions of positive ions and bound electrons are negligible, answer the following questions.



(a) The current density and the electric field have the following relation: $\mathbf{J} = \sigma \mathbf{E}$, where σ is the conductivity. Determine the expression for the conductivity of the electron gas as a function of frequency ω.

(b) The electric displacement $\mathbf{D}$ is defined as $\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}$, where $\mathbf{P}$ is the polarization and ε is the electric permittivity. What is the electric permittivity of the electron gas?

(c) Suppose that the xy plane forms the boundary between vacuum and the electron gas and the electro-magnetic wave is traveling in the z direction. Find $r \equiv E_R / E_I$ and $t \equiv E_T / E_I$ using the electrodynamic boundary conditions. The magnetic permittivity of the electron gas is close to the value in vacuum: $\mu = \mu_0$.

(d) Find the dispersion relation of the medium, i.e., express the wave vector $k'(\omega)$ as a function of angular frequency ω.

(e) Find the reflection and transmission coefficients (R and T), which are the ratio of the reflected and transmitted intensity to the incident intensity, respectively. In what frequency range will electromagnetic waves be transmitted through the medium?

P5. (a) The equation of motion of a free electron is

$$m\frac{d\vec{v}}{dt} = -e\vec{E}, \quad \text{where } \vec{v} = \vec{v}_0 e^{-i\omega t}$$

$$\text{and } \vec{E} = \vec{E}_0 e^{-i\omega t}$$

$$\Rightarrow -i\omega m\vec{v} = -e\vec{E}$$

$$\Rightarrow \vec{v} = \frac{e\vec{E}}{im\omega}$$

The current density

$$\vec{J} = -Ne\vec{v} = \frac{iNe^2}{m\omega}\vec{E} = \sigma\vec{E}$$

Thus, $\boxed{\sigma = \frac{Ne^2 i}{m\omega}}$

(b) The equation of motion for an electron displacement

$$m\frac{d^2\vec{x}}{dt^2} = -e\vec{E}, \quad \text{where } \vec{x} = \vec{x}_0 e^{-i\omega t}$$

$$\Rightarrow -m\omega^2\vec{x} = -e\vec{E} \Rightarrow \vec{x} = \frac{e\vec{E}}{m\omega^2}$$

The dipole moment is $\vec{p} = -e\vec{x} = \frac{-e^2\vec{E}}{m\omega^2}$

The polarization is $\vec{P} = N\vec{p} = -\frac{Ne^2}{m\omega^2}\vec{E}$

The electric displacement

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \left(-\frac{Ne^2}{m\omega^2}\right)\vec{E}$$

$$= \varepsilon_0 \left(1 - \frac{Ne^2}{\varepsilon_0 m\omega^2}\right)\vec{E}$$

Thus $$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{Ne^2}{\varepsilon_0 m\omega^2}\right)$$

$$= \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right), \text{ where } \omega_p^2 \equiv \frac{Ne^2}{\varepsilon_0 m\omega^2}$$

plasma frequency

(C) $\vec{E}$ field and $\vec{H}$ field must be continuous.

$$\begin{cases} E_I + E_R = E_T & -① \\ H_I + H_R = H_T & -② \end{cases}$$

$$\vec{H}_I = \frac{E_I}{\mu_0 C}\hat{y}, \quad \vec{H}_R = -\frac{E_R}{\mu_0 C}\hat{y}$$

$$\vec{H}_T = \frac{E_T n}{\mu_0 C}\hat{y}, \quad n \equiv \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}} \text{ index of refraction}$$

$$② \Rightarrow \frac{E_I}{\mu_0 C} - \frac{E_R}{\mu_0 C} = \frac{n E_T}{\mu_0 C}$$

$$\Rightarrow E_I - E_R = \sqrt{\varepsilon}\, E_T = n E_T, \quad -③$$

From ① and ③

$$E_I = \frac{n+1}{2} E_T \Rightarrow \boxed{t = \frac{E_T}{E_I} = \frac{2}{n+1}}$$

and $(n-1) E_I + (n+1) E_R = 0 \Rightarrow \boxed{r = \frac{E_R}{E_I} = \frac{1-n}{1+n}}$

(d) $k' = n(\omega) \frac{\omega}{c} = \frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}} = \frac{1}{c}\sqrt{\omega^2-\omega_p^2}$

(e) When $\omega > \omega_p$, k' and $n(\omega)$ are real.

$$I = \frac{1}{2}|EH| = \frac{n}{2\mu_0 c} E^2 \Leftarrow (\vec{S} = \vec{E} \times \vec{H})$$

Poynting vector

$$R = \frac{I_R}{I_I} = \left(\frac{E_R}{E_I}\right)^2 = r^2 = \boxed{\left(\frac{1-n}{1+n}\right)^2}$$

$$T = \frac{I_T}{I_I} = \frac{n E_T^2}{E_I^2} = n(t)^2 = \boxed{\frac{4n}{(n+1)^2}}$$

When $\omega < \omega_p$, $\varepsilon(\omega) < 0$, thus $n(\omega)$ is pure imaginary.

$$\Rightarrow k' = n(\omega) \frac{\omega}{c} = i\,|n| \frac{\omega}{c}$$

$$\Rightarrow \vec{E}_T = E_T e^{-|k|z} e^{-i\omega t}$$: exponentially decay.

$\Rightarrow$ standing wave

$\Rightarrow$ $\vec{S} = 0$ There is no energy flow, thus $\boxed{T=0}$ no propagation of EM wave.

$$R = \left|\frac{E_R}{E_I}\right|^2 = \left|\frac{1-n}{1+n}\right|^2 = \left|\frac{1-i|n|}{1+i|n|}\right|^2$$

$$= \frac{1+|n|^2}{1+|n|^2} = \boxed{1} \text{ : perfect reflection.}$$

Problem 6

A retired physicist decides to spend her retirement developing an extra-efficient engine. Being of the opinion that steam engines are too old-fashioned, she decides to create an *ice* engine.

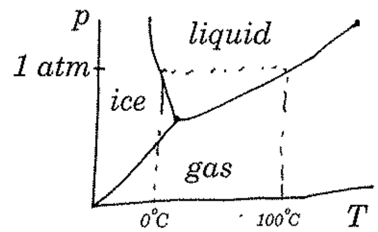
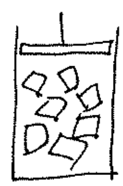


Figure 1: Phase diagram of water. *Not to scale!*

After first carefully consulting the phase diagram of water (see Figure 1), she designes her apparatus as sketched below. A piston chamber is filled with ice water. Heat baths and thermal insulation are arranged to allow both isothermal and adiabatic operation.



Having recognized that provided the volume remains intermediate between that of ice and liquid water, at constant temperature the pressure remains constant, she proposes to *reversibly* follow the loop shown in Figure 2, which is composed of two isotherms (one at 0°C and one at -1°C) and two adiabats.

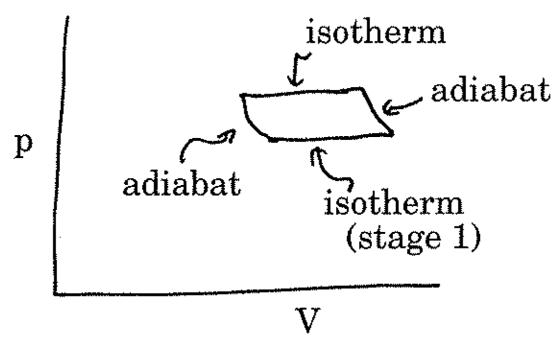


Figure 2: Path to be taken by the ice engine.

(a) In order to operate this device as a heat engine, in what direction on this cycle should the piston be operated? Draw a diagram which is a copy of Figure 2 and number the four stages in order, starting with the bottom stage as stage 1.

(b) Label the 0°C and -1°C isotherms on a copy of Figure 2. How can you tell which isotherm is which?

(c) For each of the four stages, what is the sign of the work done by the ice water (or is it zero), and what is the sign of the heat transfered to or from the ice water (or is it zero)?

(d) How does the net work in a single cycle done relate to the heat transfered in the four stages?

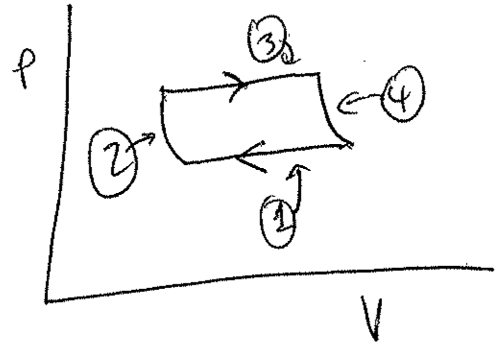
(e) What is the change of the entropy of the ice water over one cycle? Why?

(f) What is the change of entropy of the heat baths for one cycle? Why?

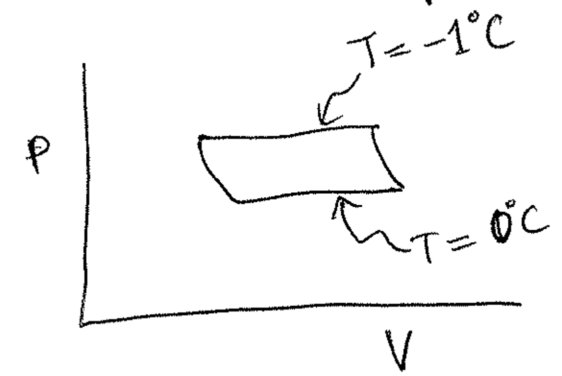
(g) What is the efficiency of this ice engine, defined as the ratio of the net work done to the heat transfered during stage 1?

(h) Would the efficiency have been different if this were a reversible steam engine?

6 (a)



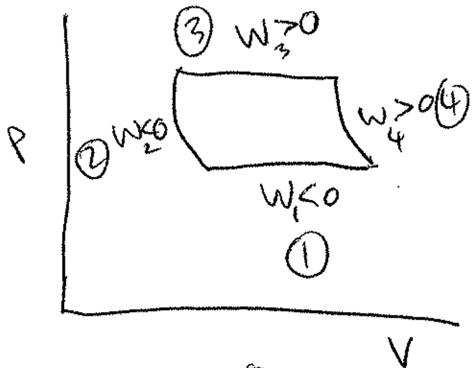
(b)



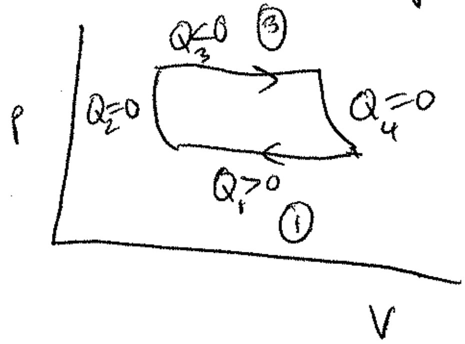
We can see this from the slope of the ice/liquid coexistence line of the phase diagram. At lower temperatures, ice and water coexist at higher pressures.

Note: that this is unusual — normally higher temperatures correspond to higher pressures, but in water the solid is less dense than the liquid, leading to some unusual behavior.

(c)



In stages 3 and 4, work is done by the ice water (i.e. $W > 0$), and in stages 1 and 2 work is done on the ice water ($W < 0$).



In stages 2 and 4, no heat is transferred.

In stage 1, heat is transferred to the ice water, causing some ice to melt.

In stage 3, heat is trasfered from the ice water ($Q_3 < 0$), as some of the water freezes.

(d) The net work done by the ice water is equal to the net heat transferred to the ice water

$$\sum_i W_i = \sum_i Q_i$$

(e) The change of entropy of the ice water over one cycle is zero, because entropy is a state function, i.e. only depends on the state of the system, which is unchanged for any cycle.

(f) The change of entropy of the heat baths is also zero because the cycle is performed reversibley. Reversible operation implies that the change of entropy of system plus surroundings is zero, since otherwise running the cycle in reverse would violate the second law.

(g) The total change of entropy for the heat baths (which is zero by the answer to f) is

$$\Delta S_{baths} = \frac{Q_1}{T_1} + \frac{Q_3}{T_3} = 0 \quad \cancel{Q_1 = \frac{T_1}{T_3} Q_3} \quad Q_3 = -\frac{T_3}{T_1} Q_1$$

$$W_{net} = Q_1 + Q_3 \quad \text{(recall that } Q_3 < 0\text{)}$$

$$\text{efficiency} = \frac{W_{net}}{Q_1} = \frac{Q_1 - \frac{T_3}{T_1} Q_1}{Q_1} = 1 - \frac{T_3}{T_1}$$

$$\boxed{\text{efficiency} = 1 - \frac{T_3}{T_1}} \quad \text{(recall that } T_3 < T_1\text{)}$$

(h) No, any reversible heat engine will have the same Carnot efficiency, which is a simple result of the first and second laws of thermodynamics.

Problem 7

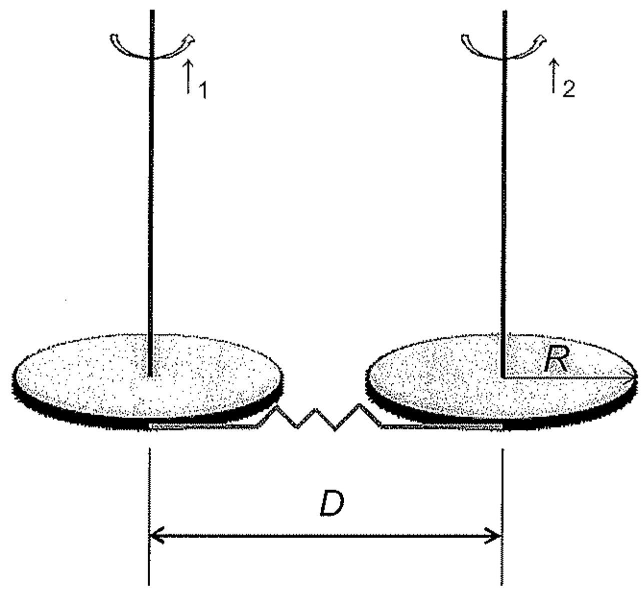
A torque pendulum comprises two identical disks, mass M and radius R each, connected to two identical massless rods with torque constants k separated by the distance $D > 2R$. The disks are also connected to each other by a massless spring with spring constant ε and rest length $l = D$ (Figure shows the system at equilibrium when rods and spring are not deformed).

1. Find normal modes of small oscillations ($|\varphi_{1,2}| << 1$) and frequencies of normal modes

2. Consider the evolution of modes and frequencies in the limits $\varepsilon \to 0$ and $\varepsilon \to \infty$.

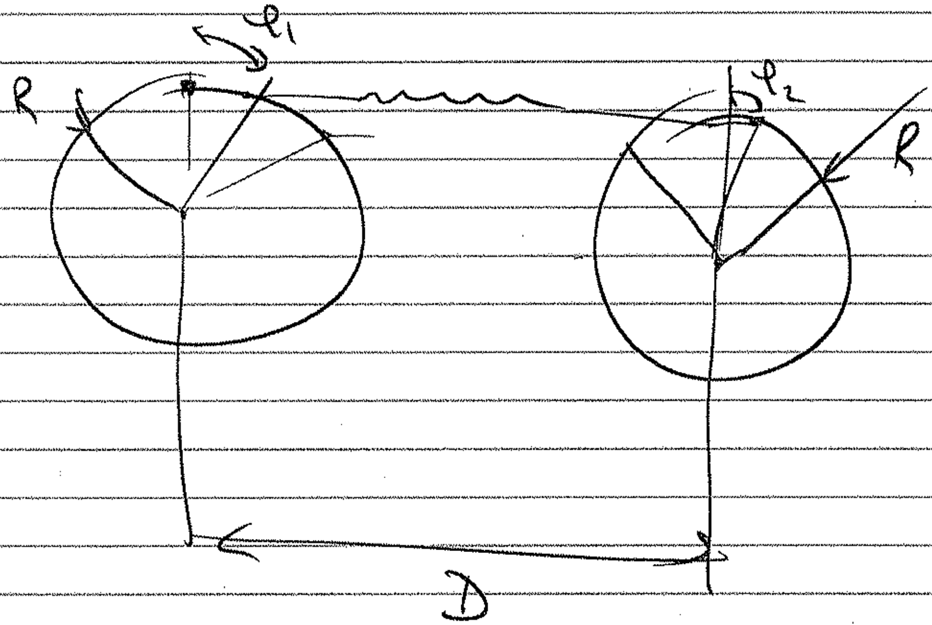
3. At $t = 0$ initial velocities are zero, and initial displacements are
   $\varphi_1(t=0) = \varphi_{01}$; $\varphi_2(t=0) = 0$.
   Solve for motion of the system.

Hint: disks can only rotate around rods; disks cannot slide horizontally or move vertically.



Comp Exam 2007- CM (grad. problem)

Note Title 3/29/2007



In the limit $|\varphi_{1,2}| \ll 1$, length of a spring:

$$\lambda \simeq D + R\varphi_1 + R\varphi_2$$

$$T = \frac{\epsilon R(\varphi_1 - \varphi_2)^2}{2} + \frac{k\varphi_1^2}{2} + \frac{k\varphi_2^2}{2}$$

$$K = \frac{I\dot{\varphi}_1^2}{2} + \frac{I\dot{\varphi}_2^2}{2} \; ; \; I = \frac{MR^2}{2}$$

$$L = K - T = \frac{I}{2}\left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2\right) - \frac{k}{2}\left(\varphi_1^2 + \varphi_2^2\right) - \frac{\epsilon}{2}R(\varphi_1 - \varphi_2)^2$$

$$\frac{\partial L}{\partial \varphi_1} = -k\varphi_1 - \epsilon R(\varphi_1 - \varphi_2) \; ; \; \frac{\partial L}{\partial \dot{\varphi}_1} = I\dot{\varphi}_1$$

$$\frac{\partial L}{\partial \varphi_2} = -k\varphi_2 + \epsilon R(\varphi_1 - \varphi_2) \; ; \; \frac{\partial L}{\partial \dot{\varphi}_2} = I\dot{\varphi}_2$$

Equations of motion:

$$\begin{cases} I\ddot{\varphi}_1 + k\varphi_1 + \epsilon R(\varphi_1 - \varphi_2) = 0 \\ I\ddot{\varphi}_2 + k\varphi_2 - \epsilon R(\varphi_1 - \varphi_2) = 0 \end{cases}$$

$$\Downarrow$$

$$I(\ddot{\varphi}_1 + \ddot{\varphi}_2) = -k(\varphi_1 + \varphi_2)$$

$$I(\ddot{\varphi}_1 - \ddot{\varphi}_2) = -(\varphi_1 - \varphi_2)(k + 2\epsilon R)$$

$$\Downarrow$$

modes:

(S) $$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \omega_1^2 = \frac{k}{I}$$

(a) $$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \omega_2^2 = \frac{k + 2\epsilon R}{I}$$

2 (S) does not depend on ϵ.

$\epsilon \to 0$: $\omega_2 \to \omega_1$; (a) and (s) are degenerate

$\epsilon \to \infty$ $\quad \omega_2 \gg \omega_1$

3 $$\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(A_1 \cos\omega_1 t + B_1 \sin\omega_1 t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}(A_2 \cos\omega_2 t + B_2 \sin\omega_2 t)$$

$$\left.\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\right|_{t=0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_1 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} A_2 = \begin{pmatrix} \varphi_{01} \\ 0 \end{pmatrix}$$

$$\left.\begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix}\right|_{t=0} = \omega_1 B_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \omega_2 B_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

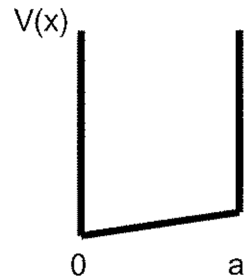
$$\Rightarrow \quad A_1 = A_2 = \varphi_{01}/2 \, ;$$

$$B_1 = B_2 = 0$$

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \varphi_{01} \cos \omega_1 t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \varphi_{01} \cos \omega_2 t$$

Problem 8

This problem concerns a particle of mass m in a one-dimensional potential well where the potential $V(x) = \lambda E_1^0 \frac{x}{a}$ for $0 \le x \le a$, and V is infinite elsewhere. $E_1^0 = \frac{\pi^2\hbar^2}{2ma^2}$ is the ground state energy of the infinite <u>square</u> well ($\lambda = 0$) and you may assume $\lambda << 1$.



(i) Find an expression for the energy of the n^{th} eigenstate to first order in λ.

(ii) Find the ground state wave function to first order in λ.

(iii) Again, to first order in λ, find the expectation value of the particle's position in the ground state.

Some of these integrals may be useful:

$$\int x\sin^2 x\,dx = \frac{x^2}{4} - \frac{x\sin 2x}{4} - \frac{\cos 2x}{8} \qquad \int x^2\sin^2 x\,dx = \frac{x^3}{6} - \left(\frac{x^2}{4} - \frac{1}{8}\right)\sin 2x - \frac{x\cos 2x}{4}$$

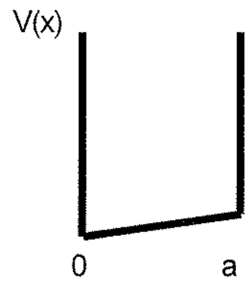
$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x \qquad \int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x \qquad \int \sin x\cos x\,dx = \frac{1}{2}\sin^2 x$$

$$\int \sin mx\,\sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \qquad (m^2 \neq n^2)$$

$$\int \sin mx\,\cos nx\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} \qquad (m^2 \neq n^2)$$

Solution to Problem 8

This problem concerns a particle of mass m in a one-dimensional potential well in which the potential $V(x) = \lambda E_1^0 \frac{x}{a}$ for $0 \le x \le a$, and V is infinite elsewhere. $E_1^0 = \frac{\pi^2 \hbar^2}{2ma^2}$ is the ground state energy of the infinite square well ($\lambda = 0$) and you may assume $\lambda << 1$.



(i) Find an expression for the n^{th} energy level of the given well to first order in λ.

Because λ is small, we can treat the potential, $V(x) = \lambda E_1^0 \frac{x}{a}$, as a perturbation with respect to the infinite square well. Using first-order, time-independent perturbation theory, the energy of the n^{th} level is

$E_n^1 = E_n^0 + \langle n|V(x)|n\rangle$ where $E_n^0 = n^2 E_1^0$ and $|n\rangle \doteq \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ are the solutions to the infinite square well. Thus,

$$\langle n|V(x)|n\rangle = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\frac{\lambda E_1^0 x}{a}\sin\left(\frac{n\pi x}{a}\right)dx = \frac{2}{a}\frac{\lambda E_1^0}{a}\left(\frac{a}{n\pi}\right)^2\int_0^{n\pi}\sin^2 u\, u\, du = \lambda\frac{E_1^0}{2}$$

$$E_n^1 = \left(n^2 + \frac{1}{2}\lambda\right)E_1^0$$

(ii) Find the ground state wave function to first order in λ.

Again, using first-order perturbation theory,

$$\psi_n^1(x) = \psi_n^0(x) + \sum_{m\neq n}\frac{\langle m|V(x)|n\rangle}{E_n^0 - E_m^0}\psi_m^0(x)$$

$$= \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) + \sum_{m=2}^{\infty}\frac{\langle m|V(x)|n\rangle}{E_1^0\left(n^2 - m^2\right)}\cdot\sqrt{\frac{2}{a}}\sin\left(\frac{m\pi x}{a}\right)$$

Evaluate the matrix element for $n = 1$:

$$\langle m|V(x)|1\rangle = \left(\frac{2}{a}\right)\int_0^a \sin\left(\frac{m\pi x}{a}\right)\frac{\lambda E_1^0 x}{a}\sin\left(\frac{\pi x}{a}\right)dx \;=\frac{2}{a}\frac{\lambda E_1^0}{a}\left(\frac{a}{\pi}\right)^2\int_0^\pi \sin mu \sin u \; u\,du$$

Using the given integral, $\int \sin mx \,\sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \qquad (m^2 \neq n^2)$, integrate by parts:

$$\int_0^\pi \sin mu \sin u\, u\,du = u\cdot\left(\frac{\sin(m-1)u}{2(m-1)} - \frac{\sin(m+1)u}{2(m+1)}\right)\Bigg|_0^\pi - \int_0^\pi\left(\frac{\sin(m-1)u}{2(m-1)} - \frac{\sin(m+1)u}{2(m+1)}\right)du$$

$$= 0 + \frac{\cos(m-1)u}{2(m-1)^2}\Bigg|_0^\pi - \frac{\cos(m+1)u}{2(m+1)^2}\Bigg|_0^\pi = \begin{cases} -\frac{1}{(m-1)^2} + \frac{1}{(m+1)^2} = -\frac{4m}{(m-1)^2(m+1)^2} & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd}\end{cases}$$

$$\langle m|V(x)|1\rangle \;= -\lambda\frac{8E_1^0}{\pi^2}\frac{m}{(m+1)^2(m-1)^2} \text{ and since } E_1^0 - E_m^0 = E_1^0\left(1-m^2\right),$$

$$\psi_1^1(x) = \sqrt{\frac{2}{a}}\left[\sin\left(\frac{\pi x}{a}\right) + \lambda \sum_{\substack{m=0\\ m\,\text{even}}}^{\infty} a(m)\,\sin\left(\frac{m\pi x}{a}\right)\right]$$

where $a(m) = \frac{8}{\pi^2}\frac{m}{(m+1)^2(m-1)^2(m^2-1)}$

(iii) Again, to first order in λ, find the expectation value of the particle's position in the ground state.

$$\langle x\rangle = \langle 1|x|1\rangle = \int_0^a \psi_1^1(x)x\,\psi_1^1(x)\,dx$$

$$= \left(\frac{2}{a}\right)\int_0^a \sin\left(\frac{\pi x}{a}\right)x\sin\left(\frac{\pi x}{a}\right)dx + \lambda\, 2\left(\frac{2}{a}\right)\sum_{\substack{m=0\\ m\,\text{even}}}^{\infty} \mathrm{a(m)}\int_0^a \sin\left(\frac{\pi x}{a}\right)x\sin\left(\frac{m\pi x}{a}\right)dx + \lambda^2(\cdots)$$

$$\langle x\rangle = \left(\frac{2}{a}\right)\left(\frac{a}{\pi}\right)^2\int_0^\pi \sin^2 u\, u\,du + \lambda\left(\frac{4}{a}\right)\left(\frac{a}{\pi}\right)^2\sum_{\substack{m=0\\ m\,\text{even}}}^{\infty}\mathrm{a(m)}\int_0^\pi \sin u \sin mu\, u\,du$$

$$\langle x\rangle = \left(\frac{2}{a}\right)\left(\frac{a}{\pi}\right)^2\cdot\frac{\pi^2}{4} + \lambda\left(\frac{4}{a}\right)\left(\frac{a}{\pi}\right)^2\sum_{\substack{m=0\\ m\,\text{even}}}^{\infty} a(m)\cdot\frac{-4m}{(m+1)^2(m-1)^2}$$

$$= \frac{a}{2} - \lambda\frac{16\,a}{\pi^2}\sum_{\substack{m=0\\ m\,\text{even}}}^{\infty} a(m)\cdot\frac{m}{(m+1)^2(m-1)^2} = \left(\frac{a}{2}\right)\left[1-\lambda\frac{32}{\pi^2}\sum_{\substack{m=0\\ m\,\text{even}}}^{\infty} a(m)\cdot\frac{m}{(m+1)^2(m-1)^2}\right]$$

The shift from the middle of the well is very small. Taking just the $m = 2$ term,

$$\langle x\rangle = \frac{a}{2}\left(1-\lambda\frac{1024}{243\,\pi^4}\right) = \frac{a}{2}(1-0.04\lambda)$$

Some of these integrals may be useful:

$$\int x\sin^2 x\,dx = \frac{x^2}{4}-\frac{x\sin 2x}{4}-\frac{\cos 2x}{8} \qquad \int x^2\sin^2 x\,dx = \frac{x^3}{6}-\left(\frac{x^2}{4}-\frac{1}{8}\right)\sin 2x-\frac{x\cos 2x}{4}$$

$$\int \sin^2 x\,dx = \frac{1}{2}x-\frac{1}{4}\sin 2x \qquad \int \cos^2 x\,dx = \frac{1}{2}x+\frac{1}{4}\sin 2x \qquad \int \sin x\cos x\,dx = \frac{1}{2}\sin^2 x$$

$$\int \sin mx\,\sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)}-\frac{\sin(m+n)x}{2(m+n)} \qquad (m^2\neq n^2)$$

$$\int \sin mx\,\cos nx\,dx = -\frac{\cos(m-n)x}{2(m-n)}-\frac{\cos(m+n)x}{2(m+n)} \qquad (m^2\neq n^2)$$