OSU PHYSICS DEPARTMENT
COMPREHENSIVE EXAMINATION #101

September 25 and 26, 2006

Comprehensive examination for Fall 2006

PART I, Monday September 25, 9:00 am

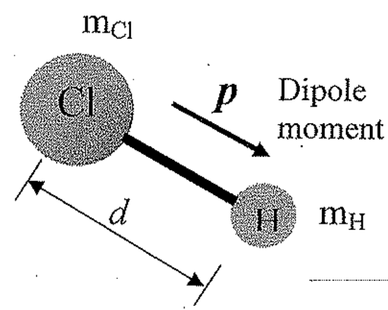
General Instructions

This Comprehensive Examination for Fall 2006 consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, September 25, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, September 26, at 9:00 am and 1:30 pm.
Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit - especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.
If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.
Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Problem 1.

The rotational energy levels of molecules fall into microwave and far-infrared spectral range. The absorption or emission of light due to rotational motion is allowed only for polar molecules, such as HCl, H_2O, NH_3, etc. HCl is a diatomic molecule which consists of a hydrogen atom (H) and a chlorine atom (Cl) connected with a covalent single bond. HCl has a large dipole moment due to the large electronegativity of the chlorine atom.

(a) Imagine a HCl molecule as two atoms (H and Cl) connected by a rigid rod and moving freely in vacuum. The Schrödinger equation is exactly solvable for the rigid rotor problem. Construct the Hamiltonian of a HCl molecule and find the rotational energy levels.



(b) The Stark effect is the splitting and shift of a spectral line into several components in the presence of an electric field. Calculate the Stark shift for the ground state of the HCl rotational motion when a static electric field ($\mathbf{E} = E_0\hat{z}$) is applied along the z-axis.

Problem 2

The meaning of the symbols used: B – magnetic field, S – the spin number of an atom, m – the quantum magnetic number, which can be thought of as the projection of the spin on the B direction (see the figure), μ_B – the Bohr magneton.

If an atom with spin S and zero orbital momentum is placed in a magnetic field B, its magnetic quantum number m is capable of assuming any one of the discrete values $m = -S, -S+1, -S+2, \ldots, S-1, S$ (in the figure below, possible m values for $S = 3$ are shown as an example). The magnetic energy of the atom is $\epsilon(m) = -2\mu_B mB$.

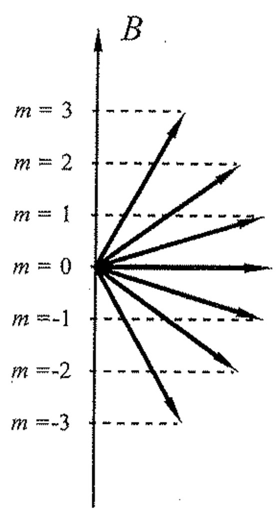
Consider a crystal containing N atoms with spin S. Assume that the interactions between the magnetic of the atoms are negligible.

1. Find the partition function of the system. *Hint:* You may want to use the relation:

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$$

2. Find the entropy of the system.

3. Suppose that the system is cooled down to temperature T_i in a strong magnetic field B_i. Next, the system is thermally insulated (no heat is allowed to flow into or out of it) and the field is slowly reduced. When the field reaches the value of B_f, the sample temperature T_f is measured. Find T_f (you are not allowed to use any integration).



Problem 3

A sphere of radius R carries the total electric charge Q. The sphere is immersed in a dielectric medium with the dielectric constant $\epsilon(r)$ depending on the distance r from the center of the sphere. Find the electric displacement, electric field, electrostatic potential, polarization, volume and surface distribution of bound charges everywhere for the following two cases:

1. The sphere is made of a metal.
2. The sphere is made of a dielectric with the dielectric constant ϵ_o, and the free charge Q is uniformly distributed over its volume. Apply your general results to the specific case $\epsilon(r) = 1 + a/r$.

Problem 4

A small bead of mass m is threaded on a thin, rigid and very long rod of negligible mass. The bead is free to slide with negligible friction along the rod. One end of the rod is hinged to an axis which is rotating freely in the horizontal plane, with a fixed angular velocity ω. Neglecting the effects related to special relativity,

1. Find the position of the bead as function of time.
2. Find an expression for the radial displacement of the bead after a long time.
3. Find the total energy of the bead.
4. Find the amount of work done per unit time on the bead.

Problem 5

The equation describing a planar sound wave propagating in the x direction in a gas is:

$$\rho \frac{\partial^2 \xi}{\partial t^2} = B_s \frac{\partial^2 \xi}{\partial x^2} \tag{1}$$

where ξ is the displacement of the gas molecules about their mean positions, ρ is the equilibrium density of the gas, and B_s is the adiabatic bulk modulus of the gas, defined as:

$$B_s = -V \left(\frac{\partial p}{\partial V} \right)_s ,$$

where p is the gas pressure, V is the gas volume, and S is the entropy.

(a) Find the expression for the speed of sound v in terms of the quantities used in Equation (1).

(b) The equation for v in terms of ρ and B_s is not particularly useful in practical applications because the B_s moduli for gases are seldom used in thermodynamic calculations, and their values are not listed in standard tables. Express v in terms of C_p, C_V and other physical parameters and constants that are typically used for describing gases in thermodynamics.

(c) Find the ratio of the velocities

$$\frac{v_{He^4}}{v_{He^3}}$$

where v_{He^4} is the sound velocity for bosonic helium He^4 and v_{He^3} is the sound velocity for fermionic helium He^3.

(d) Find the ratio of the velocities

$$\frac{v_{He^4}}{v_{N_2}}$$

where N_2 is diatomic nitrogen gas composed of its most common isotope.

Assume *room temperature* and treat the gases as ideal.

Problem 6

Two spinless identical particles are trapped in a one-dimensional harmonic oscillator, and the mutually attractive interaction is described as a Gaussian. The Hamiltonian for this quantum system is written as

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2}m\omega^2 x_2^2 - V_0 e^{-(x_1-x_2)^2/\alpha^2}.$$

(a) Find the energy levels and corresponding degeneracies when the interaction is negligible.

(b) Calculate the ground state energy to the first order assuming the Gaussian interaction term as a perturbation. Normalized wavefunction of the ground state is

$$\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \mathbf{exp}\left(-\frac{m\omega}{2\hbar}x^2\right).$$

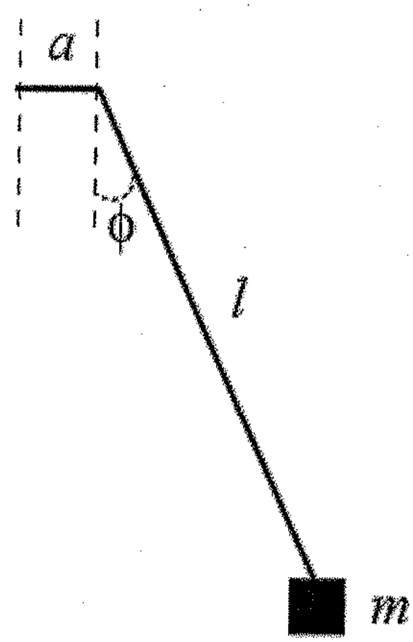
(c) Suppose that the interaction term is strong enough to dominate over the oscillator terms ($\frac{V_0}{\alpha^2} >> m\omega^2$), at least for low-lying states. Discuss the behavior of the system and estimate the ground state energy.

Problem 7

In a broad class of phenomena the behavior of a mechanical system can be related to two distinct excitation forces – a relatively slow one, and another one, rapidly changing in time. In this case, the effect of the fast force can be reduced to a shift of the effective potential energy. In this problem, you have to consider a pendulum with the length l, and mass m, whose support is oscillating with very high frequency ν in horizontal direction with a magnitude a. You have to prove that under some conditions, this pendulum will have non-vertical stable equilibrium.

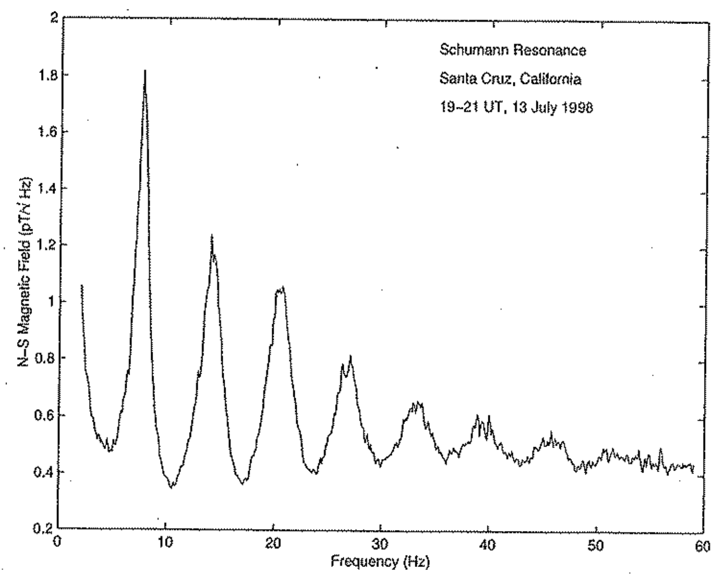
To show this unusual property of a pendulum, follow the steps below:

1. Find Lagrangian of the system, and its equations of motion
2. Represent ϕ (see Fig. below) as a sum of an "unperturbed" solution Φ (corresponding to $a=0$), and small oscillation around this solution $\xi(t)$ [due to support oscillation]. Note that $\xi(t)$ changes much faster than $\Phi(t)$; Find the equation of motion in the regime $|\xi|<<1$. Identify the "fast" and "slow" terms in this equation;
3. Write down the equation of motion for the dominant <u>fast</u> terms (neglect all higher-order corrections); solve this equation for $\xi(t)$, assuming Φ=const;
4. Plug in your solution for $\xi(t)$ into original equation of motion; average this equation over one "fast" period (assume that Φ=const at this scale); You should now be able to identify the "effective" time-independent potential for $\Phi(t)$.
5. Use this potential to find the positions of stable equilibrium. Consider two cases: $(a\nu)^2<2gl$; and $(a\nu)^2>2gl$;



Question 8

A planet, such as Earth or Jupiter, can exhibit electromagnetic propagation phenomena in the atmosphere between the core and the ionosphere. On Earth, it is particuarly interesting to analyze the propagation of extra- or ultra-low frequency (ELF or ULF) waves propagating for great distances between the surface and the F2 layer of the ionosphere which begins at 250 km. For instance, the Schumann resonances are shown below. At such low frequencies, the solid earth has a conductivity and relative dielectric constant



essentially those of the ocean, $\sigma_e = 4$ S/m and $\epsilon_e = 80$. For the ionosphere, $\sigma_i = 10^{-7}$ S/m and $\epsilon_i = 15$. For the intervening atmosphere, $\sigma_a = 0$ and $\epsilon_a = 1$.

1. Begin the analysis with the simplest model. Since the radius of the earth is about 6400 km and the distance between the surface and the F2 layer is only 250 km, consider the earth/atmosphere/ionosphere structure to constitute a planar waveguide. To simplify further, pretend that the earth and the ionosphere are both perfect conductors.

   (a) Determine the cut-off frequencies for the first two TE and TM modes.

   (b) Is there a mode with no cut-off frequency that can be responsible for ELF wave propagation?

   (c) To account for the spherical shape of the earth, impose a cyclic boundary condition on the waveguide and explain the existence of the Schumann resonances. What should be the frequencies of the first few resonances?

2. Now consider the same waveguide structure using the actual values of σ and ϵ.

   (a) Find the electric and magnetic field distribution functions for the first Schumann resonance mode.

   (b) What are the penetration depths for $\vec{E}$ and $\vec{B}$ into the solid earth and the ionosphere?

   (c) What is the loss per km for this mode of propagation?

P1. (a) In free space, the total energy of a HCl molecule is

$$E = \frac{P_H^2}{2m_H} + \frac{P_{Cl}^2}{2m_{Cl}}$$

The rigid body approximation makes it easier to describe the energy with center of mass coordinate and relative coordinate.

$$\begin{cases} \vec{R} = \dfrac{m_H \vec{r}_H + m_{Cl} \vec{r}_{Cl}}{m_H + m_{Cl}}, & M = m_H + m_{Cl} \\ \vec{r} = \vec{r}_H - \vec{r}_{Cl}, & \mu = \dfrac{m_H m_{Cl}}{m_H + m_{Cl}} \end{cases}$$

$$\Rightarrow \quad E = \frac{P_{CM}^2}{2M} + \frac{P_r^2}{2\mu}$$

Therefore, Hamitonian of the system can be described by

$$H = \underbrace{\frac{P_{CM}^2}{2M}}_{\text{free particle Kinetic energy of the molecule}} + \underbrace{\frac{P_r^2}{2\mu}}_{\text{rotational energy}} = H_{CM} + H_{Rot}$$

We are interested in the second term.

$$H_{Rot} = \frac{P_r^{\,2}}{2\mu} = -\frac{\hbar^2}{2\mu}\nabla_r^{\,2} = -\frac{\hbar^2}{2\mu}\left(\cancel{\frac{\partial^2}{\partial r^2}} + \cancel{\frac{2}{r}\frac{\partial}{\partial r}} - \frac{L^2}{r^2\hbar^2}\right)$$

where $r = d$: constant.

Therefore
$$H_{Rot} = \frac{L^2}{2\mu d^2} = \frac{L^2}{2I}$$

whose $I \equiv \mu d^2$: HCl moment of inertia.

The square of the angular momentum operator satisfies the following eigenvalue equation.
$$L^2 |\,l m\rangle = \hbar^2 l(l+1)\,|l m\rangle.$$

Thus,
$$\boxed{E_{Rot} = \frac{\hbar^2 l(l+1)}{2I}}$$

(b) The interaction Hamiltonian is
$$H_1 = -\vec{p}\cdot\vec{E} = -p E_0 \cos\theta$$

(i) First order correction

$$E^{(1)} = \langle 00|\,H_1\,|00\rangle$$
$$= -pE_0 \langle 00|\underbrace{\cos\theta}_{\text{odd function}}|00\rangle = 0$$

(ii) Second order correction

$$E^{(2)} = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \frac{|\langle lm|\,H_1\,|00\rangle|^2}{E_l - E_0}$$

where $\langle lm | H_1 | 00 \rangle = -pE_0 \langle lm | \cos\theta | 00 \rangle$

$$= -pE_0 \int Y_{lm}(\theta,\phi) \underbrace{\cos\theta}_{= \sqrt{\frac{4\pi}{3}} Y_{10}} \underbrace{Y_{00}(\theta,\phi)}_{= \frac{1}{\sqrt{4\pi}}} d\Omega$$

$$= -\frac{pE_0}{\sqrt{3}} \int Y_{lm} Y_{10} \, d\Omega$$

$$= -\frac{pE_0}{\sqrt{3}} \delta_{l1} \delta_{m0}$$

Therefore, $\underbrace{\Delta E}_{\text{Stark shift}} = E^{(2)} = \dfrac{\left| -\frac{pE_0}{\sqrt{3}} \right|^2}{E_1 - E_0}$

$$= \frac{p^2 E_0^2}{3} \frac{1}{\frac{\hbar^2}{I} - 0} = \boxed{\frac{p^2 E_0^2}{3\hbar^2}}$$

Problem 2 - solution:

(a) Assuming a plane wave solution, $\xi = A e^{i(kx-\omega t)}$ we obtain:

$$\rho \cdot \omega^2 A e^{i(kx-\omega t)} = B_s \cdot k^2 \cdot A e^{i(kx-\omega t)}$$

from which it follows that: $\frac{\omega}{k} = \sqrt{\frac{B_s}{\rho}}$, and $\frac{\omega}{k} = v$ is the speed of the sound wave.

(b) Applying the cyclic chain rule to the adiabatic modulus B_s, and then the ordinary chain rule, we get:

$$B_s = -V\left(\frac{\partial p}{\partial V}\right)_S = +V\frac{\left(\frac{\partial S}{\partial V}\right)_p}{\left(\frac{\partial S}{\partial p}\right)_V} = V\frac{\left(\frac{\partial S}{\partial T}\right)_p\left(\frac{\partial T}{\partial V}\right)_p}{\left(\frac{\partial S}{\partial T}\right)_V\left(\frac{\partial T}{\partial p}\right)_V} = V\frac{C_p\left(\frac{\partial T}{\partial V}\right)_p}{C_V\left(\frac{\partial T}{\partial p}\right)_V}$$

Again applying the cyclic chain rule "backwards"

$$\frac{\left(\frac{\partial T}{\partial V}\right)_p}{\left(\frac{\partial T}{\partial p}\right)_V} = -\left(\frac{\partial p}{\partial V}\right)_T \implies B_s = -V\frac{C_p}{C_V}\left(\frac{\partial p}{\partial V}\right)_T$$

For ideal gas, $pV = Nk_BT \implies \left(\frac{\partial p}{\partial V}\right)_T = -\frac{Nk_BT}{V^2}$

so: $B_s = \frac{C_p}{C_V}\frac{Nk_BT}{V}$

Now, with $\rho = \frac{m}{V} = \frac{\mu \cdot N}{V}$ where μ is the molecular mass

we obtain the speed of sound as: $\boxed{v = \sqrt{\frac{C_p}{C_V}\frac{kT}{\mu}}}$

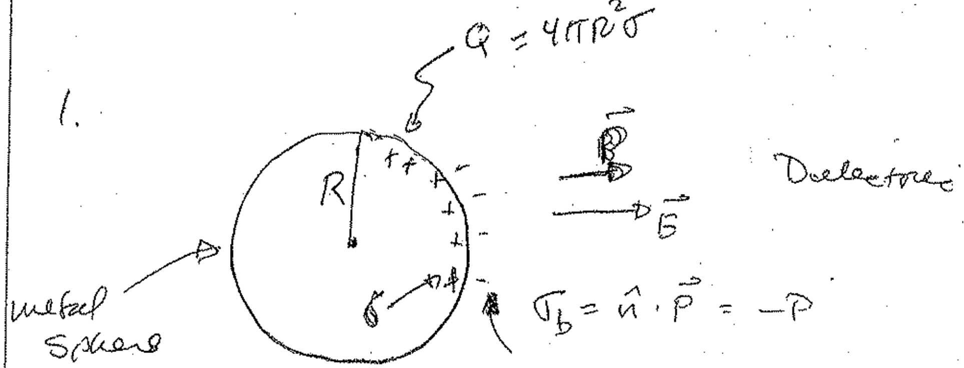
(c)
For both He^4 and He^3 $C_p/C_V = \frac{5}{3}$, so $\frac{v_{He^4}}{v_{He^3}} = \sqrt{\frac{\mu_{He}}{\mu_{He}}} = \sqrt{\frac{3}{4}} \approx 0.87$

(d) For N_2, $C_p/C_V = 7/5$, and $\mu_{N_2} = 28$

$$\frac{v_{He^4}}{v_{N_2}} = \sqrt{\frac{5/3 \cdot 28}{7/5 \cdot 4}} = \sqrt{\frac{25}{3}} \approx 2.89$$

Prob 3 UG EM

1.



$\vec{E} = \dfrac{Q \hat{r}}{4\pi \epsilon r^2}$ for $r > R$ $\qquad \vec{P} = \epsilon_0 \chi \vec{E}$, $\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi \vec{E}$

$\vec{D} = \epsilon \vec{E} = Q \hat{r} / 4\pi r^2 \qquad \chi = \epsilon/\epsilon_0 - 1$

$\vec{P} = (\epsilon - \epsilon_0)\vec{E} = Q(\epsilon - \epsilon_0)\hat{r}/4\pi\epsilon r^2 = \dfrac{Q}{4\pi r^2}\left(1 - \dfrac{\epsilon_0}{\epsilon}\right)\hat{r}$

$\sigma_b = -|\vec{P}| \qquad \rho_b = -\vec{\nabla}\cdot\vec{P} = \dfrac{Q\epsilon_0}{4\pi r^2} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(\dfrac{r^2}{\epsilon(r)}\right)$

$\Phi(r) = Q/4\pi\epsilon r$

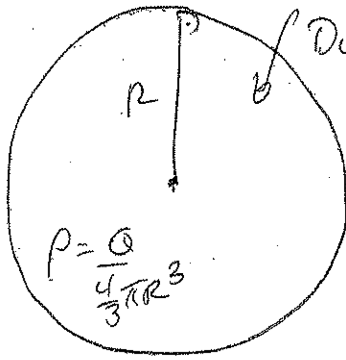
On the outer surface of the dielectric for some large value of r, $\sigma_b = +|\vec{P}| = \dfrac{Q}{4\pi r^2}\left(1 - \dfrac{\epsilon_0}{\epsilon}\right)$

Beyond the dielectric, $\vec{E} = Q\hat{r}/4\pi\epsilon_0 r^2$, $\vec{P} = 0$, $\vec{D} = \epsilon_0 \vec{E}$

$$\Phi(r) = Q/4\pi\epsilon_0 r$$

Inside sphere, $\vec{E} = 0$, $\vec{P} = 0$, $\vec{D} = 0$, $\Phi = \displaystyle\int \dfrac{\sigma\, da}{4\pi\epsilon_0)} = \dfrac{Q}{4\pi\epsilon_0 R}$

2.



Dielectric sphere $\epsilon = \epsilon_0$

$$\Phi(r<R) = \frac{\rho \frac{4}{3}\pi r^3}{4\pi\epsilon_0 r} + \int_r^R \frac{\rho}{4\pi\epsilon_0 r} r^2 \sin\theta \, d\theta \, d\varphi \, dr$$

$$= \frac{\rho r^2}{3\epsilon_0} + \frac{\rho}{2\epsilon_0}(R^2 - r^2) = \frac{\rho}{\epsilon_0}\left(\frac{R^2}{2} - \frac{r^2}{6}\right)$$

$$\vec{E}(r<R) = -\vec{\nabla}\Phi = \frac{\rho r}{3\epsilon_0}\hat{r}$$

$$\vec{D} = \epsilon_0 \vec{E}, \quad \vec{P} = 0, \quad \sigma_b = 0, \quad \rho_b = 0$$

Outside the sphere:

Same $\vec{E}$, $\vec{P}$, $\vec{D}$, ~~and~~ Φ as in part 1
Same σ_b, ρ_b

Specific case $\epsilon(r) = \epsilon_0 (1 + a/r)$, perhaps a dielectric of decreasing density

The interesting quantity is $\rho_b = -\vec{\nabla}\cdot\vec{P}$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \left(\frac{1}{1+a/r}\right) = \frac{1}{r^2}\left(\frac{2r}{1+a/r} + \frac{a}{(1+a/r)^2}\right)$$

which $\neq 0$ except as $r \to \infty$

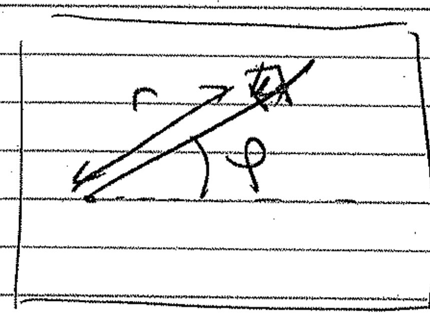
2006. Fall comp. undergraduate mech (problem #4)

Note Title 8/17/2006

(1) Starting from rotating ref. frame: ($\dot{\varphi} = \omega = \text{const}$)

$\ddot{r} = r\omega^2$

$\Downarrow$



(5') $\begin{cases} r(t) = r_1 e^{\omega t} + r_2 e^{-\omega t} \\ \varphi(t) = \omega t \end{cases}$

(2) (2) $r(t) \sim r_1 \cdot e^{\omega t}$ $[\omega t \gg 1]$

(3) $T = \frac{m v^2}{2} = \frac{m}{2}(\dot{r}^2 + r\dot{\varphi}^2) = \frac{m}{2}\Big[(r_1 \omega e^{\omega t} - r_2 \omega e^{-\omega t})^2 +$

$+ (r_1 e^{\omega t} + r_2 e^{-\omega t})^2 \omega\Big] =$

(3) $T = m\omega^2 [r_1^2 e^{2\omega t} + r_2^2 e^{-2\omega t}]$

(4) $A = \int \vec{F} d\vec{l}$; $P = \frac{dA}{dt} = \vec{F} \cdot \vec{v} = F_\varphi \cdot v_\varphi$

Note: there is no radial force in inertial ref. frame!

(5) $P = dT/dt = 2m\omega^2 (r_1^2 e^{2\omega t} - r_2^2 e^{-2\omega t})$

Solution, Probl. 5:

1. Partition function:

For an individual spin:

$$Z_{ind} = \sum_{m=-S}^{S} e^{-\varepsilon(m)/kT} = \sum_{m=-S}^{S} e^{\frac{\mu_B B}{k_B T}\cdot m} = \sum_{m=-S}^{S}\left(e^{\frac{\mu_B B}{k_B T}}\right)^m$$

We use the identity given in the hint:

$$\sum_{k=-n}^{n} x^k = x^{-n}\sum_{k=-n}^{n} x^{k+n} = x^{-n}\sum_{l=0}^{2n} x^l$$

$$= x^{-n}\cdot\frac{1-x^{2n+1}}{1-x}$$

which can be further processed:

$$= x^{-n-\frac{1}{2}}\cdot x^{\frac{1}{2}}\;\frac{1-x^{2n+1}}{1-x} = \frac{x^{-n-\frac{1}{2}} - x^{n+\frac{1}{2}}}{x^{-\frac{1}{2}} - x^{\frac{1}{2}}}$$

Accordingly:

$$Z_{ind} = \frac{\left[e^{\frac{\mu_B B}{k_B T}}\right]^{-S-\frac{1}{2}} - \left[e^{\frac{\mu_B B}{k_B T}}\right]^{S+\frac{1}{2}}}{\left[e^{\frac{\mu_B B}{k_B T}}\right]^{-\frac{1}{2}} - \left[e^{\frac{\mu_B B}{k_B T}}\right]^{\frac{1}{2}}}$$

$$= \frac{e^{\frac{\mu_B B}{2k_B T}(2S+1)} - e^{-\frac{\mu_B B}{2k_B T}(2S+1)}}{e^{\frac{\mu_B B}{2k_B T}} - e^{-\frac{\mu_B B}{2k_B T}}} = \frac{\sinh\left[\frac{\mu_B B(2S+1)}{2k_B T}\right]}{\sinh\left[\frac{\mu_B B}{2k_B T}\right]}$$

The partition function for the entire system:

$$\mathcal{Z}_{TOT} = (\mathcal{Z}_{ind})^N = \left\{ \frac{\sinh\left[\frac{\mu_B B}{2k_B T}(2S+1)\right]}{\sinh\left[\frac{\mu_B B}{2k_B T}\right]} \right\}^N$$

we use $\mathcal{S}$ for entropy in order not to confuse it with S, the spin number

2. In order to obtain the entropy $\mathcal{S}$ we use the standard procedure:

$$F = -k_B T \ln \mathcal{Z}_{TOT} \quad \text{and} \quad \mathcal{S} = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$$

$$F = -k_B T N \ln\left\{\sinh\left[\frac{\mu_B B}{k_B T}(2S+1)\right]\right\}$$

$$+ k_B T N \ln\left\{\sinh\left[\frac{\mu_B B}{k_B T}\right]\right\}$$

$$\mathcal{S} = -\frac{\partial F}{\partial T} = k_B T N \frac{\cosh\left[\frac{\mu_B B}{k_B T}(2S+1)\right]}{\sinh\left[\frac{g\mu_B B}{k_B T}(2S+1)\right]} \cdot \left[-\frac{\mu_B B}{k_B T^2}(2S+1)\right]$$

$$- k_B T N \frac{\cosh\left[\frac{\mu_B B}{k_B T}\right]}{\sinh\left[\frac{\mu_B B}{k_B T}\right]} \left[-\frac{\mu_B B}{k_B T^2}\right]$$

$$= -kN\left\{\coth\left[\frac{\mu_B}{k_B} \cdot \left(\frac{B}{T}\right)(2S+1)\right]\right\} \cdot \left[\frac{\mu_B}{k_B}\left(\frac{B}{T}\right)(2S+1)\right]$$

$$+ kN\left\{\coth\left[\frac{\mu_B}{k_B}\left(\frac{B}{T}\right)\right]\right\}\left[\frac{\mu_B}{k_B}\left(\frac{B}{T}\right)\right]$$

3. Introducing a "shorthand" symbol:

$A \equiv \frac{\mu_B}{k}$, we obtain:

$$\mathcal{S} = kNA\left(\frac{B}{T}\right)\left\{\coth\left[A\cdot\left(\frac{B}{T}\right)\right] - (2S+1)\coth\left[(2S+1)A\left(\frac{B}{T}\right)\right]\right\}$$

In the right-hand part, everything is constant, except $\left(\frac{B}{T}\right)$.

So, if $\left(\frac{B}{T}\right)$ remains constant, also $\mathcal{S}$ = const.

The decreasing of B is an adiabatic process, so that $\mathcal{S}$ = const, and therefore $\frac{B}{T}$ = const, In other words:

$$\frac{B_i}{T_i} = \frac{B_f}{T_f} \Rightarrow \boxed{T_f = \frac{B_f}{B_i}T_i}$$

In "the heat of the exam" the student may not notice that the entropy is effectively a function of a single variable $\frac{B}{T}$.

The student may try to solve task 3 by integration — which is also a legitimate method.

It requires more work, though.

P6. (a) The Hamiltonian is separable.

$$H = H_1 + H_2 = \left(\frac{P_1^2}{2m} + \frac{1}{2}m\omega^2 x_1^2\right)$$

$$+\left(\frac{P_2^2}{2m} + \frac{1}{2}m\omega^2 x_2^2\right)$$

$$\Rightarrow \begin{cases} H_1 |n_1\rangle = (n_1+\frac{1}{2})\hbar\omega |n_1\rangle \\ H_2 |n_2\rangle = (n_2+\frac{1}{2})\hbar\omega |n_2\rangle \end{cases}$$

Therefore

$$H |n_1 n_2\rangle = (n_1 + n_2 + 1)\hbar\omega \underbrace{|n_1 n_2\rangle}_{|n_1\rangle \otimes |n_2\rangle}$$

Let $n_1 + n_2 \equiv n$

$$H | n_1 n_2\rangle = \underbrace{(n+1)\hbar\omega}\, |n_1 n_2\rangle$$

$$\boxed{E_n = (n+1)\hbar\omega}$$

For given n,

n_1	n_2
0	n
1	$n-1$
2	$n-2$
⋮	⋮
n	0

$\boxed{(n+1)\text{-fold degeneracy}}$

(b) (i) Ground state, $n=0 \Rightarrow n_1 = n_2 = 0$

$$E^{(1)} = \langle 00| -V_0 e^{-(x_1-x_2)^2/\alpha^2} |00\rangle$$

$$E^{(1)} = \int dx_1 \int dx_2 \; \psi_0^2(x_1)\,\psi_0^2(x_2)\left\{-V_0\, e^{-(x_1-x_2)^2/\alpha^2}\right\}$$

$$= -\frac{m\omega V_0}{\hbar\pi} \iint e^{-\frac{m\omega}{\hbar}(x_1^2+x_2^2) - \frac{1}{\alpha^2}(x_1-x_2)^2} dx_1 dx_2$$

$\begin{pmatrix} x_1+x_2 \equiv X \\ x_1-x_2 \equiv Y \end{pmatrix}$ $$= -\frac{m\omega V_0}{\hbar\pi} \iint e^{-\frac{m\omega}{2\hbar}X^2 - \left(\frac{1}{\alpha^2}+\frac{m\omega}{2\hbar}\right)Y^2} dX\, dY$$

$$= -\frac{m\omega V_0}{\hbar\pi}\left(\frac{2\pi\hbar}{m\omega}\right)^{1/2}\left(\frac{2\hbar\alpha^2\pi}{m\omega\alpha^2+2\hbar}\right)^{1/2} = -2V_0\left(\frac{m\omega\alpha^2}{m\omega\alpha^2+2\hbar}\right)^{1/2}$$

(c) At low energy limit,

$$-V_0\, e^{-(x_1-x_2)^2/\alpha^2} \approx -V_0\left\{1 - \frac{1}{\alpha^2}(x_1-x_2)^2\right\}$$

$$\Rightarrow H \approx \frac{P_1^2}{2m} + \frac{P_2^2}{2m} - V_0 + \frac{V_0}{\alpha^2}(x_1-x_2)^2$$

Define $\begin{cases} x_c = \dfrac{x_1+x_2}{2} & \text{: center of mass coordinate} \\ x_R = x_1 - x_2 & \text{: relative coordinate} \end{cases}$

$$H = \underbrace{\frac{P_c^2}{2M}}_{\text{free-particle motion}} + \underbrace{\frac{P_R^2}{2\mu} + \frac{V_0}{\alpha^2}x_R^2}_{\text{harmonic oscillator}} - V_0$$

$$\Rightarrow \quad H = H_C + H_R - V_0$$

where $H_C = \frac{P_C^2}{2M}$

and $H_R = \frac{P_R^2}{2\mu} + \frac{1}{2}\mu\omega_R^2 x_R^2$,

$\omega_R = \sqrt{\frac{2V_0}{\mu\alpha^2}}$

Therefore,

$$E = \frac{\hbar^2 k_c^2}{2M} + (n+\tfrac{1}{2})\hbar\omega_R - V_0$$

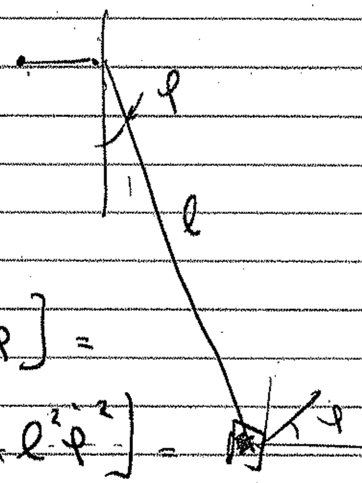
$\Rightarrow$ Grond state energy

$$\boxed{E_0 = \frac{1}{2}\hbar\omega_R - V_0, \quad \omega_R = \sqrt{\frac{2V_0}{\mu\alpha^2}}}$$

2006 Fall comp exam. graduate mechanics (Problem #7)

Note Title 8/17/200

Coordinates of the mass:



$x = l\sin\varphi + a(t)$

$y = -l\cos\varphi$

$T = \frac{m}{2}\left[(l\cos\varphi\cdot\dot\varphi - \nu a\sin\nu t)^2 + l^2\dot\varphi^2\sin^2\varphi\right] =$

$= \frac{m}{2}\left[\nu^2 a^2\sin^2\nu t - 2la\dot\varphi\nu\cos\varphi\sin\nu t + l^2\dot\varphi^2\right] =$

$= \frac{m}{2}\left[(\nu a\sin\nu t)^2 - 2la\nu\left[\frac{d}{dt}(\sin\varphi\sin\nu t) - \nu\sin\varphi\cos\nu t\right] + l^2\dot\varphi^2\right] =$

$= \frac{ml^2\dot\varphi^2}{2} + mla\nu^2\sin\varphi\cos\nu t + \frac{m}{2}\left[(\nu a\sin\nu t)^2 + \frac{2d}{dt}(\sin\varphi\sin\nu t)\right]$

$U = -mgl\cos\varphi$

Note: the terms depending only on time, as well as complete time derivatives do not enter equations of motion, and therefore can be omitted

(5) $L = T - U = \boxed{\frac{ml^2\dot\varphi^2}{2} + mla\nu^2\sin\varphi\cos\nu t + mgl\cos\varphi}$

Equation of motion:

$ml^2\ddot\varphi + mgl\sin\varphi - mla\nu^2\cos\varphi\cos\nu t = 0$

$\varphi = \Phi + \xi$;

(3) $\sin\varphi \simeq \sin\Phi + \xi\cos\Phi$; $\cos\varphi \simeq \cos\Phi - \xi\sin\Phi$

$$ml^2(\ddot{\Phi} + \ddot{\xi}) = -mgl\sin\Phi - mg\xi\cos\Phi - mla\nu^2\cos\Phi\cos\nu t + + mla\nu^2\xi\sin\Phi\cos\nu t$$

Zero-order oscillating terms give:

(3) $ml^2\ddot{\xi} = -mla\nu^2\cos\Phi\cos\nu t \Rightarrow \xi \simeq \frac{a}{l}\cos\Phi\cos\nu t$

Averaging gives:

$$ml^2\ddot{\Phi} = -mgl\sin\Phi + mla\nu^2\sin\Phi \cdot m\,\frac{a}{l}\cos\phi\,\langle\cos^2\nu t\rangle$$

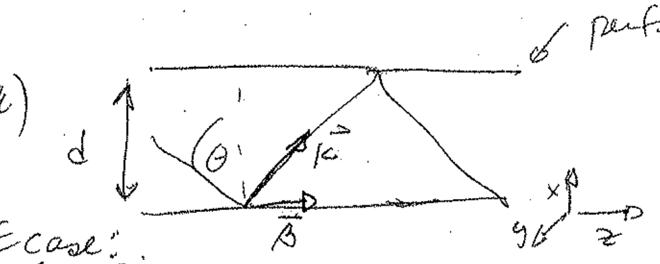
$$ml^2\ddot{\Phi} = -mgl\sin\phi + \frac{1}{2}m \quad a^2\nu^2\sin\Phi\cos\Phi$$

(5) $$ml^2\ddot{\varphi} = \frac{\partial}{\partial\Phi}\left[mgl\cos\phi - \frac{1}{4}m \quad a^2\nu^2\cos^2\Phi\right] =$$

$$= -\frac{\partial}{\partial\varphi}\underbrace{mgl\left[-\cos\varphi + \frac{a^2\nu^2}{4gl}\cos^2\varphi\right]}_{U_{eff}}$$

(4) $$\min U_{eff}: \begin{cases} 0, & a^2\nu^2 < 2gl \\ \cos\varphi = \frac{2gl}{a^2\nu^2}; & a^2\nu^2 > 2gl \end{cases}$$

Prob 8 Grad EM



1. a)

z projectn of $\vec{k}$ is $k\sin\theta$ when $k = n\omega/c = \omega/c$

TE case:

The field must be zero at both boundaries, so in general.

$$E(x,z) = \hat{y}\ \sin(x\,k\cos\theta)\, e^{i\beta z}$$

(for TE)

$$E(0,z) = 0 \text{ and } E(d,z) = 0 \Rightarrow d\,k\cos\theta = m\pi$$

$m = 1, 2, 3$

$$\text{So } \beta_{TE} = k\sin\theta = \sqrt{k^2 - k^2\cos^2\theta} =$$

$$= \sqrt{k^2 - \cancel{k}(n\pi/d)^2}$$

$\omega_{cutoff} = c\,\dfrac{m\pi}{d}$, for $m=1$, $\nu_{cutoff} = \dfrac{c}{2d} = \dfrac{3\times10^8\,m/s}{5\times10^{6}\,m} = 60\,Hz$

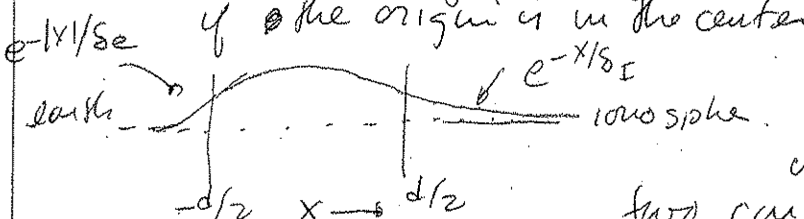
For TM, the expression for β is the same, so the cutoff frequencies are the same.

b) TEM mode has no cutoff freq.

c) Lowest resonance: $\lambda/n = 2\pi r_e$, $n = 1$ for air

$\lambda = 38{,}000\,km = 3.8\times10^7\,m \Rightarrow \nu = 3\times10^8\,m/s / 3.8\times10^7\,m$
$\simeq 7.9\,Hz$

2. a) TEM $\vec{E}$ is $\hat{y}$-polarized. $E(x) \cong \Sigma \cos\alpha x$ inside if the origin is in the center of the wave guide.



Need the penetration or skin depth in the two conductive regions

$$\delta = \sqrt{\frac{2}{\omega\sigma\mu}}$$

Must make E continuous across the boundary since $\vec{E}$ is parallel to the interface.

For the earth, $\delta_e = \sqrt{\dfrac{2}{48 \times 4 \times 4\pi \times 10^{-7}}} \approx \sqrt{\dfrac{1}{1200 \times 10^{-7}}}$

$$\approx \frac{1}{1.1 \times 10^{-2}} \cancel{\times 10} = 90 \text{ m}$$

For the ionosphere $\delta_I = \sqrt{\dfrac{2}{48 \times 10^{-7} \times 4\pi \times 10^{-7}}} \approx \dfrac{1}{\sqrt{600 \times 10^{-14}}}$

$$= \frac{1}{\sqrt{6}} \times 10^{7} \text{ m}$$

c) Power loss per meter: In both conductors, in both y and z directions

Power dissipated $= \vec{J} \cdot \tilde{E} = \sigma E^2$ at each point in space.

Integrate over x: $P = \displaystyle\int_{x=-d/2}^{-\infty} \sigma_e E_e^2(x)\,dx$

$$+ \int_{x=d/2}^{\infty} \sigma_I E_I(x)\,dx$$