OSU PHYSICS DEPARTMENT
COMPREHENSIVE EXAMINATION #100

April 03 and 04, 2006

Comprehensive examination for Spring 2006

PART 1, Monday April 03, 9:00 am

General Instructions

This Comprehensive Examination for Spring 2006 consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, April 03, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, April 04, at 9:00 am and 1:30 pm.
Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.
If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.
Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Problem 1

A shallow donor state in a GaAs/AlGaAs quantum dot is doubly degenerate. The basis vectors of the quantum system are represented by $|1\rangle$ and $|2\rangle$. When an electric field $\mathbf{E} = (E_x, 0, E_z)$ is applied, the electric dipole interaction induces Stark effect, which can be described as the model Hamiltonian

$$H = -\mathbf{p}\cdot\mathbf{E} \approx \alpha\left(E_x|1\rangle\langle 1| - E_x|2\rangle\langle 2| + E_z|1\rangle\langle 2| + E_z|2\rangle\langle 1|\right).$$

(a) Compute the eigenvalues and the eigenstates in the presence of the electric field **E**.

(b) Assume that the state is stationary when $t < 0$. At $t = 0$, the x-axis electric field is turned off abruptly ($E_x \to 0$). Compute the time evolution of the state after $t = 0$. Evaluate the expectation values of energy and x-axis dipole moment, $p_x = -\alpha\left(|1\rangle\langle 1| - |2\rangle\langle 2|\right)$, at t (> 0).

Problem 2. A solid and a vapor made up of the same kind of atoms are in equilibrium in a closed vessel of volume V at temperature T. Assume that the vapor is a monoatomic classical ideal gas, and the solid can be described in terms of the Einstein model – i.e., each atom in the solid is represented by a three-dimensional quantum harmonic oscillator performing vibration with a frequency ω about its equilibrium position independently of others. Evaluate the vapor pressure as a function of temperature.

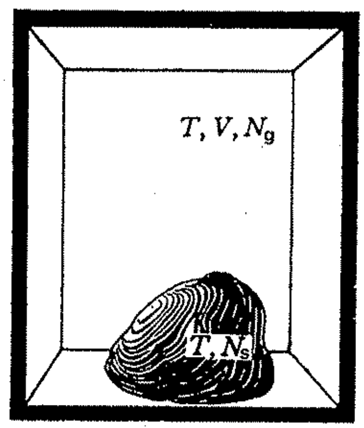
For simplicity, you may assume that the solid volume is negligibly small compared with V (i.e., that the volume of the gas $V_g \approx V$; however, at the end you will see that such an assumption isn't really necessary). For the gas, you may use without deriving:

$$\mu_g = \left(\frac{\partial F_g}{\partial N_g}\right)_{T,V} = \frac{\partial}{\partial N_g}\left[-\frac{1}{\beta}\ln(Z_g)\right]_{T,V} = \frac{1}{\beta}\ln\left(\frac{n_g}{n_Q}\right)$$

where

$$n_Q = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{3}{2}},$$

$\beta = 1/(k_B T)$, and $n_g = N_g/V_g$ is the gas atom concentration.



Question 3

Rain drops scatter and absorb microwave radiation. A rain drop can be modeled as a dielectric sphere of radius $R = 1$ mm with a dielectric function of approximately $\epsilon = \epsilon_o(60 + i3)$ at a frequency of 1 GHz.

1. Assume that the electromagnetic field at 1 GHz is a plane wave, and find the time-dependent polarization within the drop.
2. Determine the power dissipated in the drop due to absorption.
3. Derive an expression for the power radiated by the drop.

Problem 4

When launching missions to outer planets of the Solar system, like Saturn or Jupiter, it is not uncommon to initially launch the spacecraft towards the inner planets, like Venus, and use their gravitational field to accelerate the spacecraft. In this problem you are asked to develop a quantitative description of this phenomenon.

1. Consider the problem in the reference frame of a moving planet (neglect the curvature of planet's orbit to assume that this reference frame is inertial). Starting from gravitational potential, find the minimum velocity v_0 a spacecraft must have to leave the planet as a function of planet's mass M, mass of a spacecraft m, and smallest distance between the spacecraft and the planet's center of mass r_{min}.
2. Assuming that initial velocity of a spacecraft $v_i > v_0$, qualitatively describe the trajectory of a spacecraft (in the planet's reference frame); find the magnitude of the final velocity of a spacecraft v_f and its smallest separation from the planet r_{min} as a function of its initial velocity and impact parameter b (see Fig.1).
3. The trajectory of a particle in a potential $U = -\alpha / r$ in cylindrical coordinates can be described by the following relation:

$$\phi(r) = \text{ArcCos}\left[\frac{(L/r) - (m\alpha/L)}{\sqrt{2mE + m^2\alpha^2/L^2}}\right] + \text{const}$$

   with m, E, and L being mass of a particle, its total energy, and its angular momentum respectively. Use this relation to find the scattering angle of the spacecraft $\Delta\phi$ (see Fig.1)
4. Now, assume that a planet moves with velocity V_{pl} opposite to v_i. Find the final velocity of a spacecraft relative to the Sun (consider V_{pl}=const) when
   a. $v_i \gg \sqrt{\gamma M / b}$, and
   b. $v_i \ll \sqrt{\gamma M / b}$

   with γ being gravitational constant

   Sketch the trajectories of the spacecraft in both these cases; comment on the effectiveness of the gravitation-assisted acceleration.

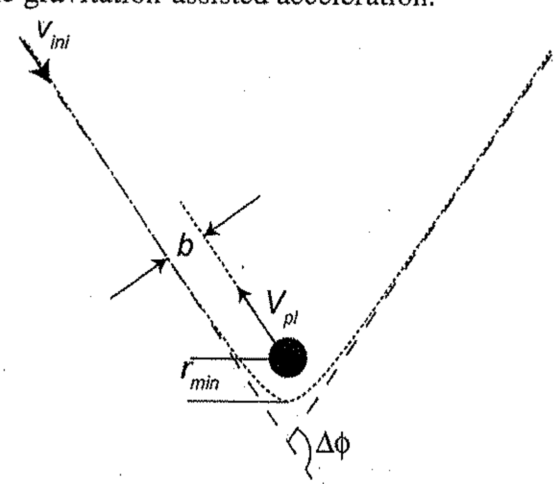


Fig.1. Schematic trajectory of spacecraft moving around the planet.

Problem 5

A sodium ion (Na^+) is confined in a micro ion trap, which can be approximated as a spherically symmetric 3-dimentional harmonic potential

$$V(r) = \frac{1}{2} m\omega^2 r^2 .$$

Disregard any electronic transitions within the ion and treat the ion as a positively charged spinless particle. The mass and charge of the ion are m and e respectively.

(a) Describe the energy spectrum of this quantum system. If any energy level is degenerate, calculate its degeneracy.

(b) Now let's subject the ion to a magnetic field. Assuming the field is weak, uniform, and aligned with the z-axis of the system, this amount to perturbing the Hamiltonian by

$$H_1 = -\frac{eB}{2mc} L_z \text{ where } L_z = xp_y - yp_x .$$

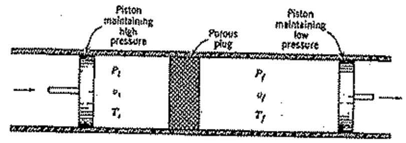
Note that position and momentum operators can be represented by annihilation and creation operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+) \text{ and } p = i\sqrt{\frac{m\hbar\omega}{2}}(a^+ - a) .$$

Calculate the energy shifts of all the states occupying the ground and the first excited states of the unperturbed oscillator when the magnetic field is turned on. For simplicity, limit your analysis to the first order of perturbation theory.

Problem 6

In the classical version of the Joule-Thompson experiment a gas is allowed to pass through a porous plug from a region of high pressure (p_i) to a region of low pressure (p_f). The p_i and p_f pressures are maintained at constant values throughout the duration of the experiment by the two pistons shown in the schematic diagram below.



Schematic representation of the Joule-Thomson process.

The cylinder walls and the pistons are made of a thermally insulating (adiabatic) material.

(a) Show that in the Joule-Thompson process the system enthalpy H remains constant ($H \equiv U + pV$).

(b) Assume that the changes in the gas parameters in the process are sufficiently small that one can treat them as differentials: $(p_f - p_i) \to dp$, and $(T_f - T_i) \to dT$. Using differential analysis, calculate the temperature change dT for a process in which the pressure change is dp, and express the result in terms of T_i, v_i, and standard thermodynamic "response functions".
Hint: you may want to use the Maxwell relation:

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$$

(c) Suppose that gas processed in the Joule-Thompson apparatus obeys the van der Waals equation of state

$$p = \frac{RT}{v-b} - \frac{a}{v^2}$$

where a and b are known parameters. The value of the constant volume molar heat capacity c_v for this gas is also known, and it does not depend on pressure and temperature in the p and T region of interest. The pressure change in the process is dp. Find the temperature change dT in terms of T_i, v_i, a, b and c_v.

Hints: If you need the expression for $c_p - c_v$ for the gas, a good way of obtaining it seems to be to express $c_p - c_v$ in terms of the $(\partial v/\partial p)_T$ and $(\partial p/\partial T)_v$ derivatives which can be readily calculated from the equation of state (however, this is not the only possible method). You may also want to recall the following differentiation rule: if there are two functions of different sets of variables, $f_1(x, y, z)$, and $f_2(x, \xi(x, y, z), z)$, and they are related by:

$$f_1(x, y, z) = f_2(x, \xi(x, y, z), z)$$

then:

$$\left(\frac{\partial f_1}{\partial x}\right)_{y,z} = \left(\frac{\partial f_2}{\partial x}\right)_{\xi,z} + \left(\frac{\partial f_2}{\partial \xi}\right)_{x,z}\left(\frac{\partial \xi}{\partial x}\right)_{y,z}$$

Question 7

When a capacitor is connected to a battery and given time to fully charge, a force between the two conducting surfaces exists. Consider a system consisting of parallel square plates of area $A = L^2$ separated by distance $d << L$ and a battery of potential V_o.

1. Begin with a capacitor with only a vacuum or single dielectric material between the plates.

   (a) Suppose the space within the capacitor is the vacuum. When the capacitor is connected to the battery and charged to potential V_o, what is the force between the plates? What is the electrostatic energy stored in the capacitor?

   (b) Now, disconnect the battery from the capacitor and fill the capacitor with a fluid of dielectric constant ϵ_1. What is the force? What is the energy stored in the capacitor?

   (c) Suppose that the capacitor was filled with the fluid first and then charged to V_o with battery. What is force? What is the stored energy?

2. Instead of a single fluid, consider the case of a capacitor filled with two immisicible liquids of dielectric constants $\epsilon_1 > \epsilon_2$. Fluid 1 is more dense than fluid 2. The plates of the capacitor are parallel to the surface of the earth, and each fluid fills one half of the available space.

   (a) When this capacitor is charged to potential V_o, what are the energies stored in the two regions?

   (b) What is the force between the plates? Is there a force between the two fluids?

Problem 8

Consider a physical pendulum shown in Fig.1. The pendulum is a parabola-shaped disk of thickness Δ ($\Delta \ll R$) with uniform density ρ. The pendulum is fixed at the origin of Cartesian coordinate system, and can freely rotate around the axes x and y. Gravity is directed along (-z) axis

1. Find the moment of inertia of the disk rotating around x axis.
2. Find the moment of inertia of the disk rotating around y axis.
3. Find the mass of the disk, and the position of its center of mass (assume that the disk is at rest)
4. Assuming small oscillations around x and y axes, find the Lagrangian of the disk and its equations of motion.
5. Find the frequencies of small oscillations of the disk around x and y axes.

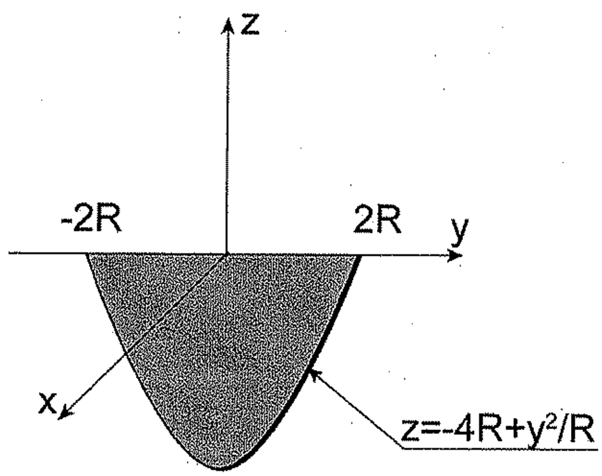


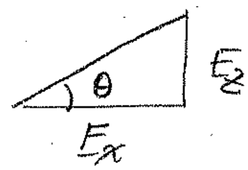
Fig.1

Comp Exam Problem #1 spring 2006; Solution

1. In matrix formalism $|1\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix}$, $|2\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$.

$$H \doteq \alpha E_x \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} + \alpha E_z \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

$$= \alpha \begin{pmatrix}E_x & E_z\\E_z & -E_x\end{pmatrix} = \alpha E_x \begin{pmatrix}1 & \frac{E_z}{E_x}\\ \frac{E_z}{E_x} & -1\end{pmatrix}$$



Let $\alpha E_x \equiv E_0$, $\frac{E_z}{E_x} = \tan\theta$

$$\Rightarrow H \doteq E_0 \begin{pmatrix}1 & \tan\theta\\ \tan\theta & -1\end{pmatrix}$$

(1) Eigen values

$$|H - \lambda I| = 0 \Rightarrow \begin{vmatrix}E_0 - \lambda & E_0\tan\theta\\ E_0\tan\theta & -E_0 - \lambda\end{vmatrix} = 0$$

$$\Rightarrow -(E_0 - \lambda)(E_0 + \lambda) - E_0^2\tan^2\theta = 0$$

$$\Rightarrow \lambda^2 - E_0^2 - E_0^2\tan^2\theta = 0$$

$$\Rightarrow \lambda = \pm E_0\sqrt{1+\tan^2\theta} = \pm E_0\sec\theta = \boxed{\pm\alpha\sqrt{E_x^2+E_z^2}}$$

(2) Eigen states

(i) When $\lambda = E_+ = +\alpha\sqrt{E_x^2+E_z^2} = E_0\sec\theta$

$H|+\rangle = E_+|+\rangle$, $|+\rangle = \begin{pmatrix}a\\b\end{pmatrix}$

$$\Rightarrow E_0\begin{pmatrix}1 & \tan\theta\\ \tan\theta & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = E_0\sec\theta\begin{pmatrix}a\\b\end{pmatrix}$$

$$\Rightarrow a + b\tan\theta = a\sec\theta$$

$\Rightarrow \quad a\cos\theta + b\sin\theta = a$

$\Rightarrow \quad a(1-\cos\theta) = b\sin\theta$

$\Rightarrow \quad 2a\sin^2\frac{\theta}{2} = 2b\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\Rightarrow \quad a\sin\frac{\theta}{2} = b\cos\frac{\theta}{2}$

$\Rightarrow \quad \frac{b}{a} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$, $|a|^2+|b|^2=1$: normalization

$$\Rightarrow \begin{cases} a = \cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\frac{E_x}{\sqrt{E_x^2+E_z^2}}}{2}} = \sqrt{\frac{\sqrt{E_x^2+E_z^2}+E_x}{2\sqrt{E_x^2+E_z^2}}} \\ b = \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{\sqrt{E_x^2+E_z^2}-E_x}{2\sqrt{E_x^2+E_z^2}}} \end{cases}$$

$$\Rightarrow |+\rangle = \boxed{\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}} = \frac{1}{\sqrt{2}(E_x^2+E_z^2)^{1/4}} \begin{pmatrix} \sqrt{\sqrt{E_x^2+E_z^2}+E_x} \\ \sqrt{\sqrt{E_x^2+E_z^2}-E_x} \end{pmatrix}$$

(ii) when $\lambda = E_- = -\alpha\sqrt{E_x^2+E_z^2} = -E_0\sec\theta$

$H|-\rangle = E_-|-\rangle, \quad |-\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

Since $\langle +|-\rangle = 0$: orthogonality,

$\left(\cos\frac{\theta}{2} \quad \sin\frac{\theta}{2}\right)\begin{pmatrix} a \\ b \end{pmatrix} = a\cos\frac{\theta}{2} + b\sin\frac{\theta}{2} = 0$, $|a|^2+|b|^2=1$

$$\Rightarrow \begin{array}{l} a = +\sin\frac{\theta}{2} \\ b = -\cos\frac{\theta}{2} \end{array} \quad \Rightarrow |-\rangle = \boxed{\begin{pmatrix} +\sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}}$$

(b) when $t > 0$, the new Hamiltonian

$$H' = \alpha E_z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Energy eigenvalues : $E' = \pm \alpha E_z$

Energy eigenstates : $\begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{for } E_+' = +\alpha E_z \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \text{for } E_-' = -\alpha E_z \end{cases}$

Before $t = 0$, the state is stationary. thus,

$|\psi(0)\rangle = |+\rangle$ or $|-\rangle$.

(i) when $|\psi(0)\rangle = |+\rangle$

$$|\psi_+(t)\rangle = e^{-i\frac{H'}{\hbar}t}|+\rangle = e^{-i\frac{H'}{\hbar}t} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= e^{-i\frac{H'}{\hbar}t} \left\{ \frac{1}{\sqrt{2}} \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

$$= \frac{1}{\sqrt{2}} \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2} \right) \frac{e^{-i\frac{E_+'}{\hbar}t}}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2} \right) \frac{e^{-i\frac{E_-'}{\hbar}t}}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \boxed{\omega \equiv \frac{\alpha E_z}{\hbar}}$$

$$= \frac{1}{2} \begin{pmatrix} \cos\frac{\theta}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) - \sin\frac{\theta}{2}\left(e^{i\omega t} - e^{-i\omega t}\right) \\ -\cos\frac{\theta}{2}\left(e^{i\omega t} - e^{-i\omega t}\right) + \sin\frac{\theta}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2}\cos\omega t - i\sin\frac{\theta}{2}\sin\omega t \\ -i\cos\frac{\theta}{2}\sin\omega t + \sin\frac{\theta}{2}\cos\omega t \end{pmatrix}$$

$\langle H \rangle = \langle \psi_+(t) | H | \psi_+(t) \rangle$: time-independent

$$= \left(\cos\frac{\theta}{2}\cos\omega t + i\sin\frac{\theta}{2}\sin\omega t,\ \sin\frac{\theta}{2}\cos\omega t + i\cos\frac{\theta}{2}\sin\omega t\right)$$

$$\cdot \alpha E_z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2}\cos\omega t - i\sin\frac{\theta}{2}\sin\omega t \\ \sin\frac{\theta}{2}\cos\omega t - i\cos\frac{\theta}{2}\sin\omega t \end{pmatrix}$$

$$= \alpha E_z \left(\cos\frac{\theta}{2}\cos\omega t + i\sin\frac{\theta}{2}\sin\omega t,\ \sin\frac{\theta}{2}\cos\omega t + i\cos\frac{\theta}{2}\sin\omega t\right) \begin{pmatrix} \sin\frac{\theta}{2}\cos\omega t - i\cos\frac{\theta}{2}\sin\omega t \\ \cos\frac{\theta}{2}\cos\omega t - i\sin\frac{\theta}{2}\sin\omega t \end{pmatrix}$$

$$= \alpha E_z \Big(\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos^2\omega t + \cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin^2\omega t \cancel{- i\cos^2\frac{\theta}{2}\cos\omega t\sin\omega t}$$
$$\cancel{+ i\sin^2\frac{\theta}{2}\sin\omega t\cos\omega t} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos^2\omega t + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin^2\omega t$$
$$\cancel{+ i\cos^2\frac{\theta}{2}\cos\omega t\sin\omega t} \cancel{- i\sin^2\frac{\theta}{2}\cos\omega t\sin\omega t} \Big)$$

$$= \alpha E_z \left(2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right) = \boxed{\alpha E_z \sin\theta}$$ time-independent

$\langle P_x \rangle = \langle \psi_+(t) | P_x | \psi_+(t) \rangle$

$$= -\alpha\left(\cos\frac{\theta}{2}\cos\omega t + i\sin\frac{\theta}{2}\sin\omega t,\ \sin\frac{\theta}{2}\cos\omega t + i\cos\frac{\theta}{2}\sin\omega t\right)\begin{pmatrix} \cos\frac{\theta}{2}\cos\omega t - i\sin\frac{\theta}{2}\sin\omega t \\ -\sin\frac{\theta}{2}\cos\omega t + i\cos\frac{\theta}{2}\sin\omega t \end{pmatrix}$$

$$= -\alpha\Big(\cos^2\frac{\theta}{2}\cos^2\omega t + \sin^2\frac{\theta}{2}\sin^2\omega t \cancel{- i\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\omega t\sin\omega t} \cancel{+ i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\omega t\cos\omega t}$$
$$-\sin^2\frac{\theta}{2}\cos^2\omega t - \cos^2\frac{\theta}{2}\sin^2\omega t \cancel{+ i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\omega t\sin\omega t} \cancel{- i\cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\omega t\cos\omega t}\Big)$$

$$\Rightarrow \langle p_x \rangle = -\alpha \left(2\cos\theta \cos^2\omega t - 2\cos\theta \sin^2\omega t \right)$$

$$= \boxed{-2\alpha \cos\theta \cos 2\omega t}$$

(ii) when $|\psi(0)\rangle = |-\rangle$

$$|\psi_-(t)\rangle = e^{-i\frac{H'}{\hbar}t}|-\rangle = e^{-i\frac{H'}{\hbar}t}\begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

$$= e^{-i\frac{H'}{\hbar}t}\left\{ \frac{1}{\sqrt{2}}\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

$$= \frac{1}{\sqrt{2}}\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)\frac{e^{-i\omega t}}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)\frac{e^{i\omega t}}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\frac{\theta}{2}\cos\omega t + i\cos\frac{\theta}{2}\sin\omega t \\ -\cos\frac{\theta}{2}\cos\omega t - i\sin\frac{\theta}{2}\sin\omega t \end{pmatrix}$$

$$\langle H \rangle = \langle \psi_-(t)| H |\psi_-(t)\rangle \quad : \text{time independent}$$

$$= \langle \psi_-(0)| H |\psi_-(0)\rangle$$

$$= \left(\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}\right)\alpha E_z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

$$= \alpha E_z \left(\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}\right)\begin{pmatrix} -\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} = \boxed{-\alpha E_z \sin\theta}$$

(P.5)

$$\langle P_x \rangle = \langle \psi_{\perp}(t) | P_x | \psi_{\perp}(t) \rangle$$

$$= -\alpha \left(\sin\tfrac{\theta}{2}\cos\omega t - i\cos\tfrac{\theta}{2}\sin\omega t,\ -\cos\tfrac{\theta}{2}\cos\omega t + i\sin\tfrac{\theta}{2}\sin\omega t \right)$$

$$\cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sin\frac{\theta}{2}\cos\omega t + i\cos\frac{\theta}{2}\sin\omega t \\ -\cos\frac{\theta}{2}\cos\omega t - i\sin\frac{\theta}{2}\sin\omega t \end{pmatrix}$$

$$= -\alpha\left(\sin^2\tfrac{\theta}{2}\cos^2\omega t + \cos^2\tfrac{\theta}{2}\sin^2\omega t \right)$$

$$+\alpha\left(\cos^2\tfrac{\theta}{2}\cos^2\omega t + \sin^2\tfrac{\theta}{2}\sin^2\omega t \right)$$

$$= \alpha\cos\theta\cos^2\omega t - \alpha\cos\theta\sin^2\omega t$$

$$= \boxed{2\alpha\cos\theta\cos 2\omega t}$$

Problem 2, thermal:

For an Einstein solid $Z = \left(Z_{\text{single oscill}}\right)^{3N_s}$

$$\epsilon(n) = \frac{\hbar\omega}{2} + n\hbar\omega$$

$$Z_{\text{single}} = \sum_{n=0}^{\infty} e^{-\frac{1}{kT}\left(\frac{\hbar\omega}{2} + n\hbar\omega\right)} = e^{-\frac{\hbar\omega}{2kT}} \sum_{n=0}^{\infty}\left(e^{-\frac{\hbar\omega}{kT}}\right)^n$$

$$= e^{-\frac{\hbar\omega}{2kT}} \frac{1}{1 - e^{-\frac{\hbar\omega}{kT}}} = e^{-\frac{\hbar\omega}{2kT}} \frac{1}{e^{-\frac{\hbar\omega}{2kT}}\left(e^{\frac{\hbar\omega}{2kT}} - e^{-\frac{\hbar\omega}{2kT}}\right)}$$

$$= \frac{1}{2\sinh\left(\frac{\hbar\omega}{2kT}\right)}$$

So, $Z = \left[2\sinh\left(\frac{\hbar\omega}{2kT}\right)\right]^{-3N_s}$

From Z we get the free energy F_s of the solid

$$F_s = -kT\ln Z = 3NkT\ln\left[2\sinh\left(\frac{\hbar\omega}{2kT}\right)\right]$$

And chemical potential:

$$\mu_s = \frac{\partial F_s}{\partial N} = 3kT\ln\left[2\sinh\left(\frac{\hbar\omega}{2kT}\right)\right]$$

In diffusive equilibrium, it must be that: $\mu_{\text{solid}} = \mu_{\text{gas}}$

i.e.: $3kT\ln\left[2\sin\left(\frac{\hbar\omega}{2kT}\right)\right] = kT\ln\left(\frac{n_g}{n_Q}\right)$

so: $\left[2\sin\left(\frac{\hbar\omega}{2kT}\right)\right]^3 = \frac{n_g}{n_Q} = \frac{N_g}{V}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}$

$$\frac{N_g}{V} = \left(\frac{2\pi\hbar^2}{mkT}\right)^{3/2}\cdot\left[2\sinh\left(\frac{\hbar\omega}{2kT}\right)\right]^3$$

but $pV_g = N_g kT$ (gas law)

so $p = \frac{N_g}{V_g}kT$

accordingly:

$$p_{\text{gas}} = kT\left(\frac{2\pi\hbar^2}{mkT}\right)^{3/2}\left[2\sinh\left(\frac{\hbar\omega}{2kT}\right)\right]^3$$

Comp. Exam #100; Problem 3; solution

1\. a) $\lambda = \frac{c}{\nu} = \frac{3\times 10^8}{10^9} m = 0.3m >> R = 1\,mm$

So, use the solution for a dielectric sphere in a uniform static field. The solution yields a polarization $\vec{P} = (\epsilon - \epsilon_0)\vec{E}$

and field inside the drop $\vec{E} = \frac{3}{\epsilon/\epsilon_0 + 2}\vec{E}_0$

$|E_0|$ is determined from the ~~[illegible]~~ intensity of the beam.

$I = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\,E_0^2$. Answer: $\vec{P}(t) = 3\epsilon_0 \frac{(\epsilon/\epsilon_0 - 1)}{(\epsilon/\epsilon_0 + 2)} \vec{E_0}\, e^{-i\omega t}$

2\. The time-averaged power dissipated in the drop is the Real part of

$P = \frac{1}{2}\int \vec{J}^{*}\cdot\vec{E}\,dV$. Use $\vec{J} = \frac{\partial}{\partial t}\vec{D} = \epsilon\frac{\partial}{\partial t}\vec{E} = \frac{-i\omega 3\epsilon}{\epsilon/\epsilon_0 + 2}\vec{E_0}$

$\vec{J}^{*}\cdot\vec{E} = i\omega\epsilon^{*}\left(\frac{9}{\epsilon^{*}/\epsilon_0 + 2}\right)\frac{E_0^2}{(\epsilon/\epsilon_0 + 2)} = i\omega\epsilon_0 9 \frac{\epsilon_r^{*}}{(\epsilon_r^{*}+2)^{*}(\epsilon_r + 2)} E_0^2$

where $\epsilon_r = a + ib$

$\vec{J}^{*}\cdot\vec{E} = 9\omega\epsilon_0 \frac{(b + ia)\,E_0^2}{(a^2 + b^2 + 4a + 4)}$

Power dissipated $= \frac{4}{3}\pi R^3 \frac{9\,\omega\epsilon_0\, b\, E_0^2}{(a^2+b^2+4a+4)}$

3\. Since $\lambda >> R$, consider radiation from a point dipole

$\vec{p} = \int \vec{P}\,dV = \frac{4\pi}{3}R^3\vec{P}$. $|\vec{E}_{rad}| \sim |\vec{p}|\frac{e^{ikr}}{r}k^2$, $k = \frac{\omega}{c}$

$\frac{dP}{d\Omega} \sim Re(\vec{E}_{rad}\times\vec{H}^{*}_{rad}) \sim k^4|p|^2\sin^2\theta$

Total Power radiated $\sim k^4|p|^2 = \frac{\omega^4}{c^4}\left(\frac{4\pi R^3}{3}\right)^2 9\epsilon_0^2\left|\frac{\epsilon_r - 1}{\epsilon_r + 2}\right|^2 E_0^2$

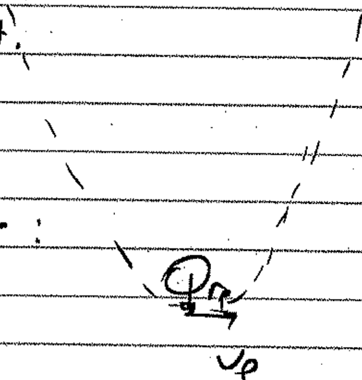
2006 Spring Comprehensive exam. #4 - solution

Note Title 3/28/2006

(1) The energy of the satellite is conserved:

$$E = -\gamma \frac{Mm}{r} + \frac{m v^2}{2} = \text{const}; \quad \gamma = \text{grav. const.}$$

to leave the gravitational pull of a planet, the satellite must have non-zero v @ $r \to \infty$:



(3pt) $$-\frac{\gamma M m}{r_p} + \frac{m v_p^2}{2} = \frac{m v_\infty^2}{2}$$

$$v_p^2 = \frac{2\gamma M}{r_p} + v_\infty^2 \geq \frac{2\gamma M}{r_p}$$

minimum terminal velocity is

$$\boxed{v_p^{min} = \sqrt{\frac{2\gamma M}{r_p}}}$$

(2) If the spacecraft comes from infinity, its initial velocity $V_i > 0$, and therefore its energy is

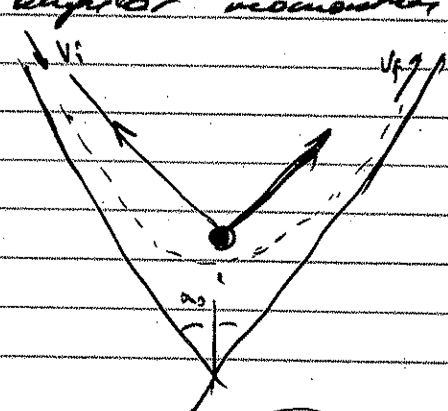
$$E = \frac{m v_i^2}{2} \Rightarrow \boxed{v_f^2 = v_i^2}$$

in gravitational field energy and angular momentum are conserved.

(4pt)

$$L = m r \cdot v \cdot \sin\alpha = \text{const} = m v_\infty b$$

$$\alpha_f = -\alpha_i;$$



The minimum radius is found from the condition that $\vec{r}_{min} \perp \vec{v}_{max}$:

$$\left.\begin{aligned} v_p^2 &= v_i^2 + \frac{2\gamma M}{r_{min}} \\ \cancel{m}\, r_{min} v_p &= \cancel{m}\, v_i \cdot b \end{aligned}\right\} \Rightarrow \quad v_p = \frac{v_i b}{r_{min}}$$

$$\Rightarrow \frac{v_i^2 b^2}{r_{min}^2} = v_i^2 + \frac{2\gamma M}{r_{min}}$$

$$v_i^2 r_{min}^2 + 2\gamma M\, r_{min} - v_i^2 b^2 = 0$$

$$r_{min} = \frac{\sqrt{\gamma^2 M^2 + v_i^4 b^2} - \gamma M}{v_i^2} = b\left[\sqrt{1 + \frac{\gamma^2 M^2}{v_i^4 b^2}} - \frac{\gamma M}{v_i^2 b}\right]$$

(3) Using hint:

$$\varphi(r) = \mathrm{ArcCos}\left[\frac{L/r - m\alpha/L}{\sqrt{2mE + \frac{m^2\alpha^2}{L^2}}}\right] + \text{const.}$$

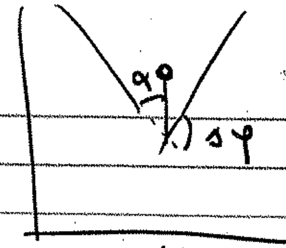
10 pt

$$\alpha = mM\gamma \;; \quad E = \frac{m v_i^2}{2} \;; \quad L = m v_i b \;;$$

$$\varphi(r) = \mathrm{ArcCos}\left[\left(\frac{m v_i b}{r} - \frac{m^2 M\gamma}{m v_i b}\right) \Big/ \sqrt{m^2 v_i^2 + \frac{m^4 M^2 \gamma^2}{m^2 v_i^2 b^2}}\right] + \text{const} =$$

$$= \mathrm{ArcCos}\left[\left(\frac{\cancel{m} v_i b}{r} - \frac{\cancel{m} M\gamma}{v_i b}\right) \Big/ \sqrt{\cancel{m^2} v_i^2 + \frac{\cancel{m^2} M^2 \gamma^2}{v_i^2 b^2}}\right] + \text{const}$$

$$\boxed{\Delta\varphi = \pi - 2\alpha}$$



$$\alpha = -\text{Arccos}\left[\left(\frac{b}{r_{min}} - \frac{\mu\gamma}{v_i^2 b}\right) \Big/ \sqrt{1 + \frac{\mu^2\gamma^2}{v_i^4 b^2}}\right] - \text{Arccos}\left[\frac{\mu\gamma}{v_i^2 b} \Big/ \sqrt{1 + \frac{\mu^2\gamma^2}{v_i^4 b^2}}\right];$$

or, introducing $\zeta = \frac{\mu\gamma}{v^2 b}$, obtain:

$$\alpha = \text{Arccos}\left[\zeta \big/ \sqrt{1+\zeta^2}\right] - \text{Arccos}\left[\left(\frac{1}{\sqrt{1+\zeta^2} - \zeta} - \zeta\right) \Big/ \sqrt{1+\zeta^2}\right] =$$

$$= \text{Arccos}\left[\frac{\zeta}{\sqrt{1+\zeta^2}}\right] - \text{Arccos}\left[\frac{1}{1+\zeta^2 - \zeta\sqrt{1+\zeta^2}} - \frac{\zeta}{\sqrt{1+\zeta^2}}\right] =$$

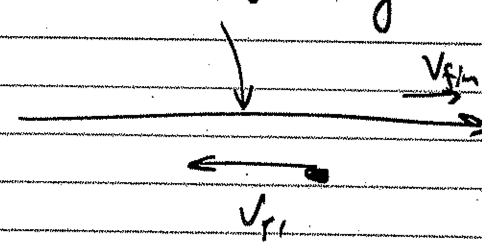
$$= \text{Arccos}\left[\frac{\zeta}{\sqrt{1+\zeta^2}}\right]$$

(4) In the case $v_i^2 \gg \frac{\mu\gamma}{b}$, we have $\zeta \approx 0$

$$\alpha = \text{Arccos}[0] = \frac{\pi}{2}$$

3pt $\Delta\varphi = 0$

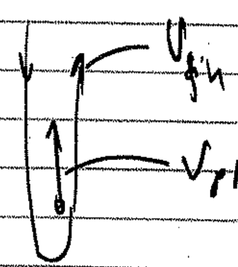
$$\boxed{v_f^{(final)} = v_i^{(final)} = v_i - V_{pl}}$$



in the opposite case $v_i \ll \frac{\mu^2\gamma^2}{b^2}$ $\zeta = 1$

$$\alpha = \; = \text{Arccos}\left[\frac{1}{2}\right] = 0$$

$$\Delta\varphi = \pi$$



$$\boxed{v_i^{(final)} = v_i - V_{pl} \; ; \; v_{f_1}^{(final)} = v_i + V_{pl}}$$

Comp Exam Problem #5 spring 2006

1. (a) Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}kr^2$

$$= \left(\frac{p_x^2}{2m} + \frac{1}{2}kx^2\right) + \left(\frac{p_y^2}{2m} + \frac{1}{2}ky^2\right) + \left(\frac{p_z^2}{2m} + \frac{1}{2}kz^2\right) = \underbrace{H_x + H_y + H_z}_{\text{3 1-D SHO Hamiltonian}}$$

Eigen values and eigen states of H_x, H_y, and H_z

$$\begin{cases} H_x|n_x\rangle = (n_x+\frac{1}{2})\hbar\omega|n_x\rangle \\ H_y|n_y\rangle = (n_y+\frac{1}{2})\hbar\omega|n_y\rangle \\ H_z|n_z\rangle = (n_z+\frac{1}{2})\hbar\omega|n_z\rangle \end{cases}$$

where $n_x, n_y, n_z = 0, 1, 2, \ldots$

Thus,

$$H|n_x n_y n_z\rangle = (n_x+n_y+n_z+\frac{3}{2})\hbar\omega|n_x n_y n_z\rangle$$

$$= (n+\frac{3}{2})\hbar\omega|n_x n_y n_z\rangle$$

where $n = n_x + n_y + n_z$

and $|n_x n_y n_z\rangle = |n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle$

For given n,

n_x	n_y	n_z	# of states
0	$n, n-1, \ldots, 0$	$0, \ldots, n$	$n+1$
1	$n-1, n-2, \ldots, 0$	$0, 1, \ldots, n-1$	n
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	0	0	1

Then, degeneracy $g_n = 1+2+\cdots+n+n+1 = \sum_{i=1}^{n+1} i = \boxed{\frac{1}{2}(n+1)(n+2)}$

(b) $H_1 = -\frac{eB}{2mc} L_z$

$L_z |n_x\, n_y\, n_z\rangle = (x P_y - y P_x)|n_x\, n_y\, n_z\rangle$

$= \Big\{ \sqrt{\frac{\hbar}{2m\omega}}(a_x + a_x^+)\, i\sqrt{\frac{m\hbar\omega}{2}}(a_y^+ - a_y)$

$- \sqrt{\frac{\hbar}{2m\omega}}(a_y + a_y^+)\, i\sqrt{\frac{m\hbar\omega}{2}}(a_x^+ - a_x)\Big\} |n_x\, n_y\, n_z\rangle$

$= \frac{i\hbar}{2}\Big\{ a_x a_y^+ - \cancel{a_x a_y} + \cancel{a_x^+ a_y^+} - a_x^+ a_y$

$- a_y a_x^+ + \cancel{a_y a_x} - \cancel{a_y^+ a_x^+} + a_y^+ a_x \Big\} |n_x\, n_y\, n_z\rangle$

$= i\hbar (a_x a_y^+ - a_x^+ a_y)|n_x\, n_y\, n_z\rangle$

Since $a|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$,

$L_z |n_x\, n_y\, n_z\rangle = i\hbar\Big(\sqrt{n_x}\sqrt{n_y+1}\,|n_x-1\;\, n_y+1\;\, n_z\rangle$

$- \sqrt{n_x+1}\sqrt{n_y}\,|n_x+1\;\, n_y-1\;\, n_z\rangle\Big)$

(i) For the ground state, $n=0 \Rightarrow |n_x\, n_y\, n_z\rangle$
$= |0\,0\,0\rangle$

$L_z|0\,0\,0\rangle = 0 \Rightarrow E^{(1)} = \langle 000|H_1|000\rangle = \boxed{0}$

(i) For the 1st excited state : $n=1$ (3-fold degeneracy)

$$|n_x n_y n_z\rangle = |100\rangle, |010\rangle, |001\rangle$$

$$\begin{cases} L_z|100\rangle = i\hbar|010\rangle \\ L_z|010\rangle = -i\hbar|100\rangle \\ L_z|001\rangle = 0 \end{cases}$$

$$\Rightarrow \quad H_1 = -\frac{eB\hbar}{2mc}\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ where $|100\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|010\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|001\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad |H_1 - \lambda I| = 0 \Rightarrow \lambda = 0, \pm\frac{eB\hbar}{2mc}$$

Therefore,

$$\boxed{E^{(1)} = \pm\frac{eB\hbar}{2mc},\ 0}$$

Problem 6 – Solution:

(a) We start with V_i, p_i in the left chamber, and no gas in the right;

We end with no gas in the left chamber, and V_f, p_f in the right.

The internal energy at the initial state is U_i, and at the final state is U_f.

Since $\Delta Q = 0$ (adiabatic insulation), from the I law we obtain:

$$U_f - U_i = (\text{work input}) - (\text{work output})$$

Work input – work done by pushing the left piston

$\Delta W_{inp} = p_i V_i$

Work output – work done by the gas on the right piston,

$\Delta W_{out} = p_f V_f$

So:

$$U_f - U_i = p_i V_i - p_f V_f$$

$$U_f + p_f V_f = U_i + p_i V_i$$

But since $H = U + pV$, $H_f = H_i$, i.e,

$$H = \text{const.}$$

(b)

$$dT = \left(\frac{\partial T}{\partial p}\right)_H dp = -\frac{\left(\frac{\partial H}{\partial p}\right)_T}{\left(\frac{\partial H}{\partial T}\right)_p} dp \quad (\text{"chain rule"})$$

$$dH = d(U + pV) = dU + d(pV) = TdS - pdV + pdV + VdP$$
$$= TdS + VdP$$

Let's take $S = S(T, P) \Rightarrow dS = \left(\frac{\partial S}{\partial T}\right)_p dT + \left(\frac{\partial S}{\partial P}\right)_T dP$

$\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$ (Maxwell relation)

But since: $T\left(\frac{\partial S}{\partial T}\right)_p \equiv C_p \Rightarrow \left(\frac{\partial S}{\partial T}\right)_p = \frac{C_p}{T}$

So: $dS = \frac{C_p}{T} dT - \left(\frac{\partial V}{\partial T}\right)_p dp$

Furthermore: $\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \equiv \alpha$ (coeff. of thermal expansion)

So:

$$dS = \frac{C_p}{T} dT - V\alpha\, dp$$

Plugging into the expression for dH:

$$dH = T\left(\frac{C_p}{T}\right)dT - TV\alpha\, dP + V dp = C_p dT + V(1 - T\alpha) dP$$

On the other hand: if $H = H(T, P)$

$$dH = \left(\frac{\partial H}{\partial T}\right)_p dT + \left(\frac{\partial H}{\partial P}\right)_T dp$$

By comparing: $\left(\frac{\partial H}{\partial T}\right)_p = C_p$ and $\left(\frac{\partial H}{\partial P}\right)_T = V(1 - T\alpha)$

Hence:

$$dT = -\frac{\left(\frac{\partial H}{\partial P}\right)_T}{\left(\frac{\partial H}{\partial T}\right)_p} dp = -\frac{V(1 - T\alpha)}{C_p} dp$$

Since $V = N \cdot v$, and $C_p = N c_p$

$$\boxed{dT = \frac{(T\alpha - 1)v}{c_p} dp}$$ solution, (b)

$\boxed{(c):}$

For van der Waals gas, we have to find α:

$$\alpha = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p = \frac{1}{v}\left(\frac{\partial T}{\partial v}\right)_p^{-1}$$

$$p = \frac{RT}{v-b} - \frac{a}{v^2} \Rightarrow T = \frac{1}{R}\left(p + \frac{a}{v^2}\right)(v-b) = \frac{1}{R}\left(vp - bp + \frac{a}{v} - \frac{ab}{v^2}\right)$$

$$\left(\frac{\partial T}{\partial v}\right)_p = \frac{1}{R}\left(p - \frac{a}{v^2} + \frac{2ab}{v^3}\right) = \frac{1}{R}\left(\frac{RT}{v-b} - \frac{a}{v^2} - \frac{a}{v^2} + \frac{2ab}{v^3}\right)$$

$$= \frac{1}{R}\left[\frac{RT}{v-b} - \frac{2a}{v^2}\left(1 - \frac{b}{v}\right)\right] = \frac{1}{R}\left[\frac{RT}{v-b} - \frac{2a}{v^3}(v-b)\right] = \frac{1}{R}\left[\frac{RTv^3 - 2a(v-b)^2}{(v-b)v^3}\right]$$

$$\left(\frac{\partial v}{\partial T}\right)_p = R\frac{(v-b)v^3}{RTv^3 - 2a(v-b)^2}; \quad \alpha = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p = R\frac{(v-b)v^2}{RTv^3 - 2a(v-b)^2}$$

$$\boxed{\alpha = R\frac{(v-b)v^2}{RTv^3 - 2a(v-b)^2}}$$

We don't have c_p, only c_v – but one can calculate $c_p - c_v$ in the standard way; considering the entropy:

$$S_1(T,V,N) = S_2(T, p(T,V,N), N); \text{ then}$$

$$\left(\frac{\partial S_1}{\partial T}\right)_{V,N} = \left(\frac{\partial S_2}{\partial T}\right)_{p,N} + \left(\frac{\partial S}{\partial p}\right)_{T,N} \cdot \left(\frac{\partial p}{\partial T}\right)_{V,N} \quad \Big| \text{ multiply the eq. by } T$$

$$C_V = C_p + T\left(\frac{\partial S}{\partial p}\right)_{T,N} \cdot \left(\frac{\partial p}{\partial T}\right)_{V,N} = C_p - \left(\frac{\partial V}{\partial T}\right)_{p,N} \cdot \left(\frac{\partial p}{\partial T}\right)_{V,N}$$

$$\left(\frac{\partial V}{\partial T}\right)_{p,N} = -\frac{\left(\frac{\partial p}{\partial T}\right)_{V,N}}{\left(\frac{\partial p}{\partial V}\right)_{T,N}} = -\left(\frac{\partial p}{\partial T}\right)_{V,N} \cdot \left(\frac{\partial V}{\partial p}\right)_{T,N}$$

So, we get:
$$C_p = C_V - T\left(\frac{\partial p}{\partial T}\right)_{V,N}^2 \left(\frac{\partial V}{\partial p}\right)_{T,N}$$

or, for molar coordinates: $c_p = c_v - T\left(\frac{\partial p}{\partial T}\right)_v^2\left(\frac{\partial v}{\partial p}\right)_T = c_v\left(\frac{\partial p}{\partial T}\right)_v^2\left(\frac{\partial p}{\partial v}\right)_T^{-1}$

$$p = \frac{RT}{v-b} - \frac{a}{v^2}$$

$$\left(\frac{\partial p}{\partial T}\right)_v = \frac{R}{v-b}; \quad \left(\frac{\partial p}{\partial v}\right)_T = -\frac{RT}{(v-b)^2} + \frac{2a}{v^3}$$

So:
$$c_p = c_v - \frac{T\frac{R^2}{(v-b)^2}}{-\frac{RT}{(v-b)^2} + \frac{2a}{v^3}} = c_v + \frac{R}{1 - \frac{2a(v-b)^2}{RTv^3}}$$

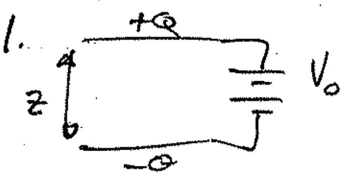
so: $$dT = \frac{v}{c_p}(T\alpha - 1)\,dp$$

With:
$$c_p = c_v + \frac{R^2Tv^3}{RTv^3 - 2a(v-b)^2}$$

$$\alpha = \frac{R(v-b)v^2}{RTv^3 - 2a(v-b)^2}$$

Comp. Exam # 100; Problem 7; Solution.

1.



a) $Q_0 = \sigma_0 A \quad E_0 = V_0/z$ and $E_0 = \frac{\sigma_0}{\epsilon_0}$

so, $Q_0 = C_0 V_0 \Rightarrow A\epsilon_0 E_0 = C_0 V_0 \Rightarrow A\epsilon_0 \frac{V_0}{z} = C_0 V_0$

Ignore edge effects.

and $C_0 = \epsilon_0 A/z$

$u_0 = \frac{\epsilon_0}{2} E_0^2 \Rightarrow U_0 = \text{volume} \times u_0 = Az \frac{\epsilon_0}{2} \frac{V_0^2}{z^2}$

or $U_0 = \frac{\epsilon_0 A}{2} \frac{V_0^2}{z}$

To calculate the force use either $F = -\frac{d}{dz} U_0$ or use the fact that the field at the lower plate is $\sigma/2\epsilon_0$. Then the force per unit area is $\sigma_0 \sigma_0/2\epsilon_0 = \epsilon_0 V_0^2/2z^2$

b) This can be a difficult problem. What is conserved? Energy might not appear to be conserved because work is done on the fluid as it enters. So, conserve charge

$Q = Q_0$ or $CV = C_0 V_0$. We know that $C = \epsilon A/z$,
so $V = \epsilon_0 V_0/\epsilon$. Energy density is $u = \frac{\epsilon}{2}\frac{V^2}{z^2} = \frac{1}{2}\frac{\epsilon_0^2}{\epsilon}\left(\frac{V_0}{z}\right)^2$

and $U = Azu = \frac{A}{2}\frac{\epsilon_0^2}{\epsilon}\frac{V_0^2}{z}$

Force $= -\frac{d}{dz} U \Rightarrow |F| = \frac{A}{2}\frac{\epsilon_0^2}{\epsilon}\frac{V_0^2}{z^2}$

or $|F| = A\sigma_0\sigma_0/2\epsilon = \frac{A}{2}\frac{\epsilon_0^2}{\epsilon}\frac{V_0^2}{z^2}$

c) Same as (a) except that ϵ replaces ϵ_0

2. a) Using the boundary condition that normal D is continuous,
$E_1 = \epsilon_0 E_0/\epsilon_1$ and $E_2 = \epsilon_0 E_0/\epsilon_2$ where $E_0 = V_0/z$

$u_1 = \frac{1}{2}\frac{\epsilon_0^2}{\epsilon_1}\frac{V_0^2}{z^2}$ and $u_2 = \frac{1}{2}\frac{\epsilon_0^2}{\epsilon_2} V_0^2/z^2$

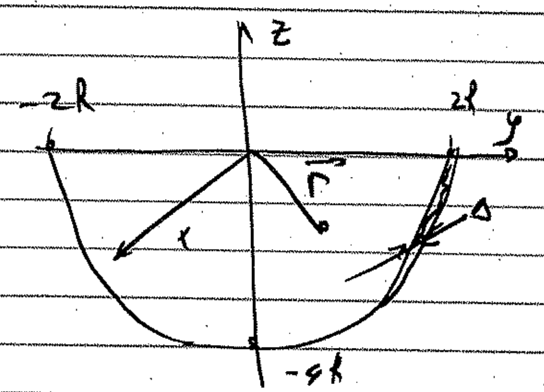
b) $|F| = \frac{d}{dz}\left(A\frac{z}{2}u_1 + A\frac{z}{2}u_2\right) = \frac{A\epsilon_0^2 V_0^2}{4z^2}\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)$
There is no force on the fluids, ~~since~~ — all fields cancel on surfaces.

Spring 2006. Comprehensive exam. # 8

Note Title

3/28/2006

(1) Moment of inertia around x axis.



$$I_x = \int dz\, dy \cdot \Delta \cdot \rho \left(z^2 + y^2\right) =$$

$$= \rho \cdot \Delta \cdot \int_{-2R}^{2R} dy \cdot \int_{-4R + y^2/R}^{0} dz \left(z^2 + y^2\right) =$$

$$= \rho \cdot \Delta \int_{-2R}^{2R} dy \left[y^2 \left(4R - \frac{y^2}{R}\right) + \frac{1}{3}\left(4R - \frac{y^2}{R}\right)^3 \right] =$$

$$= \rho \cdot \Delta \cdot \int_{-2R}^{2R} dy \left[4R y^2 - \frac{y^4}{R} + \frac{1}{3}\left(64R^3 - \frac{y^6}{R^3} + 12R \cdot \frac{y^4}{R^2} - 48R^2 \cdot \frac{y^2}{R}\right)\right] =$$

$$= \rho \cdot \Delta \int_{-2R}^{2R} dy \left[-\frac{1}{3}\frac{y^6}{R^3} + \frac{3y^4}{R} - 12y^2 \cdot R + \frac{64}{3}R^3 \right] =$$

$$= 2\rho \cdot \Delta \cdot \left[-\frac{(2R)^7}{21R^3} + \frac{3(2R)^5}{5R} - \frac{12(2R)^3}{3}R + \frac{64}{3} \cdot 2R \cdot R^3 \right] =$$

$$= 2\rho \cdot \Delta \cdot R^4 \left[\frac{2 \cdot 64}{3} - \frac{12 \cdot 8}{3} + \frac{32 \cdot 3}{5} - \frac{128}{21} \right] =$$

(5pts)

$$= 2\rho \Delta R^4 \left[\frac{2^7}{3} - \frac{3 \cdot 2^5}{3} + \frac{2^5 \cdot 3}{5} - \frac{2^7}{21} \right] =$$

$$= 2^6 \cdot \rho \Delta R^4 \left[\frac{4}{3} - \frac{3}{3} + \frac{3}{5} - \frac{4}{21} \right] =$$

n	2^n
1	2
2	4
3	8
4	16
5	32
6	64
7	128