OSU Physics Department
Comprehensive Examination #135
Solutions

Wednesday, September 25 and Thursday, September 26, 2019

Fall 2019 Comprehensive Examination

Quantum Mechanics	9 AM-12 PM	Wednesday, September 25
Electricity and Magnetism	1 PM-4 PM	Wednesday, September 25
Statistical Mechanics	9 AM-12 PM	Thursday, September 26
Classical Mechanics	1 PM-4 PM	Thursday, September 26

General Instructions

This Fall 2019 Comprehensive Examination consists of four separate parts of two problems each, and you have three hours to work on each part. Each problem caries equal weight (20 points). Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work on the provided pages, work each problem in its own labeled pages, and be certain that your chosen student letter (but not your name) is on the header of each page of your exam, including any unused pages. If you need additional paper for your work, use the blank pages provided. Each page of work should include the problem number, a page number, your chosen student letter, and the total number of pages actually used. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam and the collection of formulas distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please staple and return all pages of your exam—including unused pages—at the end of the exam.

Let the matrix representation of the Hamiltonian of a three-state system be

$$H \doteq \begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}$$

using the basis states $|1\rangle$, $|2\rangle$, and $|3\rangle$.

a) If the state of the system at time $t = 0$ is $|\psi(0)\rangle = |2\rangle$, what is the probability that the system is in state $|2\rangle$ at time t?

b) If, instead, the state of the system at time $t = 0$ is $|\psi(0)\rangle = |3\rangle$, what is the probability that the system is in state $|3\rangle$ at time t?

First we need to find the energy eigenvalue and eigenstates. Diagonalizing H yields the eigenvalues

$$\begin{pmatrix} E_0-\lambda & 0 & A \\ 0 & E_1-\lambda & 0 \\ A & 0 & E_0-\lambda \end{pmatrix} = 0 \;\Rightarrow\; (E_0-\lambda)^2(E_1-\lambda) - A^2(E_1-\lambda) = 0$$

$$\Rightarrow\; (E_1-\lambda)\left\{(E_0-\lambda)^2 - A^2\right\} = 0 \;\Rightarrow\; \lambda = E_1, E_0+A, E_0-A$$

and the eigenvectors

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = E_1\begin{pmatrix} u \\ v \\ w \end{pmatrix} \;\Rightarrow\; \begin{matrix} E_0u + Aw = E_1u \\ E_1v = E_1v \\ Au + E_0w = E_1w \end{matrix} \;\Rightarrow\; u = w = 0$$

$$|u|^2 + |v|^2 + |w|^2 = 1 \;\Rightarrow\; |v|^2 = 1 \;\Rightarrow\; u = 0, v = 1, w = 0 \;\Rightarrow\; |E_1\rangle = |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = (E_0 \pm A)\begin{pmatrix} u \\ v \\ w \end{pmatrix} \;\Rightarrow\; \begin{matrix} E_0u + Aw = (E_0 \pm A)u \\ E_1v = (E_0 \pm A)v \\ Au + E_0w = (E_0 \pm A)w \end{matrix} \;\Rightarrow\; v = 0, u = \pm w$$

$$|u|^2 + |v|^2 + |w|^2 = 1 \;\Rightarrow\; 2|u|^2 = 1 \;\Rightarrow\; u = \tfrac{1}{\sqrt{2}}, v = 0, w = \pm\tfrac{1}{\sqrt{2}}$$

$$\Rightarrow\; |E_0 \pm A\rangle = \tfrac{1}{\sqrt{2}}|1\rangle \pm \tfrac{1}{\sqrt{2}}|3\rangle \doteq \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ \pm 1 \end{pmatrix}$$

(a) The initial state is

$$|\psi(0)\rangle = |2\rangle = |E_1\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The time evolved state is

$$|\psi(t)\rangle = e^{-iE_1t/\hbar}|E_1\rangle \doteq \begin{pmatrix} 0 \\ e^{-iE_1t/\hbar} \\ 0 \end{pmatrix}$$

The probability of measuring the system to be in state $|2\rangle$ is

$$\mathcal{P}_2 = \left|\langle 2|\psi(t)\rangle\right|^2 = \left|\langle 2|e^{-iE_1t/2\hbar}|2\rangle\right|^2 = \left|e^{-iE_1t/2\hbar}\right|^2 = 1$$

(b) The initial state is

$$|\psi(0)\rangle = |3\rangle = \tfrac{1}{\sqrt{2}}\left(|E_0 + A\rangle - |E_0 - A\rangle\right) \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The time-evolved state is

$$\begin{aligned} |\psi(t)\rangle &= \tfrac{1}{\sqrt{2}}\left(e^{-i(E_0+A)t/\hbar}|E_0 + A\rangle - e^{-i(E_0-A)t/\hbar}|E_0 - A\rangle\right) \\ &\doteq \tfrac{1}{\sqrt{2}}e^{-i(E_0+A)t/\hbar}\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \tfrac{1}{\sqrt{2}}e^{-i(E_0-A)t/\hbar}\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &\doteq \tfrac{1}{2}e^{-iE_0t/\hbar}\begin{pmatrix} -2i\sin(At/\hbar) \\ 0 \\ 2\cos(At/\hbar) \end{pmatrix} \end{aligned}$$

The probability of measuring the system to be in state $|3\rangle$ is

$$\mathcal{P}_3 = \left|\langle 3|\psi(t)\rangle\right|^2 = \left|\tfrac{1}{2}e^{-iE_0t/\hbar}2\cos(At/\hbar)\right|^2 = \cos^2(At/\hbar) = \tfrac{1}{2}\left(1+\cos(2At/\hbar)\right)$$

Problem 2

Consider the semi-infinite square potential energy well:

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \le x \le a \\ V_0 & x > a \end{cases}$$

a) Find the transcendental equation for the bound state energies for a particle of mass m in this potential well.
b) Sketch the ground-state wave function. Describe its key features.

a) The energy eigenvalue equation is

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V\right)\varphi_E(x)=E\varphi_E(x)$$

It is useful to define a wave vector k inside the well

$$k=\sqrt{\frac{2mE}{\hbar^2}}$$

and a similar constant outside the well ($x > a$)

$$q=\sqrt{\frac{2m}{\hbar^2}(V_0-E)}\ .$$

For bound states, $0<E<V_0$, and therefore both k and q are real. We use these two constants to rewrite the energy eigenvalue equation:

$$\frac{d^2\varphi_E(x)}{dx^2}=-k^2\varphi_E(x)\quad \textit{inside box}$$
$$\frac{d^2\varphi_E(x)}{dx^2}=q^2\varphi_E(x)\quad \textit{outside box}$$

For $x < 0$, the wave function must be zero. The energy eigenstates must be constructed by connecting solutions in the three regions. We write the general solution as

$$\varphi_E(x)=\begin{cases} 0,\ \text{for } x<0 \\ C\sin kx+D\cos kx,\ \text{for } 0<x<a \\ Fe^{qx}+Ge^{-qx},\ \text{for } x>a \end{cases}$$

The solutions in the three regions must satisfy boundary conditions where the regions connect. At $x = 0$, only the wave function continuity is required:

$$C\sin k0+D\cos k0=0\ \Rightarrow\ D=0$$

To ensure normalizability, the growing exponential is not allowed, so $F = 0$. At $x = a$, we use both the wave function and the derivative conditions:

$$C\sin ka=Ge^{-qa}$$
$$Ck\cos ka=-Gqe^{-qa}$$

Dividing the 2 equations gives

$$\tan ka=-\frac{k}{q}$$

In terms of the allowed energies, this transcendental equation is

$$\tan\left(a\sqrt{\frac{2mE}{\hbar^2}}\right) = -\frac{\sqrt{\frac{2mE}{\hbar^2}}}{\sqrt{\frac{2m}{\hbar^2}\left(V_0 - E\right)}} = -\sqrt{\frac{E}{V_0 - E}}$$

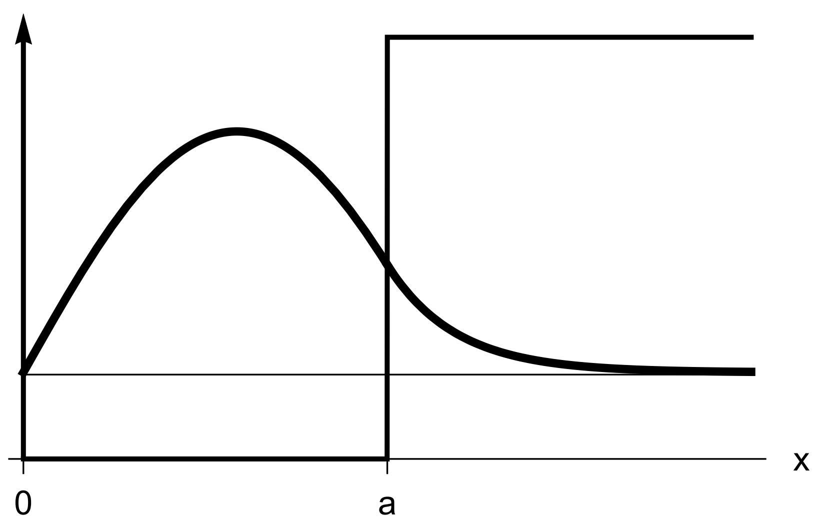
A numerical or graphical solution is required. That exercise is left to the reader.

b) The ground state wave function is shown below. Important features are:

i) Zero at origin where potential goes to infinity (and beyond).

ii) Asymptotically approaches zero as x goes to infinity to ensure normalizability.

iii) Inflection point at $x = a$, where potential changes from below E to above E.



Problem 3

Suppose a semi-infinite slab extending from $y = -b$ to $y = b$ (and infinite in the x- and z- directions) carries a volume current density

$$\vec{J} = J_0 \frac{|y|}{b} \hat{z}.$$

(a) Find the magnetic field $\vec{B}$ everywhere, inside and outside of the slab.

(b) Sketch $\vec{B}$.

(c) Make sure your sketch makes good physical sense. Explain.

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(a) Find the magnetic field $\vec{B}$ everywhere, inside and outside of the slab.

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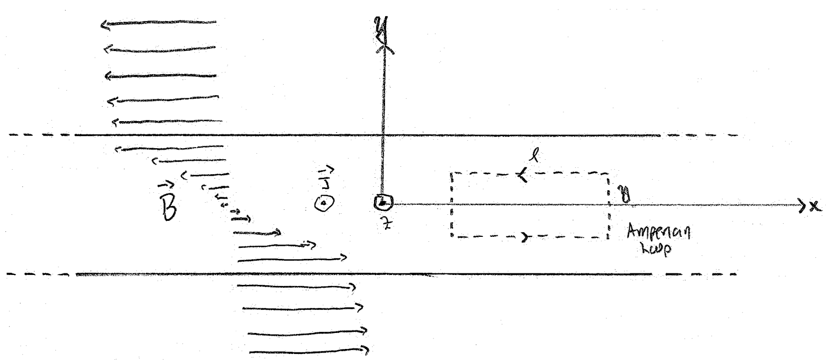
Solution:
This problem is all about Ampere 's law, $\nabla \times \vec{B} = \mu_0 \vec{J}$, which follows from the Maxwell equation $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \partial \vec{E}/\partial t$ applied to static fields. Integrating over a surface spanning a chosen "Amperian loop" and applying Stokes theorem, this gives $\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a} \equiv \mu_o I_{\mathrm{enclosed}}$, where the line integral is taken around the loop in a sense related by the right-hand rule to the normal direction chosen to calculate the integral giving the flux of current through the surface (aka the current enclosed by the loop).



(a) This is easier if you do parts (b-c) first, so read that now. You need the figure above to understand how to do the maths properly. Once you have some intuition for the shape of the resulting field, you should be able to recognize that the proper thing to do is apply Ampere's law by integrating it over a rectangular surface with sides parallel to the x- and y-axes and symmetric about the x-axis, and apply Stokes' theorem to turn the integral of the curl over the surface to a line integral – the "Amperian loop" – around its boundary. With the normal to the rectangle chosen outward along the z-axis, that line integral is counterclockwise around the loop.[1] Since the field is purely in the x-direction, only the integral of $\vec{B}$ along the top and bottom contributes, and the integral is trivial

[1] Since you know the field is zero for $y = 0$, you could just as well choose an Amperian loop half the size, one side of which runs along $y = 0$ instead of at negative y.

because the field does not vary in the x-direction. Thus, integrating over the surface shown and applying Stokes' theorem,

$$
\begin{aligned}
\nabla \times \vec{B} &= \mu_0 \vec{J} \\
\int \nabla \times \vec{B} \cdot d\vec{a} &= \mu_0 \int \vec{J} \cdot d\vec{a} \\
\oint \vec{B} \cdot d\vec{l} &= \mu_0 \int \vec{J} \cdot d\vec{a} \quad (\equiv \mu_0 I_{\text{enc}}) \\
-B_x(y) \cdot l + B_x(-y) \cdot l &= \frac{\mu_0 J_0}{b} \int_0^l dx \int_{-y}^{y} dy\, |y| \qquad (\text{where } y \le b) \\
-2B_x(y) \cdot l &= 2\mu_0 J_0 \frac{l}{b} \int_0^y dy\, y \qquad (y \le b) \\
B_x(y) &= -\mu_0 J_0 \frac{1}{b} \begin{cases} \frac{1}{2} y^2 & y \le b \quad (\text{inside the slab}) \\ \frac{1}{2} b^2 & y > b \quad (\text{outside the slab}) \end{cases} .
\end{aligned}
$$

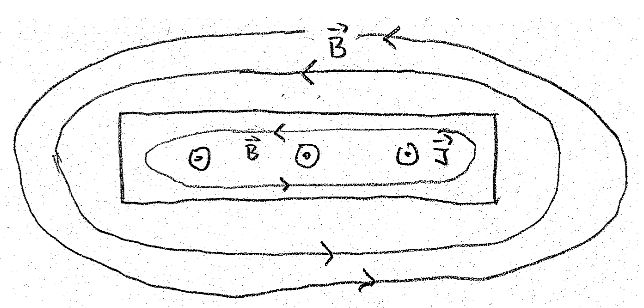
and $B_x(-y) = -B_x(y)$. Or,

$$
\vec{B}(y) = \frac{1}{2}\mu_0 J_0 b \begin{cases} -1 & y > b \\ -\left(\frac{y}{b}\right)^2 & 0 \le y \le b \\ +\left(\frac{y}{b}\right)^2 & -b \le y \le 0 \\ 1 & y < -b \end{cases} \quad \hat{\text{x}} .
$$

This field is entirely consistent with our reasoning in part (c), below.

(b) See the figure. The figure and the discussion below assumes $J_0 > 0$, i.e. that the current is out of the page.

(c) The essential intuition is that magnetic fields wrap around the current that is generating them according to the right-hand rule. (This follows directly from Ampere's law, which tells you that the curl of a magnetic field points in the direction of its source current.) Since the current is flowing out of the page (see the figure) the magnetic field wants to wrap around it counter-clockwise. However, since the slab is infinite in the x-direction, the field lines can't close around the edges of the slab like they would if the slab were finite in extent in that dimension, and end up being parallel to the x-axis. (It's a limiting case of a finite slab:



Thought of another way, the infinite slab is an approximation to a finite slab close to the surface of the slab, close to its center.)

Anyway, this intuition tells you further that the field lines must point to the left (in the negative-x direction) above the center-line of the slab (i.e. the x-y plane), and to the right (the positive-x direction) below it. By the translational symmetry in the x- and z-directions, the magnitude of the field must be constant in those directions, so the only possible variation allowed must be in the y-direction. Additionally, by the reflection symmetry through the x-y plane, it therefore must be that $B_x(-y) = -B_x(y)$. Moreover, because the field changes direction across the x-y plane, the field must be zero right on the plane ($y = 0$). Further, Ampere's law tells you that the more current is enclosed by the Amperian loop, the stronger the field, so the field strength must increase as you get further from the x-y plane. Once you get outside the slab, the field can't keep increasing because you aren't enclosing any more current, so it must remain constant outside of the slab all the way to $y = \pm\infty$. (This isn't of course realistic, but that is because slabs that are infinite in extent aren't either. It's the same situation with an infinite sheet of charge: Gauss' law tells you that the electric field above such a sheet is also constant out to infinity away from the sheet. This is the magnetic version of the same problem.)

All of that intuition and reasoning yields the figure shown.

TLDR: If you really want to get fancy, we can even mostly work out the value of the field strength from dimensional analysis. Ampere's law tells you that the magnitude of $\vec{B}$ has dimensions of $\mu_0 \times J_0 \times$ [length], and the only parameter in the problem with dimensions of length is the (half-)thickness of the slab, b. Thus, since outside the slab the field strength is constant, we must have $B_x \propto \mu_0 J_0 b$. Inside, since it must go to zero for $y = 0$, and must join smoothly to the field outside, we might guess something like $B_x \propto \mu_0 J_0 b \times (y/b)$. This guess would be correct were the current density inside the slab spatially constant, so that the current *enclosed* by the Amperian loop increases linearly in y. However, since the magnitude of $\vec{J}$ is not constant but rather increases linearly with y, the enclosed current increases *quadratically* and the y-dependence of the field is $(y/b)^2$ rather than simply (y/b). The work in part (a) is just to nail down details such as any possible overall numeric factors.

Problem 4

A spatially uniform magnetic field $\vec{B} = B_z(t)\hat{z}$ is confined to a circular region of radius a_B, outside of which the field is 0. Looping around this magnetic field (and concentric with it) is a metal hoop of radius $a_H > a_B$ with a resistance R. Suppose that at $t = 0$ the strength of the magnetic field begins to change:

$$B_z(t) = B_0 - K \cdot t^2,$$

where K is a constant.

(a) Find the current (magnitude and direction) in the hoop for $t > 0$.

(b) Note the magnetic field eventually reverses direction. Does the current do the same? Explain physically.

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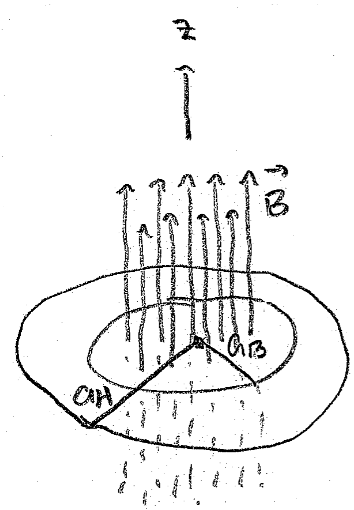
where K is a constant.

(a) Find the current (magnitude and direction) in the hoop for $t > 0$.

(b) Note the magnetic field eventually reverses direction. Does the current do the same? Explain physically.

Solution:

This problem is all about Faraday's 's law, $\mathcal{E} = -d\Phi_B/dt$, which follows from the Maxwell equation $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$ integrated over a surface and applying Stokes' theorem. Here, the flux $\Phi_B = \int \vec{B} \cdot d\vec{a}$ is calculated by integrating $\vec{B}$ over the surface and $\mathcal{E} = \oint \vec{E} \cdot d\vec{l}$ is integrated around the boundary of that surface in a direction related by the right-hand rule to the chosen normal to said surface.



(a) The idea is to recognize that because the flux of magnetic field through the hoop is changing, there is an induced electric field (yielding the "emf" $\mathcal{E}$) that induces a current I in the hoop, which we find by setting $\mathcal{E} = I \cdot R$. The "surface" over which we are integrating is the circle of radius a_H enclosed by the hoop, with normal $+\hat{z}$. (This is because that will tell us the electric field, hence $\mathcal{E}$, induced at the boundary of that surface, namely, inside the hoop.) Intuitively, assuming that $K > 0$, the magnetic flux Φ_B up through the hoop is *decreasing* – worded more carefully, becoming *more negative* – with time. According to Lenz' law (aka the $-$ sign in Faraday's law), the induced current *resists* the change in flux, so a current will be induced **counterclockwise** around the hoop (as shown) in order to create a positive magnetic flux up through the hoop to counter the decrease in flux from the external field.

Here's the math:

$$\begin{aligned}
\mathcal{E} &= -\frac{d}{dt}\int_{\text{hoop}} \vec{B}\cdot d\vec{a} \\
&= -\frac{d}{dt}(B_z(t)\cdot \pi a_B^2) \qquad \text{(note } a_B\text{, not } a_H\text{, because } \vec{B} \neq 0 \text{ only for } r < a_B) \\
&= +2\pi a_B^2 K t \\
&= I \cdot R \\
I &= \frac{2\pi a_B^2 K t}{R} \qquad \text{(counterclockwise around } z\text{-axis)}\ .
\end{aligned}$$

The flux integral is trivial because the magnetic field is spatially constant. Note the current $I > 0$, which tells us that it is counterclockwise around the hoop. (That's because with $\hat{z}$ as the chosen normal to the surface over which we are integrating, we were integrating $\mathcal{E} = \oint \vec{E}\cdot d\vec{l}$ counterclockwise around the hoop, and our calculation shows that with these choices $\mathcal{E}$, hence I, is positive, consistent with Lenz' law. Lenz' law, of course, is just a way of putting words and intuition into a $-$ sign that is built into Maxwell's equations.)

(b) See the end of the first paragraph of the answer to (a). Note that argument holds regardless of the sign of B_0 or of $B_z(t)$; what *matters* is that $\partial B_z(t)/\partial t < 0$, so that $d\Phi_B/dt < 0$. That is the change the induced current is resisting, and the *change* doesn't change. Therefore, the induced current I does not change direction.

Problem 5

Adiabatic diatomic gas Consider an insulated cylinder which is full of a diatomic gas, which is governed by the following equations of state:

$$U = \frac{5}{2} N k_B T + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} \tag{5.1}$$

$$pV = N k_B T \tag{5.2}$$

The initial temperature is T_0 and the volume is V_0. The gas is slowly compressed until it is twice its original temperature. Solve for the final volume. You may express your final answer as an equation involving an integral.

Note that the second term in the internal energy above is frequently ignored under the assumption that $k_B T \ll \hbar\omega$. In this problem you should not make that approximation.

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The initial temperature is T_0 and the volume is V_0. The gas is slowly compressed until it is twice its original temperature. Solve for the final volume. You may express your final answer as an equation involving an integral.

Note that the second term in the internal energy above is frequently ignored under the assumption that $k_B T \ll \hbar\omega$. In this problem you should not make that approximation.

Solution:

Firstly, I will mention there was an error in the problem. The second term in U should have a factor of N. This meant that the internal energy was not extensive, which undermines sense-making about this problem.

We begin with the thermodynamic identity:

$$dU = TdS - pdV \tag{5.5}$$

and note that because we are adiabatically compressing, the entropy remains constant, so we can say that

$$dU = -pdV \tag{5.6}$$

$$= -\frac{N k_B T}{V} dV \tag{5.7}$$

Now we can use the other equation of state, and take a derivative to find a relation between dU and dT:

$$dU = \frac{5}{2} N k_B dT + \frac{\hbar\omega}{\left(e^{\frac{\hbar\omega}{k_B T}} - 1\right)^2} e^{\frac{\hbar\omega}{k_B T}} \frac{\hbar\omega}{k_B T} \frac{1}{T} dT \tag{5.8}$$

Now we can set the two expressions for dU equal:

$$-\frac{N k_B T}{V} dV = \left(\frac{5}{2} N k_B + \frac{\hbar\omega}{\left(e^{\frac{\hbar\omega}{k_B T}} - 1\right)^2} e^{\frac{\hbar\omega}{k_B T}} \frac{\hbar\omega}{k_B T} \frac{1}{T} \right) dT \tag{5.9}$$

Now we put all the V on one side and all the T on the other, and we can integrate from the initial state to the final state.

$$-\frac{Nk_B}{V}dV = \left(\frac{5}{2}Nk_B + \frac{\hbar\omega}{\left(e^{\frac{\hbar\omega}{k_BT}}-1\right)^2}e^{\frac{\hbar\omega}{k_BT}}\frac{\hbar\omega}{k_BT^2}\right)\frac{1}{T}dT \tag{5.10}$$

$$-\int_{V_0}^{V_f}\frac{Nk_B}{V}dV = \int_{T_0}^{2T_0}\left(\frac{5}{2}Nk_B + \frac{\hbar\omega}{\left(e^{\frac{\hbar\omega}{k_BT}}-1\right)^2}e^{\frac{\hbar\omega}{k_BT}}\frac{\hbar\omega}{k_BT^2}\right)\frac{1}{T}dT \tag{5.11}$$

$$-Nk_B\ln\frac{V_f}{V_0} = \frac{5}{2}Nk_B\ln 2 + \int_{T_0}^{2T_0}\frac{\hbar\omega}{\left(e^{\frac{\hbar\omega}{k_BT}}-1\right)^2}e^{\frac{\hbar\omega}{k_BT}}\frac{\hbar\omega}{k_BT^3}dT \tag{5.12}$$

At this point, we essentially have an answer, but we're supposed to solve for V_f, and it also seems like it would be nice to simplify the integral by taking out the dimensions.

$$\tag{5.13}$$

$$u = \frac{\hbar\omega}{kT} \qquad\qquad du = -\frac{\hbar\omega}{kT^2}dT \tag{5.14}$$

$$-Nk_B\ln\frac{V_f}{V_0} = \frac{5}{2}Nk_B\ln 2 + \int_{\frac{\hbar\omega}{k_BT_0}}^{\frac{\hbar\omega}{2k_BT_0}}\frac{e^u}{(e^u-1)^2}k_Bu(-du) \tag{5.15}$$

$$= \frac{5}{2}Nk_B\ln 2 + k_B\int_{\frac{1}{2}\frac{\hbar\omega}{k_BT_0}}^{\frac{\hbar\omega}{k_BT_0}}\frac{e^u}{(e^u-1)^2}udu \tag{5.16}$$

Sadly, we can't take the physics out of the limits of the integral, which is why you weren't asked to do so. But we can still solve for V_f in terms of this integral.

$$V_f = V_0 2^{-\frac{5}{2}}e^{-\frac{1}{N}\int_{\frac{1}{2}\beta_0}^{\beta_0}\frac{e^u}{(e^u-1)^2}udu} \tag{5.17}$$

Unknown system Consider a system with the following internal energy:

$$U = \frac{E_1 2e^{-\beta E_1}}{1 + 2e^{-\beta E_1}} \tag{6.1}$$

where the energy $E_1 > 0$ and as usual $\beta = \frac{1}{k_B T}$. You are told by a reliable source that the ground state of the system is non-degenerate.

(a) Solve for the heat capacity of this system at constant volume.

(b) How many microstates does the system have, and what are their energies?

(c) Solve for the limiting value of the entropy at high and low temperatures. You need to retain any temperature dependence in your answer.

Unknown system Consider a system with the following internal energy:

$$U = \frac{E_1 2e^{-\beta E_1}}{1 + 2e^{-\beta E_1}} \tag{6.2}$$

where the energy $E_1 > 0$ and as usual $\beta = \frac{1}{k_B T}$. You are told by a reliable source that the ground state of the system is non-degenerate.

(a) Solve for the heat capacity of this system at constant volume.

Solution:
This part does not require that you solve for the entropy, although most students ended up solving part (b) first and then finding the entropy and computing the heat capacity from that. That approach is completely correct, but is a bit more work than what I show here..

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V \tag{6.3}$$

$$= \left(\frac{\partial U}{\partial T}\right)_V \tag{6.4}$$

This bit comes from the thermodynamic identity $dU = TdS - pdV$, and recognizing that since V is held constand during the derivative $dU = TdS$. Solving for the entropy (once you realize that it is a three-state system, see below) isn't too bad, but is a bit more algebraically messy.

$$C_V = 2E_1\left(-E_1\frac{e^{-\beta E_1}}{1 + 2e^{-\beta E_1}} + 2E_1\frac{e^{-\beta E_1}}{(1 + 2e^{-\beta E_1})^2}e^{-\beta E_1}\right)\frac{\partial \beta}{\partial T} \tag{6.5}$$

$$= \left(\frac{e^{-\beta E_1}}{1 + 2e^{-\beta E_1}} - 2\frac{e^{-2\beta E_1}}{(1 + 2e^{-\beta E_1})^2}\right)\frac{2E_1^2}{k_B T^2} \tag{6.6}$$

$$= \left(\frac{e^{-\beta E_1} + 2e^{-2\beta E_1}}{(1 + 2e^{-\beta E_1})^2} - 2\frac{e^{-2\beta E_1}}{(1 + 2e^{-\beta E_1})^2}\right)\frac{2E_1^2}{k_B T^2} \tag{6.7}$$

$$= \frac{2E_1^2}{k_B T^2}\frac{e^{-\beta E_1}}{(1 + 2e^{-\beta E_1})^2} \tag{6.8}$$

Always check here that the dimensions are energy per temperature, which they are because one of the E_1s cancels out the $k_B T$ on the bottom.

(b) How many microstates does the system have, and what are their energies?

Solution:
There are a couple of ways to solve this problem. Perhaps the most natural

is to start with recalling that the internal energy is a Boltzmann-weighted average of the energies of all the microstates:

$$U = \frac{\sum_i^{\text{all microstates}} E_i e^{-\beta E_i}}{\sum_i^{\text{all microstates}} e^{-\beta E_i}} \tag{6.9}$$

$$= \frac{E_1 2e^{-\beta E_1}}{1 + 2e^{-\beta E_1}} \tag{6.10}$$

$$= \frac{0 \cdot e^0 + E_1 e^{-\beta E_1} + E_1 e^{-\beta E_1}}{e^0 + e^{-\beta E_1} + e^{-\beta E_1}} \tag{6.11}$$

Setting these two expressions for U equal, we can start piecing together the energies. The ground state energy looks to be zero, since the first exponential in the denominator is e^0, which is consistent with the fact that we don't have two terms in the numerator (because $0e^0 = 0$). So the ground state energy is zero. Next we consider the fact that one expression has prefactors of 2 on top and bottom, versus the other that does not have such factors. You can either recognize this as the multiplicity (particularly if you use the "sum over energies" expression rather than the "sum over microstates"), or you can just note that you can obtain a factor of two if there are two identical exponentials due to two states with energy E_1.

In any case, you can confirm that if we have one microstate with energy 0 and two microstates that have energy E_1 you will reproduce the internal energy given. It is well worth confirming that this is the case:

$$U = \frac{\sum_i E_i e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} \tag{6.12}$$

$$= \frac{0 + E_1 e^{-\beta E_1} + E_1 e^{-\beta E_1}}{1 + +e^{-\beta E_1} + e^{-\beta E_1}} \tag{6.13}$$

$$= \frac{2E_1 e^{-\beta E_1}}{1 + +2e^{-\beta E_1}} \tag{6.14}$$

which is the same as what were given.

(c) Solve for the limiting value of the entropy at high and low temperatures. You need to retain any temperature dependence in your answer.

Solution:
This requires our answer from part a. The entropy is given by

$$S = -k_B \sum_i^{\text{all microstates}} P_i \ln P_i \tag{6.15}$$

Our life is made easier by the fact that we were only asked for the high- and low-temperature values, which means that rather than solving for S for arbitrary T (using $P_{1,2} = \frac{e^{-\beta E_1}}{1+2e^{-\beta E_1}}$ and $P_0 = \frac{1}{1+2e^{-\beta E_1}}$, which is totally doable), we can find the probabilities in the high- and low-temperature limits, and use those to find the entropy in those limits.

In the limit of low temperature, the probability approaches 100% that the system is in the ground state, so $P_0 = 1$ and $P_{1,2} = 0$. This means that the entropy is zero, since each $P \ln P$ is zero (since $\ln 1 = 0$ and while $\ln 0 = -\infty$ it approaches infinity more slowly than the factor of P approaches zero).

$$S(T = 0) = 0 \tag{6.16}$$

This is a standardly common result, and even gets the name of the Third Law of Thermodynamics (when expressed correctly).

As the temperature approaches infinity, all three states will be equally likely. This means that $P_0 = P_1 = P_2 = \frac{1}{3}$, so

$$S(T = \infty) = -k_B \left(\frac{1}{3} \ln \frac{1}{3} + \frac{1}{3} \ln \frac{1}{3} + \frac{1}{3} \ln \frac{1}{3} \right) \tag{6.17}$$

$$= -k_B \ln \frac{1}{3} \tag{6.18}$$

$$= k_B \ln 3 \tag{6.19}$$

This is also just the Boltzmann expression for entropy when there are three microstates with essentially the same energy.

Since students can't necessarily be expected to see the easy solution, I'll talk through the hard way. We write down:

$$S = -k_B \left(P_0 \ln P_0 + P_1 \ln P_1 + P_2 \ln P_2 \right) \tag{6.20}$$

$$= -k_B \left(P_0 \ln P_0 + 2 P_1 \ln P_1 \right) \tag{6.21}$$

$$= -k_B \left(\frac{1}{1 + 2e^{-\beta E_1}} \ln \frac{1}{1 + 2e^{-\beta E_1}} + 2 \frac{e^{-\beta E_1}}{1 + 2e^{-\beta E_1}} \ln \frac{e^{-\beta E_1}}{1 + 2e^{-\beta E_1}} \right) \tag{6.22}$$

$$= \frac{k_B}{1 + 2e^{-\beta E_1}} \left(\ln(1 + 2e^{-\beta E_1}) + 2e^{-\beta E_1} \ln(1 + 2e^{-\beta E_1}) + 2\beta E_1 e^{-\beta E_1} \right) \tag{6.23}$$

$$= k_B \ln(1 + 2e^{-\beta E_1}) + \frac{2E_1}{T} \frac{e^{-\beta E_1}}{1 + 2e^{-\beta E_1}} \tag{6.24}$$

Once you get here, you just need to take the high- and low-temperature limits.

There is yet a third (easier than hard, but harder than the easiest) way to solve this. That is to solve for the entropy using the Helmholtz free energy and the internal energy. This involves remembering that

$$F = U - TS \tag{6.25}$$

$$S = \frac{U - F}{T} \tag{6.26}$$

and also remembering that

$$F = -k_B T \ln Z \tag{6.27}$$

so taken together, you find

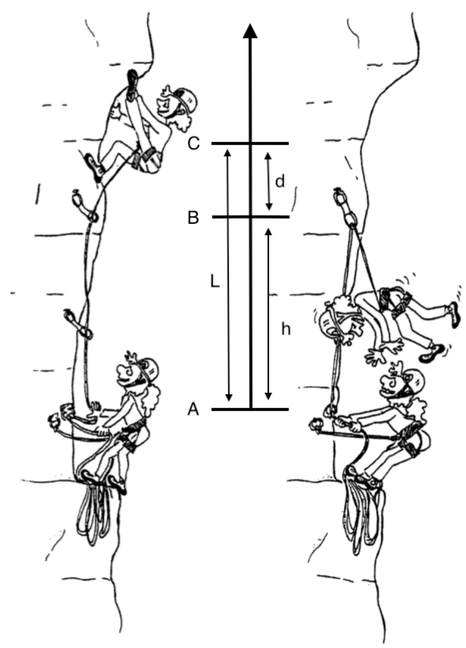
$$S = \frac{\frac{E_1 2e^{-\beta E_1}}{1+2e^{-\beta E_1}} - (-k_B T \ln(1 + 2e^{-\beta E_1}))}{T} \tag{6.28}$$

$$= \frac{2E_1}{T} \frac{e^{-\beta E_1}}{1 + 2e^{-\beta E_1}} + k_B \ln(1 + 2e^{-\beta E_1}) \tag{6.29}$$

which of course is the same answer we got when computing the entropy directly from probabilities.

A climber (mass m, (label C in the figure)) starts climbing from a point high up on a vertical wall (labeled A in the figure) and has placed his last (highest) anchor (denoted B in the figure) a distance h above point A. His (elastic) climbing rope (unstretched length L) is firmly secured (tied) to an indestructible anchor at point A a distance L below the climber. The climber has climbed a distance d above his last anchor point (a small fixed ring), when he slips and falls a distance $2d + \Delta x$, where Δx is the maximum stretching length of the elastic rope.

To answer the questions below a number of idealizations are made: You may assume that the elastic climbing rope can be treated as an ideal spring obeying Hooke's law with a spring constant $k(L) = R/L$. R is the rope modulus given by $R = YA$ with Y being Young's modulus and A the cross sectional area of the rope. You may treat the climber as a point mass and also assume that the rope is massless and moves without friction through the last anchor point B. You may further ignore all motion after the rope fully arrests the fall, i.e. the climber velocity reaches $v = 0$ for the first time.



(a) (b)

(a) Calculate the maximum force acting on the climber during the fall and show that it does not depends on the total length of the fall $(2d + \Delta x)$ but rather on the ratio $f = d/L$.

Solution:
Hooke's law: $F_{el} = -k\Delta x = -R/L\Delta x$.

Maximum force will be for maximum $\Delta x = \Delta x_{max}$. We write Δx_{max} simply as x below.

Conservation of energy determines $\Delta x_{max} = x$:

$$mg(2d + x) = \frac{1}{2}\frac{R}{L}x^2$$

and

$$\begin{aligned} x &= \frac{mg}{R}L\left(1 + \sqrt{1 + 4\frac{R}{mg}\frac{d}{L}}\right) \\ &= \frac{L}{\kappa}\left(1 + \sqrt{1 + 4\kappa\frac{d}{L}}\right), \end{aligned}$$

where we introduced the notation $\kappa = R/mg$.

The force on the climber is the sum of gravity and tension:

$$\begin{aligned} F &= mg - \frac{R}{L}x \\ &= mg\sqrt{1 + 4\kappa f}\,, \qquad f = d/L\,. \end{aligned}$$

(b) Calculate the time from the initial slip of the climber until the rope fully arrests the fall, i.e. the climber velocity reaches $v = 0$ for the first time.

Solution:
The time is the sum of free fall time t_f and forced pendulum time t_p: $t = t_f + t_p$. Free fall of distance $2d$:

$$2d = \frac{1}{2}gt_f^2 \Rightarrow t_f = 2\sqrt{d/g}\,.$$

We obtain a differential equation for the forced pendulum from Newton's 3rd law and the force acting on the climber:

$$\begin{aligned} F &= m\ddot{x} \\ mg - \frac{R}{L}x &= m\ddot{x} \\ \Rightarrow \ddot{x} + \omega^2 x &= g \quad \text{with}\,\omega^2 = \frac{R}{mL} = \frac{\kappa g}{L}\,. \end{aligned}$$

The general solution is the sum of the solution of the homogenoues ODE and a special solution, here simply $x_0 = \text{const} = g/\omega^2 = L/\kappa$:

$$\begin{aligned} x(t) &= a\sin\omega t + b\cos\omega t + L/\kappa \\ v(t) &= a\omega\cos\omega t - b\omega\sin\omega t\,. \end{aligned}$$

With initial conditions, $x(t_p = 0) = 0$ and $v(t_p = 0) = gt_f = 2\sqrt{dg}$, we obtain

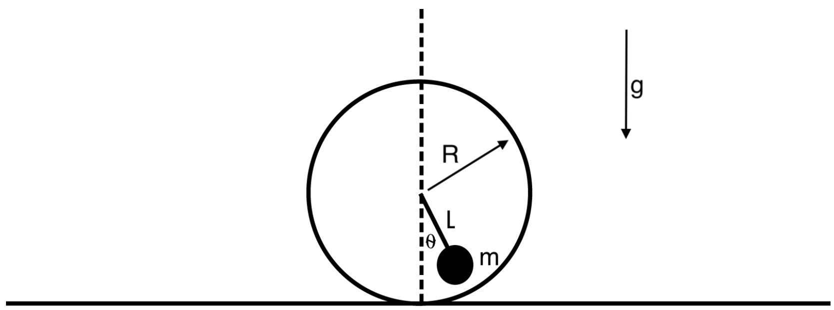
$$x(t) = \frac{L}{\kappa}\left(1 - \cos\omega t + \sqrt{4\kappa f}\sin\omega t\right)$$
$$v(t) = \sqrt{\frac{L}{g\kappa}}\left(\sin\omega t + \sqrt{4\kappa f}\cos\omega t\right) .$$

$v(t)$ becomes 0 for the first time when

$$\tan\omega t_p = -\sqrt{4\kappa f}$$

$$\Rightarrow t_p = \sqrt{\frac{L}{g\kappa}}\left(\pi - \arctan(\sqrt{4\kappa f})\right) .$$

A uniform disk, having mass M and radius R, rolls without slipping on a horizontal surface. A frictionless pendulum consisting of a point mass m and a massless rod of length L, $(L < R)$, swings from the center of the disk. The motion takes place in a vertical plane under the influence of a uniform gravitational field $\vec{g}$ (see figure).



(a) Find Lagrange's equations of motion for this system.

Solution:

$$L = T - U\,.$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}m\left[\left(\frac{d}{dt}(x + L\sin\theta)\right)^2 + \left(\frac{d}{dt}(L\cos\theta)\right)^2\right]\,.$$

With $\omega = \dot{x}/R$ (rolling) and $I = \frac{1}{2}MR^2$:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{4}M\dot{x}^2 + \frac{1}{2}m\left[\left(\dot{x} + L\dot{\theta}\cos\theta\right)^2 + \left(L\dot{\theta}\sin\theta\right)^2\right]\,.$$

$$U = -mgL\cos\theta\,,$$

and

$$L = \frac{3}{4}M\dot{x}^2 + \frac{m}{2}\left(\dot{x}^2 + 2L\dot{x}\dot{\theta}\cos\theta + L^2\dot{\theta}^2\right) + mgL\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}\left(\frac{3}{2}M\dot{x} + m\dot{x} + mL\dot{\theta}\cos\theta\right) = 0$$

$$\frac{3}{2}M\ddot{x} + m\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \frac{\partial L}{\partial \theta}$$
$$\frac{d}{dt}\left(mL\dot x\cos\theta + mL^2\dot\theta\right) = -mL\dot x\dot\theta\sin\theta - mgL\sin\theta$$
$$mL\ddot x\cos\theta - mL\dot x\dot\theta\sin\theta + mL^2\ddot\theta = -mL\dot x\dot\theta\sin\theta - mgL\sin\theta$$
$$\Rightarrow \ddot x\cos\theta + L\ddot\theta + g\sin\theta = 0$$

(b) Calculate the frequency of the pendulum for small oscillations of the pendulum: $\theta \ll 1$.

Solution:
Small angle approximation: $\theta \ll 1 \ \Rightarrow \cos\theta \approx 1$ and $\sin\theta \approx \theta$, also drop higher order terms in θ and $\dot\theta$:

$$\frac{3}{2}M\ddot x + m\ddot x + mL\ddot\theta = 0$$
$$\ddot x + L\ddot\theta + g\theta = 0$$

From first eqn:

$$\ddot x = \frac{-mL\ddot\theta}{\frac{3}{2}M + m}$$

insert into second eqn:

$$\left(\frac{-mL}{\frac{3}{2}M + m} + L\right)\ddot\theta + g\theta = 0$$
$$\ddot\theta + \left(\left(\frac{2}{3}\frac{m}{M} + 1\right)\frac{g}{L}\right)\theta = 0$$

and

$$\omega_0 = \sqrt{\left(\frac{2}{3}\frac{m}{M} + 1\right)\frac{g}{L}}$$