OSU Physics Department
Comprehensive Examination #125

Monday, March 28 & Tuesday, March 29

Spring 2016 Comprehensive Examination

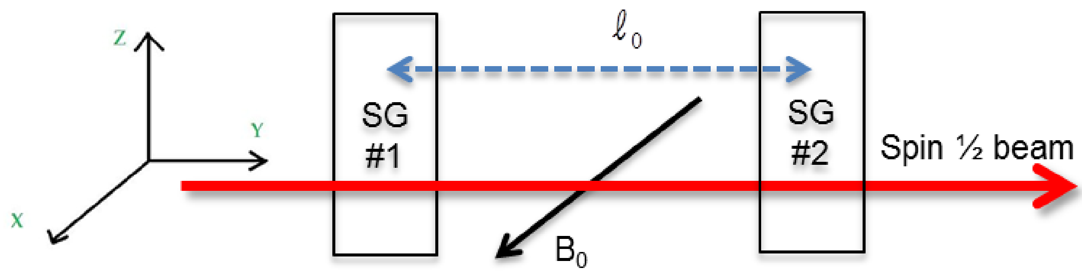
PART 1, 2, 3 & 4

General Instructions

This Spring 2016 Comprehensive Examination consists of four separate parts of two problems each. Each problem caries equal weight (20 points each). The first part (Quantum Mechanics) is handed out at 9:00 am on Monday, March 28 , and lasts three hours. The second part (Electricity and Magnetism) will be handed out at 1:00 pm on the same day and will also last three hours. The third (Statistical Mechanics) and fourth (Classical Mechanics) parts will be administered on Tuesday, March 29, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

If you submit blue books for any given section, that section will be graded as part of your cumulative score. Unless you are taking the exam for practice, all sections not previously passed need to be attempted and submitted.



A beam of neutral particles with spin $\frac{1}{2}$, and no other angular momentum, travels in the y-direction (horizontal in the figure above). The beam passes through two Stern-Gerlach (SG) magnets. A SG magnet is a device that allows particles to pass only if they have a particular spin state. In this case, both SG magnets allow passage of particles with spin projection up along the z-axis (vertical in the figure above).

Between the SG magnets, there is a uniform magnetic field B_0 in the x-direction (perpendicular to the propagation direction and perpendicular to the SG analyzer magnets).

(a) First explain qualitatively: what are the results of measurement after SG2 when there is no field at all between SG1 and SG2, and how does the application of the B field in the x-direction affect the results? What is the underlying physics?

Now be quantitative. The particles travel at speed v_o in the y-direction and the SG magnets are a distance ℓ_0 apart. The gyromagnetic ratio is γ so that the magnetic moment is $\vec{\mu} = \gamma \vec{S}$. The spin matrices are:

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}; S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}; S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}$$

(b) Find the energy eigenvalues and eigenstates in the region between the SG analyzers.

(c) Find the state of the particles (i) immediately after passing through SG 1 and (ii) immediately before passing through SG 2.

(d) Find a value of ℓ_0 (in terms of the other parameters given in statement of the problem) that results in a 25% probability that particles that have passed through SG 1 also pass though SG2.

Comprehensive Exam, Spring 2016 QM Graduate (Solution)

(a) With no field between the analyzers, all particles that pass through SG#1 pass through SG#2 (100% transmission barring non-ideal conditions such as scattering etc.) With the application of a field, the magnetic moment precesses about the field direction and therefore will not be aligned along $+z$ at all times. In QM language, the quantum state evolves in time and accesses different eigenstates of the Hamiltonian, or superpositions of eigenstates, as time progresses. Therefore just before SG#2, the system state will have some contribution from the spin up state w.r.t. z, but not necessarily 100% (it depends on the time taken to travel between the two magnets). Only some fraction of the particles will therefore go through SG#2.

(b) To establish notation, note that with spin matrices as given, the eigenvectors of S_z (the diagonal matrix) are $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, with eigenvalues $\frac{\hbar}{2}$; $-\frac{\hbar}{2}$, for "spin-up" and "spin down" in the z-direction, respectively. These are the eigenstates if the field is applied in the z-direction. In this basis, the S_x matrix is has off-diagonal elements and zeroes along the diagonal. The Hamiltonian in the region between the SG elements is:

$$H = -\vec{\mu} \cdot \vec{B} = -\gamma S_x B_0 = -\frac{\hbar \gamma B_0}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues and eigenstates of this Hamiltonian are:

$$E^+ = \frac{\hbar \gamma B_0}{2}; \quad |+\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad E^- = -\frac{\hbar \gamma B_0}{2}; \quad |-\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(c) The **initial state just after SG#1**, is a state that is 100% polarized in the z-direction – i.e. an eigenstate of the S_z operator. $|\varphi_{init}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv |+\rangle$

In terms of the eigenstates of the Hamiltonian described in (b), the initial state is

$$|\varphi_{init}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] = \frac{1}{\sqrt{2}} \left[|+\rangle_x + |-\rangle_x \right]$$

This state evolves in time

$$|\varphi(t)\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle_x e^{-iE^+ t/\hbar} + |-\rangle_x e^{-iE^- t/\hbar} \right] = \frac{1}{\sqrt{2}} \left[|+\rangle_x e^{-\frac{i\gamma B_0}{2} t} + |-\rangle_x e^{+i\frac{\gamma B_0}{2} t} \right]$$

The particles take time $t = \frac{\ell_0}{v_0}$ to traverse ℓ_0, **so just before SG#2,**

$$|\varphi(t)\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle_x e^{-\frac{i\gamma B_0 \ell_0}{2 v_0}} + |-\rangle_x e^{+i\frac{\gamma B_0 \ell_0}{2 v_0}} \right]$$

(d) After passage through SG#2, the component that is spin-up along z has been selected, so project along the $+z$-direction:

$$\langle + | \varphi(t) \rangle = \frac{1}{\sqrt{2}} \left[\underbrace{\langle + | + \rangle_x}_{1/\sqrt{2}} e^{-\frac{i\gamma B_0 \ell_0}{2v_0}} + \underbrace{\langle + | - \rangle_x}_{1/\sqrt{2}} e^{+i\frac{i\gamma B_0 \ell_0}{2v_0}} \right] = \frac{1}{2} \left[e^{-\frac{i\gamma B_0 \ell_0}{2v_0}} + e^{+\frac{i\gamma B_0 \ell_0}{2v_0}} \right] = \cos\left(\frac{\gamma B_0 \ell_0}{2v_0} \right)$$

Probability is square of projection and therefore need

$$\cos^2\left(\frac{\gamma B_0 \ell_0}{2v_0} \right) = 1/4 \Rightarrow \cos\left(\frac{\gamma B_0 \ell_0}{2v_0} \right) = 1/2 \Rightarrow \frac{\gamma B_0 \ell_0}{2v_0} = \frac{\pi}{3} \quad \text{so} \quad \boxed{\ell_0 = \frac{2v_0 \pi}{3\gamma B_0}}$$

The Zeeman effect is the shift in the energy of electron states of an atom as a consequence of the application of an applied external magnetic field. Use the hydrogen atom as an example and let the applied external field be $\vec{B}$. For an electron with orbital and spin angular momentum $\vec{L}$ and $\vec{S}$, the magnetic dipole moment is

$$\vec{\mu} = \frac{e\hbar}{2m}\left(\vec{L} + 2\vec{S}\right).$$

Assume a strong-field limit in which the Zeeman effect is small compared to the Coulomb energy of the hydrogen atom levels, but dominates all other corrections.

(a) In this strong-field limit, calculate the Zeeman correction to the H atom levels for the $1s$, $2p$ and $3d$ states as follows:
(i) Explicitly enumerate the relevant quantum numbers that identify the states in the appropriate basis.
(ii) Perform the calculation, explicitly addressing the energy degeneracy and its role in the calculation.
(iii) Sketch a graph of energy as a function of B that illustrates how the levels evolve and to what extent the degeneracy is lifted by the field.

(b) If the applied field is zero or weak, other important effects dominate the corrections to the Coulomb energy. List the most important effects with a very brief physical description. Make an order of magnitude estimate of the size of the applied magnetic field at which the Zeeman energy becomes important. How big is that field on a laboratory scale?

Possibly helpful numerical constants (2 significant digits)
electron mass: $m_e = 9.1 \times 10^{-31}$ kg
elementary charge: $e = 1.6 \times 10^{-19}$ C
Planck's constant: $h = 6.6 \times 10^{-34}\ J \cdot s$
speed of light: $c = 3.0 \times 10^{8}$ m/s

Comprehensive Exam, Spring 2016 QM (Solution)

(a) (i) The perturbation Hamiltonian involves the *z*-components of angular momentum (see (ii)) so the appropriate basis is $\left|n,\ell,m_\ell,s,m_s\right\rangle$ which specifies those via the *m* quantum numbers. Suppress *s* because *s* = ½ always -> $\left|n,\ell,m_\ell,m_s\right\rangle$. (Do not introduce proton spin because the hyperfine effects are too small.)

1s: $\left|1,0,0,\tfrac{1}{2}\right\rangle;\left|1,0,0,\tfrac{-1}{2}\right\rangle$ degeneracy 2

2p: $\left|2,1,1,\tfrac{1}{2}\right\rangle;\left|2,1,1,\tfrac{-1}{2}\right\rangle;\left|2,1,0,\tfrac{1}{2}\right\rangle;\left|2,1,0,\tfrac{-1}{2}\right\rangle;\left|2,1,-1,\tfrac{1}{2}\right\rangle;\left|2,1,-1,\tfrac{-1}{2}\right\rangle$ degeneracy 6

3d: $\begin{matrix}\left|3,2,2,\tfrac{1}{2}\right\rangle;\left|3,2,2,\tfrac{-1}{2}\right\rangle;\left|3,2,1,\tfrac{1}{2}\right\rangle;\left|3,2,1,\tfrac{-1}{2}\right\rangle; \\ \left|3,2,0,\tfrac{1}{2}\right\rangle;\left|3,2,0,\tfrac{-1}{2}\right\rangle;\left|3,2,-1,\tfrac{1}{2}\right\rangle;\left|3,1,-1,\tfrac{-1}{2}\right\rangle;\left|3,2,-2,\tfrac{1}{2}\right\rangle;\left|3,2,-2,\tfrac{-1}{2}\right\rangle\end{matrix}$ degeneracy 10

(a) (ii) The energy of interaction of a magnetic dipole $\vec{\mu}$ with a magnetic field $\vec{B}$ is $H'=-\vec{\mu}\cdot\vec{B}$. Choose the field in the *z*-direction so that

$$H'=-\vec{\mu}\cdot\vec{B}=-\frac{e\hbar B}{2m}\left(L_z+2S_z\right).$$

The Zeeman perturbation is *diagonal* in the basis enumerated above and

$$H'_{ij}=-\mu_B B\left\langle n',\ell,m_\ell',m_s'\right|\left(L_z+2S_z\right)\left|n,\ell,m_\ell,m_s\right\rangle$$

$$=-\mu_B B\left(m_\ell+2m_s\right)\hbar\delta_{nn'}\delta_{\ell\ell'}\delta_{\ell\ell'}\delta_{m_\ell m_\ell'}\delta_{m_s m_s'}$$

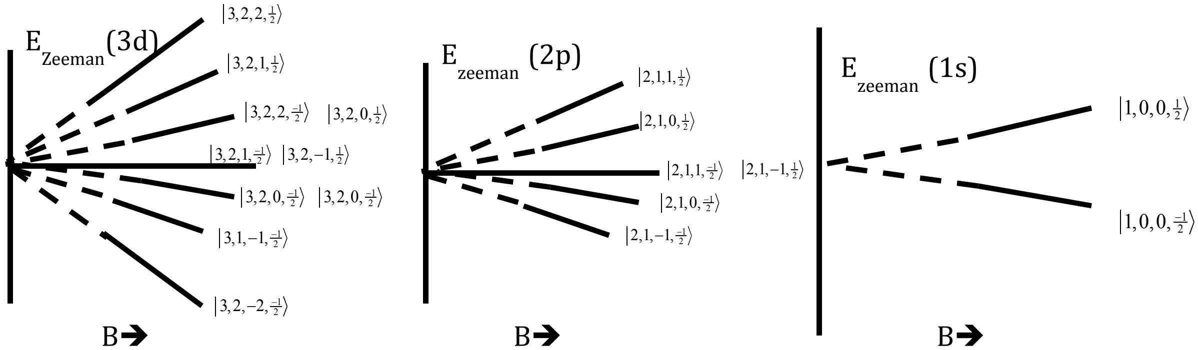
Here *i,j* are notational simplifications to represent the set of 4 quantum numbers, and $\delta_{nn'}$ is the Kronecker delta, 1 if the subscripts are the same, and zero otherwise.

Because the perturbation is diagonal in this basis, at least in each degenerate subspace, it is OK to read the corrections to the Coulomb energy from the diagonal elements of the matrix.
$E_{Zeeman}=-\mu_B\hbar B\left(m_\ell+2m_s\right)$ independent of *n* and *l* (expect insofar as they determine the allowed values of m_l).
For each state, the energy changes linearly with *B*, with slope determined by $m_\ell+2m_s$

1s	$m_\ell+2m_s$	2p	$m_\ell+2m_s$	3d	$m_\ell+2m_s$
				$\left|3,2,2,\tfrac{1}{2}\right\rangle$	3
				$\left|3,2,2,\tfrac{-1}{2}\right\rangle$	1
		$\left|2,1,1,\tfrac{1}{2}\right\rangle$	2	$\left|3,2,1,\tfrac{1}{2}\right\rangle$	2
		$\left|2,1,1,\tfrac{-1}{2}\right\rangle$	0	$\left|3,2,1,\tfrac{-1}{2}\right\rangle$	0
$\left|1,0,0,\tfrac{1}{2}\right\rangle$	1	$\left|2,1,0,\tfrac{1}{2}\right\rangle$	1	$\left|3,2,0,\tfrac{1}{2}\right\rangle$	1
$\left|1,0,0,\tfrac{-1}{2}\right\rangle$	-1	$\left|2,1,0,\tfrac{-1}{2}\right\rangle$	-1	$\left|3,2,0,\tfrac{-1}{2}\right\rangle$	-1
		$\left|2,1,-1,\tfrac{1}{2}\right\rangle$	0	$\left|3,2,-1,\tfrac{1}{2}\right\rangle$	0
		$\left|2,1,-1,\tfrac{-1}{2}\right\rangle$	-2	$\left|3,1,-1,\tfrac{-1}{2}\right\rangle$	-2
				$\left|3,2,-2,\tfrac{1}{2}\right\rangle$	-1
				$\left|3,2,-2,\tfrac{-1}{2}\right\rangle$	-3

GRAPHS (for large enough field, as shown by solid lines):

(a) (iii) 3d:



(b) The Coulomb energy of the H atom levels is $E_n = -\frac{1}{2n^2}\alpha^2 mc^2 = -\frac{13.6}{n^2}$ eV

Corrections come at the level of the fine structure,

(i) **spin-orbit interaction** :the response of the electron spin magnetic moment to the magnetic field of the proton caused by the orbital motion,

(ii) **relativistic correction** - the kinetic energy of the electron is about $0.01c$,

(iii) **Darwin** term, a quantum electrodynamic effect that applies only to *s* states.

These fine structure corrections are of order $\alpha^2 E_n = -\alpha^2 \frac{13.6}{n^2}$ eV.

Hyperfine corrections (proton-spin/electron spin interactions) and the Lamb shift (QED effect) are all much smaller than the fine structure.

A conservative (high-side) estimate of the fine structure corrections is

$$E_{fs} \approx \frac{13.6}{137 \times 137}\text{ eV} \approx \frac{1}{1370}\text{ eV} \approx 0.0007\text{ eV}$$

If the Zeeman effect is to be larger than the fine structure corrections, the order of magnitude must be $E_Z \approx \vec{\mu}\cdot\vec{B} \approx \mu_B B \approx \frac{e\hbar}{2m}B = 0.0007$ eV

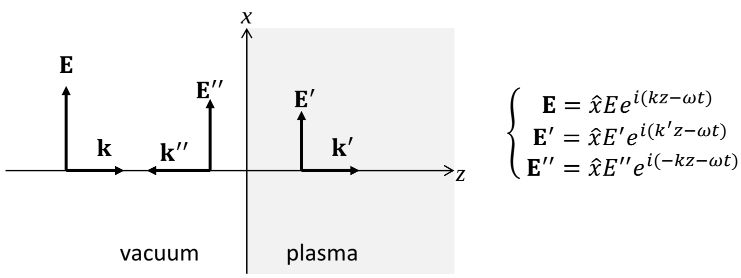
From this we deduce $B \approx 0.0007\,\frac{2m}{e\hbar}\text{eV} = 0.0007\,\frac{\pi mc^2}{ehc^2}\text{eV} = \pi\frac{0.5MeV}{hc^2}V \approx 8T$

The earth's field is about 10^{-4} T or 1 Gauss. A typical lab-scale field is 1-10 T. An electromagnet is about 2T, a commercial superconducting magnet can be up to 10-12 T. In special cases, the high field magnet lab, there are special magnets that get to 40T.

A linearly polarized monochromatic wave $\mathbf{E}(z,t) = \hat{x}Ee^{i(kz-\omega t)}$ is normally incident on a plane interface between vacuum and a plasma as shown in the figure below. The dispersion relation of the plasma is given as

$$\omega^2 = \omega_p^2 + c^2 k'^2,$$

where ω_p is the plasma frequency, c is the speed of light in vacuum, and k' is the wavenumber inside the plasma. The light frequency ω is lower than the plasma frequency ($\omega < \omega_p$).



(a) What is the refractive index of the plasma, $n(\omega)$?

(b) Find the transmitted and reflected electric field amplitudes E' and E'' at the boundary in terms of the incident field E and the plasma refractive index n.

(c) The time-averaged energy flux is expressed as the real part of the complex Poynting vector

$$\mathbf{S} = \frac{1}{2}\Re\{\mathbf{E}\times\mathbf{H}^*\} = \frac{1}{2\mu_0}\Re\{\mathbf{E}\times\mathbf{B}^*\}.$$

Calculate the time-averaged energy flux of the incident, transmitted, and reflected waves, $\mathbf{S}_i$, $\mathbf{S}_t$, and $\mathbf{S}_r$. What are the reflectance R and transmittance T?

(d) The transmitted wave exponentially decays in the plasma. The decaying electric field can be expressed as

$$\mathbf{E}'(z,t) = \hat{x}E'e^{(k'z-\omega t)} = \hat{x}|E'|e^{i\phi}e^{-z/\delta}e^{i(k_r z-\omega t)}$$

Determine the field amplitude $|E'|$, the phase ϕ, the skin depth δ, and the wave number k_r in terms of E, c, ω and ω_p.

A linearly polarized monochromatic wave $\mathbf{E}(z,t) = \hat{x}Ee^{i(kz-\omega t)}$ is normally incident on a plane interface between vacuum and a plasma as shown in the figure below. The dispersion relation of the plasma is given as

$$\omega^2 = \omega_p^2 + c^2k'^2,$$

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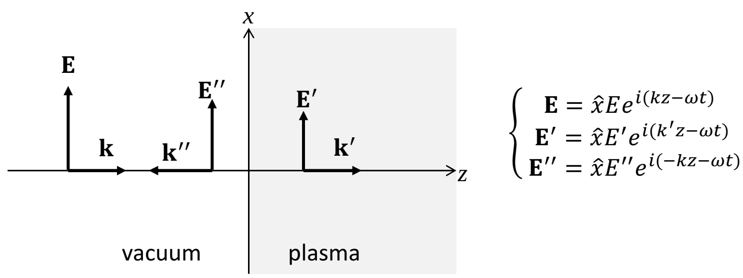


(a) What is the refractive index of the plasma, $n(\omega)$?

Solution:

Modifying the dispersion relation, we obtain

$$\omega^2 = \omega_p^2 + c^2k'^2 \Rightarrow k'^2 = \frac{1}{c^2}(\omega^2 - \omega_p^2) \Rightarrow k' = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}} = n(\omega)\frac{\omega}{c}$$

Therefore, the refractive index of the plasma is

$$n(\omega) = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \tag{1}$$

Since $n(\omega)$ is pure imaginary for $\omega < \omega_p$, we can write

$$n(\omega) = i\alpha(\omega), \text{where } \alpha(\omega) = \sqrt{\frac{\omega_p^2}{\omega^2} - 1} \text{ and } \alpha(\omega) > 0 \tag{2}$$

(b) Find the transmitted and reflected electric field amplitudes E' and E'' at the boundary in terms of the incident field E and the plasma refractive index n.

Solution:
We use the boundary condition that the electric and magnetic fields are continuous at the interface. From the Maxwell's equation, we get

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \Rightarrow i\mathbf{k} \times \mathbf{E} - i\omega\mathbf{B} = 0 \Rightarrow \mathbf{B} = \frac{1}{c}\mathbf{k} \times \mathbf{E}$$

for a monochromatic plane wave $\mathbf{E}(\mathbf{x},\mathbf{t}) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Therefore, the electric and magnetic fields at the boundary are expressed as

$$\mathbf{E} = E\hat{x} \qquad \mathbf{B} = B\hat{y} = \tfrac{E}{c}\hat{y} \tag{3}$$
$$\mathbf{E}' = E'\hat{x} \qquad \mathbf{B}' = B'\hat{y} = \tfrac{nE'}{c}\hat{y} \tag{4}$$
$$\mathbf{E}'' = E''\hat{x} \quad \mathbf{B}'' = -B''\hat{y} = -\tfrac{E''}{c}\hat{y} \tag{5}$$

Applying the boundary conditions, we get

$$E + E'' = E' \tag{6}$$

and

$$B - B'' = B' \to E - E'' = nE' \tag{7}$$

From Eq.(6) and (7), we obtain

$$E' = \frac{2}{1+n}E = \frac{2}{1+i\alpha}E \tag{8}$$
$$E'' = \frac{1-n}{1+n} = \frac{1-i\alpha}{1+i\alpha}E \tag{9}$$

(c) The time-averaged energy flux is expressed as the real part of the complex Poynting vector

$$\mathbf{S} = \frac{1}{2}\Re\left\{\mathbf{E}\times\mathbf{H}^*\right\} = \frac{1}{2\mu_0}\Re\left\{\mathbf{E}\times\mathbf{B}^*\right\}.$$

Calculate the time-averaged energy flux of the incident, transmitted, and reflected waves, $\mathbf{S}_i$, $\mathbf{S}_t$, and $\mathbf{S}_r$. What are the reflectance R and transmittance T?

Solution:

From Eqs. (3)-(6),

$$\mathbf{H} = \frac{E}{c\mu_0}\hat{y} = c\varepsilon_0 E\hat{y} \tag{10}$$
$$\mathbf{H}' = \frac{nE'}{c\mu_0}\hat{y} = ic\alpha\varepsilon_0 E'\hat{y} \tag{11}$$
$$\mathbf{H}'' = -c\varepsilon_0 E''\hat{y} \tag{12}$$

Therefore, the time-averaged energy fluxes are

$$\mathbf{S}_i = \frac{1}{2}\Re\left\{\mathbf{E}\times\mathbf{H}^*\right\} = \frac{1}{2}c\varepsilon_0|E|^2\hat{z} \tag{13}$$
$$\mathbf{S}_t = \frac{1}{2}\Re\left\{ic\alpha\varepsilon_0|E|^2\hat{z}\right\} = 0 \tag{14}$$
$$\mathbf{S}_r = -\frac{1}{2}c\varepsilon_0|E''|^2\hat{z} = -\frac{1}{2}c\varepsilon_0\left|\frac{1-i\alpha}{1+i\alpha}\right|^2|E|^2\hat{z} = -\frac{1}{2}c\varepsilon_0|E|^2\hat{z} = \mathbf{S}_i \tag{15}$$

Inside the plasma, $\mathbf{S}_t = 0$, because the EM wave forms a standing wave in which $H' = ic\alpha\varepsilon_0 E'$, i.e., $\mathbf{E}'$ and $\mathbf{B}'$ has the $\pi/2$ phase difference. In this case, the time average of the Poyinting vector vanishes because $\langle\cos(\omega t)\cos(\omega t + \pi/2)\rangle = \langle\cos(\omega t)\sin(\omega t)\rangle = 0$.

The reflectance and transmittance are

$$R = \frac{|\mathbf{S}_r|}{|\mathbf{S}_i|} = 1 \tag{16}$$

$$T = \frac{|\mathbf{S}_t|}{|\mathbf{S}_i|} = 0 \tag{17}$$

(d) The transmitted wave exponentially decays in the plasma. The decaying electric field can be expressed as

$$\mathbf{E}'(z,t) = \hat{x}E'e^{(k'z-\omega t)} = \hat{x}|E'|e^{i\phi}e^{-z/\delta}e^{i(k_r z-\omega t)}$$

Determine the field amplitude $|E'|$, the phase ϕ, the skin depth δ, and the wave number k_r in terms of E, c, ω and ω_p.

Solution:

From Eq.(8), we get

$$E' = \frac{2}{1+i\alpha}E \Rightarrow |E'| = \left|\frac{2}{1+i\alpha}\right||E| = \frac{2}{\sqrt{1+\alpha^2}}|E| \tag{18}$$

and

$$\frac{2}{1+i\alpha} = \frac{2}{1+\alpha^2}(1-i\alpha) \Rightarrow \tan\phi = -\alpha \Rightarrow \phi = -\tan^{-1}\alpha \tag{19}$$

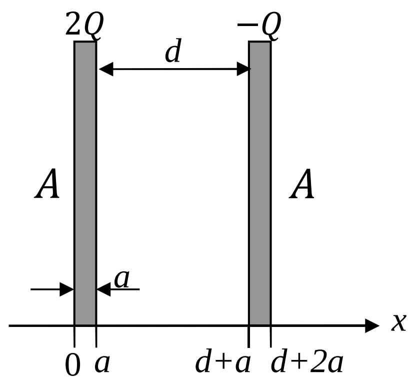
Since $e^{ik'z} = e^{-\alpha\frac{\omega}{c}z}$,

$$\delta = \frac{\omega}{c\alpha} = \frac{c}{\sqrt{\omega_p^2 - \omega^2}} \tag{20}$$

Because $k' = i\alpha\frac{\omega}{c}$ is pure imaginary,

$$k_r = 0 \tag{21}$$

Two parallel conducting plates of area A and thickness a are separated by a distance d as shown in the figure below. They carry charges, $2Q$ and $-Q$, respectively. Assume that the plates are infinitely large, i.e., $A \gg d^2, a^2$.



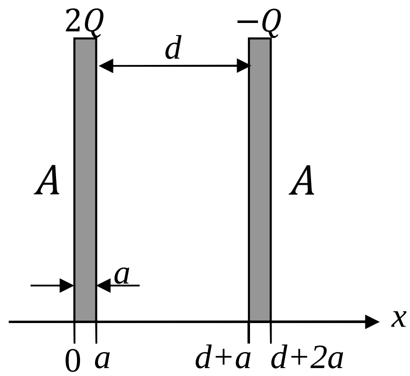
(a) What is the electric field inside the conductors for $0 < x < a$ and $d + a < x < d + 2a$?

(b) Determine the surface charge density σ on the plates at $x = 0$, a, $d + a$, and $d + 2a$. Assume that each surface carries a uniform charge density.

(c) Find the electric field $\mathbf{E}$ in the regions of (i) $x < 0$, (ii) $a < x < d + a$, and (iii) $x > d + 2a$.

(d) Find the electric potential $V(x)$ when the left conducting plate is grounded. Sketch $V(x)$ as a function of x.

Two parallel conducting plates of area A and thickness a are separated by a distance d as shown in the figure below. They carry charges, $2Q$ and $-Q$, respectively. Assume that the plates are infinitely large, i.e., $A \gg d^2, a^2$.

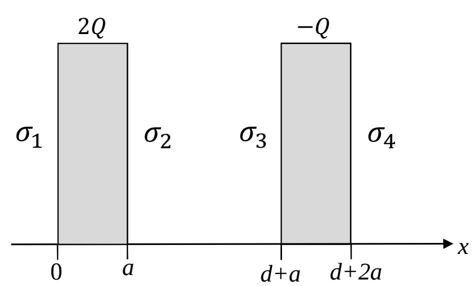


(a) What is the electric field inside the conductors for $0 < x < a$ and $d + a < x < d + 2a$?

Solution:
$\mathbf{E} = 0$ inside conductors.

(b) Determine the surface charge density σ on the plates at $x = 0$, a, $d + a$, and $d + 2a$. Assume that each surface carries a uniform charge density.

Solution:



On the first and second plates,

$$\sigma_1 + \sigma_2 = \frac{2Q}{A} \tag{22}$$

$$\sigma_3 + \sigma_4 = -\frac{Q}{A} \tag{23}$$

Using Gauss's law, we obtain the electric field of an infinite plane carrying a uniform surface charge σ,

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{n},$$

where $\hat{n}$ is a surface normal unit vector pointing away from the surface.

Because the electric field vanishes inside the first plate ($0 < x < a$),

$$\begin{aligned} \mathbf{E} &= \frac{\sigma_1}{2\varepsilon_0}\hat{x} - \frac{\sigma_2}{2\varepsilon_0}\hat{x} - \frac{\sigma_3}{2\varepsilon_0}\hat{x} - \frac{\sigma_4}{2\varepsilon_0}\hat{x} = 0 \\ \Rightarrow \sigma_1 &= \sigma_2 + \sigma_3 + \sigma_4 \end{aligned} \tag{24}$$

Similarly, because $\mathbf{E} = 0$ for $d + a < x < d + 2a$,

$$\sigma_1 + \sigma_2 + \sigma_3 = \sigma_4 \tag{25}$$

Subtracting Eq.(4) from Eq.(3),

$$\sigma_2 + \sigma_3 = 0 \tag{26}$$

From Eq.(3) and Eq.(5),

$$\sigma_1 = \sigma_4 \tag{27}$$

Since $\sigma_4 = \sigma_1$ and $\sigma_3 = -\sigma_2$, Eq.(2) becomes

$$\sigma_1 - \sigma_2 = -\frac{Q}{A} \tag{28}$$

From Eq.(1) and Eq.(7),

$$\sigma_1 = \sigma_4 = \frac{Q}{2A} \tag{29}$$

and

$$\sigma_2 = -\sigma_3 = \frac{3Q}{2A} \tag{30}$$

(c) Find the electric field $\mathbf{E}$ in the regions of (i) $x < 0$, (ii) $a < x < d + a$, and (iii) $x > d + 2a$.

Solution:

Near a conducting surface,

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{n},$$

where $\hat{n}$ is a surface normal unit vector pointing away from the surface.

(i) For $x < 0$,

$$\mathbf{E} = -\frac{\sigma_1}{\varepsilon_0}\hat{x} = -\frac{Q}{2\varepsilon_0 A}\hat{x}$$

(i) For $0 < x < d + a$,

$$\mathbf{E} = \frac{\sigma_2}{\varepsilon_0}\hat{x} = -\frac{\sigma_3}{\varepsilon_0}\hat{x} = \frac{3Q}{2\varepsilon_0 A}\hat{x}$$

(i) For $x > d + a$,

$$\mathbf{E} = \frac{\sigma_4}{\varepsilon_0}\hat{x} = \frac{Q}{2\varepsilon_0 A}\hat{x}$$

(d) Find the electric potential $V(x)$ when the left conducting plate is grounded. Sketch $V(x)$ as a function of x.

Solution:

The electric potential is

$$V(x) = -\int_{x_0}^{x} E(x)dx + V(x_0),$$

where $V(0) = V(a) = 0$ because the left conducting plate is grounded.

(i) For $x < 0$,

$$V(x) = -\int_0^x \left(\frac{Q}{2\varepsilon_0 A}\right) dx = \frac{Q}{2\varepsilon_0 A}x$$

(ii) For $0 < x < a$, $V(x) = 0$.

(iii) For $a < x < a + d$,

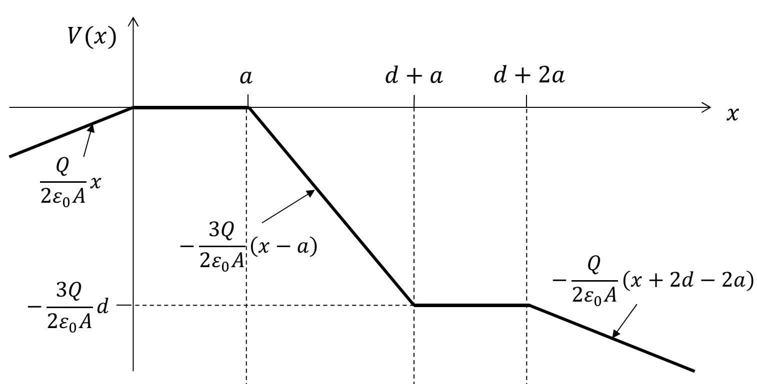
$$V(x) = -\int_a^x \left(\frac{3Q}{2\varepsilon_0 A}\right) dx + V(a) = -\frac{3Q}{2\varepsilon_0 A}(x - a)$$

(iv) For $d + a < x < d + 2a$,

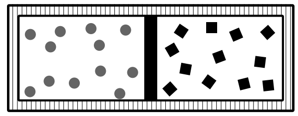
$$V(x) = V(d + a) = -\frac{3Q}{2\varepsilon_0 A}d$$

(iv) For $x > d + 2a$,

$$V(x) = -\int_{d+2a}^{x} \left(\frac{3Q}{2\varepsilon_0 A}\right) dx + V(d+2a) = -\frac{Q}{2\varepsilon_0 A}(x-d-2a) - \frac{3Q}{2\varepsilon_0 A}d = -\frac{Q}{2\varepsilon_0 A}(x+2d-2a)$$



Consider a thermally insulated box with volume $2V_0$ containing two distinct monatomic ideal gasses, separated by an impermeable barrier, as illustrated below.



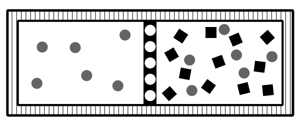
On the left of the partition are N_0 atoms of a circular atom type, stored in volume V_0 at temperature T_0. On the right side of the partition are N_0 atoms of a square atom type, which occupy volume V_0 and are in thermal equilibrium with the circular atoms on the left side of the barrier.

For this problem you may need the following equations of state defining the behavior of a mixture of two monatomic ideal gasses:

$$U = \frac{3}{2} N k_B T$$
$$p = \frac{N k_B T}{V}$$

where N is the total number of atoms (squares plus circles) in a given volume.

(a) **(3 pts)** The barrier between the two sides of the box is now made permeable to the circular atoms only, while remaining impermeable to the square atoms on the right side.

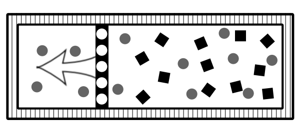


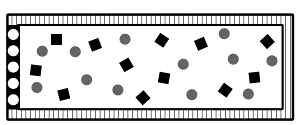
When the box has reached equilibrium, what is the pressure in each side of the box?

(b) **(3 pts)** During this process, did the entropy of the system increase, decrease or remain the same? Explain your answer.

(c) **(3 pts)** During this process, did the temperature of the system increase, decrease or remain the same? Explain your answer.

(d) **(3 pts)** Now we will slowly move the permeable partition to the left side of the box, until it reaches the left-hand wall, at which point there will be only one enclosure with volume equal to $2V_0$.

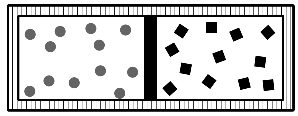




Consider the change in entropy of the system (enclosed by the box) due to moving the permeable membrane to its edge. Is this change positive, negative or zero? Explain your answer.

(e) **(8 pts)** What is the final temperature of the system?

Consider a thermally insulated box with volume $2V_0$ containing two distinct monatomic ideal gasses, separated by an impermeable barrier, as illustrated below.



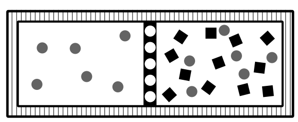
On the left of the partition are N_0 atoms of a circular atom type, stored in volume V_0 at temperature T_0. On the right side of the partition are N_0 atoms of a square atom type, which occupy volume V_0 and are in thermal equilibrium with the circular atoms on the left side of the barrier.

For this problem you may need the following equations of state defining the behavior of a mixture of two monatomic ideal gasses:

$$U = \frac{3}{2}Nk_BT$$
$$p = \frac{Nk_BT}{V}$$

where N is the total number of atoms (squares plus circles) in a given volume.

(a) **(3 pts)** The barrier between the two sides of the box is now made permeable to the circular atoms only, while remaining impermeable to the square atoms on the right side.



When the box has reached equilibrium, what is the pressure in each side of the box?

Solution:
The circles will equally fill each box, so we will end up with N_0 squares and $N_0/2$ circles on the right, and $N_0/2$ circles on the left.

To find the pressure on each side, we will also need to know the temperature. There is no work done during this process, and nor does heat leave the system, so the internal energy must be a constant. Since the number N is also a constant, based on our formula for U we can conclude that T is constant. Thus

$$p_L = \frac{1}{2}\frac{N_0kT_0}{V_0}$$
$$p_R = \frac{3}{2}\frac{N_0kT_0}{V_0}$$

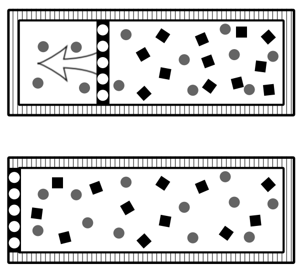
(b) **(3 pts)** During this process, did the entropy of the system increase, decrease or remain the same? Explain your answer.

Solution:
This is a spontaneous process, so the entropy of system plus surroundings increased. Since the system is isolated, the surroundings didn't change, and thus the entropy of the system increased.

(c) **(3 pts)** During this process, did the temperature of the system increase, decrease or remain the same? Explain your answer.

Solution:
(We did this earlier, but I will repeat it here for clarity.) There is no work done during this process, and nor does heat leave the system, so the internal energy must be a constant. Since the number N is also a constant, based on our formula for U we can conclude that T is constant (i.e. $T = T_0$ on both sides of the box).

(d) **(3 pts)** Now we will slowly move the permeable partition to the left side of the box, until it reaches the left-hand wall, at which point there will be only one enclosure with volume equal to $2V_0$.



Consider the change in entropy of the system (enclosed by the box) due to moving the permeable membrane to its edge. Is this change positive, negative or zero? Explain your answer.

Solution:
The entropy of the system does not change during this process, because it is slow and reversible. This means the entropy of system plus surroundings does not change (by Second Law), and since the surroundings does not change, the entropy of the system must also not change.

(e) **(8 pts)** What is the final temperature of the system?

Solution:
To find the change in temperature requires us to think about energy conservation. The square gas is clearly going to work as it expands, and that energy is going to have to come from its internal energy, since there is no heat flow from the system (assuming we view the system as the entire combined system). Let's look at the change in internal energy two ways, first as the work done,

and secondly as relates to temperature given the formula for internal energy.

$$dU = -pdV \tag{31}$$
$$dU = \frac{3}{2}N_{\text{tot}}kdT \tag{32}$$
$$= 3N_0kdT \tag{33}$$

At the last step, we noted that the total number of atoms is twice N_0. The two gases are in thermal contact, so both will be at the same temperature during this slow process. We can now set these expressions for dU equal, and then apply the ideal gas law to the square gass (since the difference in pressure is what is needed for the work done, and that is entirely due to the square gas).

$$-p_{\text{square}}dV = 3N_0kdT \tag{34}$$
$$-\frac{N_0kT}{V}dV = 3N_0kdT \tag{35}$$
$$-\frac{dV}{V}dV = 3\frac{dT}{T} \tag{36}$$

Now that we know how much the temperature changes for an infinitesimal change in volume, we just need to integrate to find the final temperature.

$$-\int_{V_0}^{2V_0}\frac{dV}{V} = 3\int_{T_0}^{T_f}\frac{dT}{T} \tag{37}$$
$$-\ln\left(\frac{2V_0}{V_0}\right) = 3\ln\left(\frac{T_f}{T_0}\right) \tag{38}$$
$$\ln\left(\frac{1}{2}\right) = \ln\left[\left(\frac{T_f}{T_0}\right)^3\right] \tag{39}$$
$$\frac{1}{2} = \left(\frac{T_f}{T_0}\right)^3 \tag{40}$$
$$T_f = 2^{-\frac{1}{3}}T_0 \tag{41}$$

So the temperature drops by a factor of $2^{-\frac{1}{3}}$. Note that blind application of pV^γ (because it is an adiabatic expansion) fails. This is because the square gas is in thermal contact with the circular gas, so the square gas treated as a separate system is *not* undergoing adiabatic expansion. Fortunately, the correct approach to this problem is identical to that for the ordinary adiabatic expansion, so if you understand one, you should understand the other.

Consider a system of N rigid rotors at temperature T. The energy eigenvalues of the rigid rotor are given by the following expression:

$$E_{lm} = \frac{\hbar^2}{2I}\ell(\ell+1)$$
$$\ell = 0, 1, 2, \cdots$$
$$m = -\ell, \cdots, 0, \cdots, \ell$$

where I is the moment of inertia, and ℓ and m are the usual angular momentum quantum numbers.

(a) **(6 pts)** Find an expression for the internal energy of this system, consisting of N rigid rotors at temperature T.

(b) **(8 pts)** Find the internal energy in the high temperature limit ($kT \gg \hbar^2/2I$). You can get two points by using physical arguments to reach the correct limit without finding the limit using math.

(c) **(6 pts)** Find the low-temperature limit ($kT \ll \hbar^2/2I$) of the internal energy, keeping some temperature dependence (i.e. an expression that is independent of T is not acceptable).

Consider a system of N rigid rotors at temperature T. The energy eigenvalues of the rigid rotor are given by the following expression:

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where I is the moment of inertia, and ℓ and m are the usual angular momentum quantum numbers.

(a) **(6 pts)** Find an expression for the internal energy of this system, consisting of N rigid rotors at temperature T.

Solution:
The internal energy is given in general by

$$U = N\frac{\sum_i^{\text{all states}} E_i e^{-\beta E_i}}{\sum_i^{\text{all states}} e^{-\beta E_i}} \tag{42}$$

which is just the Boltzmann-weighted average of the energy for a single rotor, multiplied by the total number of rotors N. In our case this comes out to:

$$U = N\frac{\sum_i^{\text{all states}} E_i e^{-\beta E_i}}{\sum_i^{\text{all states}} e^{-\beta E_i}} \tag{43}$$

$$= N\frac{\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} E_\ell e^{-\beta E_\ell}}{\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\beta E_\ell}} \tag{44}$$

$$= N\frac{\hbar^2}{2I}\frac{\sum_{\ell=0}^{\infty}(2\ell+1)(\ell^2+\ell)e^{-\frac{\hbar^2(\ell^2+\ell)}{2IiT}}}{\sum_{\ell=0}^{\infty}(2\ell+1)e^{-\frac{\hbar^2(\ell^2+\ell)}{2IkT}}} \tag{45}$$

This last expression is as simple as we can get it without making any additional assumptions.

(b) **(8 pts)** Find the internal energy in the high temperature limit ($kT \gg \hbar^2/2I$). You can get two points by using physical arguments to reach the correct limit without finding the limit using math.

Solution:
In the high-temperature limit $\frac{\hbar^2}{2IkT} \ll 1$, which means that the dimensionless quantity $\frac{\hbar^2}{2IkT}(\ell^2+\ell)$ changes only by a small amount when ℓ changes by 1, so we can approximate each sum using an integral.

Let's start with the partition function:

$$Z = \sum_{\ell=0}^{\infty}(2\ell+1)e^{-\frac{\hbar^2(\ell^2+\ell)}{2IkT}} \tag{46}$$

$$\approx \int_0^{\infty} d\ell(2\ell+1)e^{-\frac{\hbar^2(\ell^2+\ell)}{2IkT}} \tag{47}$$

We can do a u substitution with $u = (\ell^2 + \ell)\hbar^2/(2IkT)$.

$$Z \approx \frac{2IkT}{\hbar^2} \int_0^\infty e^{-u} du \tag{48}$$

$$= \frac{2IkT}{\hbar^2} \tag{49}$$

$$= \frac{2I}{\hbar^2 \beta} \tag{50}$$

Now we need to do the numerator. Fortunately, there is an extra-nice trick for doing that, which is to recognize that

$$\sum_i^{\text{all states}} E_i e^{-\beta E_i} = -\frac{dZ}{d\beta} \tag{51}$$

$$\approx \frac{2I}{\hbar^2 \beta^2} \tag{52}$$

$$= \frac{2IkT}{\hbar^2} kT \tag{53}$$

Thus we find that

$$U \approx N \frac{\frac{2IkT}{\hbar^2} kT}{\frac{2IkT}{\hbar^2}} \tag{54}$$

$$= NkT \tag{55}$$

The short way of getting to this answer is to use equipartition, and recognize that classically there are two quadratic degrees of freedom, which are the kinetic energy for rotation in two directions. However, the problem asked you to *show* this, and you only get partial credit for using equipartition.

(c) **(6 pts)** Find the low-temperature limit ($kT \ll \hbar^2/2I$) of the internal energy, keeping some temperature dependence (i.e. an expression that is independent of T is not acceptable).

Solution:
The low-temperature limit is easier than the high-temperature limit. In this case, the dimensionless quantity in the exponential is *large* rather than *small.* This means that each term in the sum is way smaller than the last, so we can approximate the energy by just keeping the first few terms.

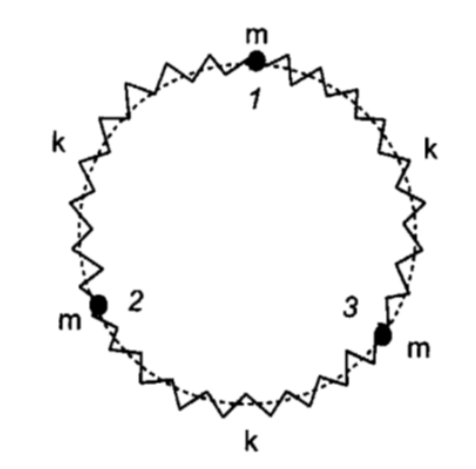
$$U = N \frac{\hbar^2}{2I} \frac{\sum_{\ell=0}^\infty (2\ell + 1)(\ell^2 + \ell) e^{-\frac{\hbar^2(\ell^2+\ell)}{2IiT}}}{\sum_{\ell=0}^\infty (2\ell+1) e^{-\frac{\hbar^2(\ell^2+\ell)}{2IkT}}} \tag{56}$$

$$\approx N \frac{\hbar^2}{2I} \frac{0 + 3 \cdot 2\, e^{-\frac{2\hbar^2}{2IkT}}}{1 + 2\, e^{-\frac{2\hbar^2}{2IkT}}} \tag{57}$$

$$\approx N \frac{\hbar^2}{2I} 6\, e^{-\frac{\hbar^2}{IkT}} \tag{58}$$

At low temperatures the internal energy is exponential in β, which is what we expect in any system with an energy gap between the ground state and first excited state.

Three coupled masses on a circle.
Three identical point masses m are constrained to move on a circle, as shown in the figure below. The masses are connected with identical springs each with spring constant, k that obey Hooke's Law on the same circle. There is no friction, gravity or motion outside the circle.

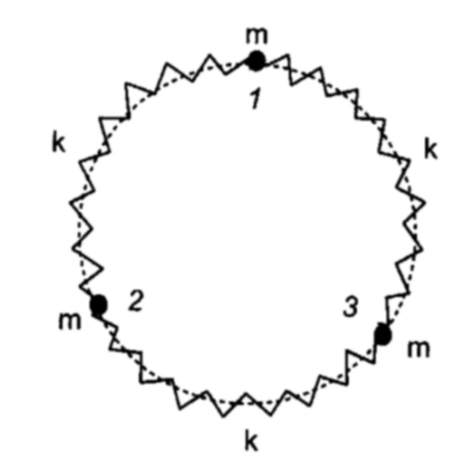


(a) Find all the natural or characteristic frequencies of oscillation for this system.

(b) Solve for the normal modes and describe the motion of the masses for each mode (with words or a rough sketch).

(c) At $t = 0$ masses at position 1 was found to be displaced by a distance δ from its equilibrium position (i.e. when all springs are unstretched as shown above). The velocities of all three masses at $t = 0$ were zero. Find the resulting equations of motion for each of the three masses as a function of time.

Three coupled masses on a circle.
Three identical point masses m are constrained to move on a circle, as shown in the figure below. The masses are connected with identical springs each with spring constant, k that obey Hooke's Law on the same circle. There is no friction, gravity or motion outside the circle.



(a) Find all the natural or characteristic frequencies of oscillation for this system.

Solution:
Let $x_{1-3}(t)$ be the circular motion displacement from equilibrium for each mass. The equations of motions can be written down by Hooke's and Newton's law by direct inspection, specifically

$$m\ddot{x}_1 = -k(x_1 - x_2) + k(x_3 - x_1) \tag{59}$$
$$m\ddot{x}_2 = k(x_1 - x_2) - k(x_2 - x_3) \tag{60}$$
$$m\ddot{x}_3 = k(x_2 - x_3) - k(x_3 - x_2) \tag{61}$$
$$\tag{62}$$

Assume the solution may be obtained with an exponential of form $x_n(t) = A_n \exp i\omega t$, and we may solve the coupled ODEs by method of determinants to give a matrix,

$$0 = [C - m\omega^2 I]A = \begin{pmatrix} 2k - m\omega^2 & -k & -k \\ -k & 2k - m\omega^2 & -k \\ -k & -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Let $\lambda = m\omega^2/k$ and define $M = \begin{pmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{pmatrix}$ We can now solve for the eigenvalues (or characteristic frequencies) of matrix C by,

$$\begin{aligned}
\det(C-\lambda I) &= (2-\lambda)^3-(2-\lambda)-1-(2-\lambda)-1-(2-\lambda) && (63)\\
&= (2-\lambda)^3-3(2-\lambda)-2 && (64)\\
&= -\lambda(\lambda^2-6\lambda+9) && (65)\\
&= -\lambda(\lambda-3)^2 && (66)\\
&= 0 && (67)
\end{aligned}$$

Therefore, $\lambda = 3$ or $\lambda = 0$. Hence characteristic frequencies are $\omega_1 = 0$ and $\omega_2 = \omega_3 = \sqrt{3k/m}$.

(b) Solve for the normal modes and describe the motion of the masses for each mode (with words or a rough sketch).

Solution:
Case 1; $\omega = 0$. By inspection of the matrix M it is readily seen that the eigen-equation is only satisfied when $A_1 = A_2 = A_3$. This corresponds to the collective circular motion of all three masses with (the springs unstretched) in either the clockwise or anti-clockwise directions.

Case 2; $\omega = \sqrt{3k/m}$. Here $M = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$ and so $-A_1 - A_2 - A_3 = 0$, and this normal modes must orthogonal to case 1, for instance if $A_1 = -A_2$ and $A_3 = 0$ masses 1 and 2 will oscillate anti-symmetrically, and mass 3 will not oscillate. Likewise if $A_1 = A_2 = -2A_3$, masses 1 and 2 will start by oscillating in phase clockwise toward mass 3, and mass 3 will oscillate anti-clockwise with half the relative amplitude.

(c) At $t = 0$ masses at position 1 was found to be displaced by a distance δ from its equilibrium position (i.e. when all springs are unstretched as shown above). The velocities of all three masses at $t = 0$ were zero. Find the resulting equations of motion for each of the three masses as a function of time.

Solution:
You are given the initial conditions that: $x_1(t=0) = \delta$ and $x_2 = x_3 = 0$, likewise since all start from rest $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$.

To get a general solution we must first construct a superposition of normal mode equations of motion. Specifically, case 1 above gives, $A_1 = A_2 = A_3$ and so the normalized $q_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(at+b)$, with since $\omega = 0$ the masses can only move (not oscillate). Under our initial conditions $a = 0$, and $b = 1$.

In case 2, we can write two solutions that oscillate at $\omega = \sqrt{3k/m}$, namely $q_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\cos(\omega t+$

$\phi)$ and $q_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}\cos(\omega t+\phi)$. Since the masses are not initially out of phase $\phi = 0$.

To get the generalized equation we can write it as superposition of normal modes (weighted by their amplitudes) under the initial condition that $q(t=0) = \delta\begin{pmatrix}1\\0\\0\end{pmatrix}$. This implies that,

$q(t) = \frac{\delta}{\sqrt{3}}q_1(t) + \frac{\delta}{\sqrt{2}}q_2(t) + \frac{\delta}{\sqrt{6}}q_3(t)$

Lastly we can evaluate $q(t)$ for $x_{1-3}(t)$ under our initial conditions as, specifically that

$$x_1(t) = \frac{1}{\sqrt{3}}\frac{\delta}{\sqrt{3}} + \frac{1}{\sqrt{2}}\frac{\delta}{\sqrt{2}}\cos(\omega t) + \frac{1}{\sqrt{6}}\frac{\delta}{\sqrt{6}}\cos(\omega t) \tag{68}$$

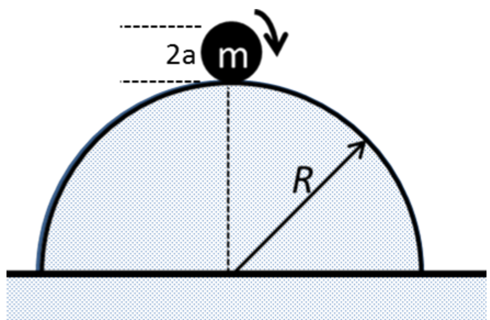
$$= \frac{\delta}{3} + \frac{2\delta}{3}\cos(\omega t) \tag{69}$$

$$x_2(t) = \frac{\delta}{3} - \frac{\delta}{3}\cos(\omega t) \tag{70}$$

$$x_3(t) = \frac{\delta}{3} - \frac{\delta}{3}\cos(\omega t) \tag{71}$$

Thus the system will oscillate at $\omega = \sqrt{3k/m}$. Masses 2 and 3 will oscillate in phase with eachother and mass 1 will have twice the amplitude and be anti-phased with respect to masses 2 and 3.

A solid cylinder of mass m and radius a rolls without slipping from rest under the influence of gravity from the top of a fixed solid half-cylinder of radius R, as shown below. Once rolling fast enough, the rolling cylinder may leave the surface of the static half-cylinder.

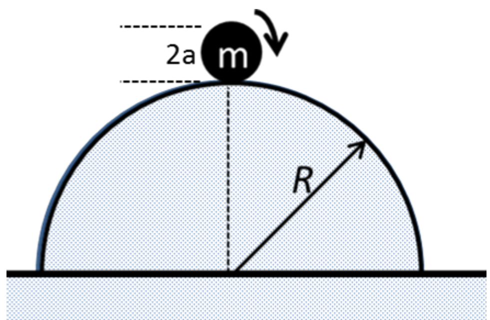


(a) *Derive* the moment of inertia for the rolling solid cylinder about its axis.

(b) Find the differential equations of motion for this system.

(c) Derive an expression for the point (e.g. angle) at which the cylinder leaves the surface of the surface.

A solid cylinder of mass m and radius a rolls without slipping from rest under the influence of gravity from the top of a fixed solid half-cylinder of radius R, as shown below. Once rolling fast enough, the rolling cylinder may leave the surface of the static half-cylinder.



(a) *Derive* the moment of inertia for the rolling solid cylinder about its axis.

Solution:
The moment of inertia is defined as $I = \int_0^m r^2 dm'$. The mass element can be expressed in terms of an infinitesimal radial thickness dr given by: $dm' = \rho dV = \rho L 2\pi r dr$ where the density is $\rho = \frac{m}{\pi a^2 L}$. Putting this in the definition we get:

$$I = 2\pi\rho L \int_0^a r^3 dr \tag{72}$$
$$= 2\pi\rho L \frac{a^4}{4} \tag{73}$$
$$= 2\pi \frac{m}{\pi a^2 L} L \frac{a^4}{4} \tag{74}$$
$$= \frac{1}{2} m a^2 \tag{75}$$

(b) Find the differential equations of motion for this system.

Solution:

The angular displacement of the system is in terms of two angles that represent the angular displacements of the cylinder rolling and its position on the surface as ϕ and θ, respectively. Let r and $\dot{r}$ be the unconstrained radial position/velocity of the center of mass of the cylinder (this values is constant at $r = R$ until the cylinder leaves the surface). The rolling of the cylinder can be described by an angular velocity $\omega_{cylinder}$. The rolling without slipping condition is then $R\theta = a\phi$.
The kinetic energy (e.g. $\frac{1}{2}I\dot{\theta}^2$) of the system is then the sum of the two angular velocity contributions and the the radial velocity (which is non-zero once the cylinder leaves the surface). The potential energy while the cylinder is on the surface is always just the gravitation component (i.e. $mgr\cos\theta$).

The Lagrangian is $\mathcal{L} = T - U$. We further evaluate the Lagrangian (under the $R\theta = a\phi$ rolling without slipping condition to eliminate the ϕ coordinate),

$$\mathcal{L} = T - U \tag{76}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2 + \frac{1}{2}I\dot{\phi}^2 - mgr\cos\theta \tag{77}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\left(mr^2 + \frac{1}{2}mR^2\right)\dot{\theta}^2 - mgr\cos\theta \tag{78}$$

To get the differential equations of motion we invoke the Euler-Lagrange equations. But, we must further include a constraint to account for the condition where the cylinder is rolling on the surface, i.e. $f(r,\theta) = r - a - R = 0$. This force constraint can be included in the Euler-Lagrange equations, by using a undetermined multiplier λ. With respect to θ we get,

$$0 = \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) + \lambda\frac{\partial f}{\partial \theta} \tag{79}$$

$$0 = mgr\sin\theta - \left(mr^2 + \frac{1}{2}mR^2\right)\ddot{\theta} - 2m\dot{r}\dot{\theta} + \lambda \cdot 0 \tag{80}$$

$$0 = g(a+R)\sin\theta - \left((a+R)^2 + \frac{1}{2}R^2\right)\ddot{\theta} \tag{81}$$

where in the last line we use $r = a + R$ and $\dot{r} = 0$ until the cylinder leaves the surface of the surface.

With respect to r we get,

$$0 = \frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) + \lambda\frac{\partial f}{\partial r} \tag{82}$$

$$0 = mr^2\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda \tag{83}$$

$$0 = m(a+R)^2\dot{\theta}^2 - mg\cos\theta + \lambda \tag{84}$$

where in the last line we use $r = a + R$ and $\ddot{r} = 0$ until the cylinder leaves the surface of the surface.

Therefore our differential equations of motion are:

$$0 = g(a+R)\sin\theta - \left((a+R)^2 + \frac{1}{2}R^2\right)\ddot{\theta} \tag{85}$$

$$0 = m(a+R)^2\dot{\theta}^2 - mg\cos\theta + \lambda \tag{86}$$

Note that both equations of motion independently describe the motion of the cylinder on the surface allowing for us to solve of λ (part c).

(c) Derive an expression for the point (e.g. angle) at which the cylinder leaves the surface of the surface.

Solution:
First, you should recognize from part b that we have two equations that depend only on θ and λ; hence if λ is known, then that angle at which the cylinder leaves the surface is also known. Solving first for $\ddot{\theta}$, and then for $\dot{\theta}$ by integration, we obtain,

$$\ddot{\theta} = \frac{g(a+R)}{(a+R)^2 + \frac{1}{2}R^2}\sin\theta \tag{87}$$

$$\dot{\theta}d\dot{\theta} = \frac{g(a+R)}{(a+R)^2 + \frac{1}{2}R^2}\sin\theta d\theta \tag{88}$$

$$\Rightarrow \frac{1}{2}\dot{\theta}^2 = -\frac{g(a+R)}{(a+R)^2 + \frac{1}{2}R^2}\cos\theta + C \tag{89}$$

$$\tag{90}$$

The constant C is found by the initial condition of rolling from rest (i.e. $\dot{\theta} = 0$ when $\theta = 0$), hence $C = \frac{g(a+R)}{(a+R)^2+\frac{1}{2}R^2}$ or $\dot{\theta}^2 = 2\frac{g(a+R)}{(a+R)^2+\frac{1}{2}R^2}(1-\cos\theta)$. We can now plug our expression for $\dot{\theta}$ in the second Euler-Lagrange equation in part b to obtain:

$$0 = m(a+R)^2\dot{\theta}^2 - mg\cos\theta + \lambda \tag{91}$$

$$0 = m(a+R)^2 2\frac{g(a+R)}{(a+R)^2 + \frac{1}{2}R^2}(1-\cos\theta) - mg\cos\theta + \lambda \tag{92}$$

$$\Rightarrow \lambda = mg\frac{3(a+R)^2 + \frac{1}{2}R^2}{(a+R)^2 + \frac{1}{2}R^2}\cos\theta - \frac{2mg(a+R)^2}{(a+R)^2 + \frac{1}{2}R^2} \tag{93}$$

The cylinder will leave the surface when $\lambda = 0$, i.e. when

$$\theta = \arccos\left[\frac{2(a+R)^2}{3(a+R)^2 + \frac{1}{2}R^2}\right] \tag{94}$$