Saint-Venant Equations

We consider the following two equations:

$$
\begin{aligned}
B\frac{\partial y}{\partial t} + \frac{\partial Q}{\partial x} &= 0, \quad x \in [a,b], t \in [t_0, t_1] \quad (1)\\
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\left(\frac{\partial y}{\partial x} + S_f - S_0\right) &= 0, \quad x \in [a,b], t \in [t_0, t_1], \quad (2)
\end{aligned}
$$

where y is a depth, Q is a streamflow, B is a width of the channel, g is an accelaration due to gravity, A is a cross-sectional area of the flow, S_f is a friction slope, S_0 is a channel bottom slope, assumed given constant and considered positive sloping downwards, $(b-a)$ is a length of the channel. y and Q are two unknowns. We prescribe initial conditions

$$
\begin{aligned}
y(x,0) &= y_0(x), \quad x \in [a,b] \quad (3)\\
Q(x,0) &= Q_0(x), \quad x \in [a,b]. \quad (4)
\end{aligned}
$$

We assume that we deal with the subcritical flow, so we need to prescribe only two boundary conditions: one on the left end and one on the right end

$$
\begin{aligned}
y(b,t) &= y_b(t), \quad t \in [t_0, t_1] \quad (5)\\
Q(a,t) &= Q_a(t), \quad t \in [t_0, t_1]. \quad (6)
\end{aligned}
$$

The formulas describing the relationship between the mentioned variables are given below:

$$
\begin{aligned}
Q &= VA, \quad \text{Discharge formula} \quad (7)\\
A &= By, \quad \text{only for rectangular channels} \quad (8)\\
S_f &= \frac{n^2|Q|Q}{k^2A^2R^{4/3}}, \quad \text{Manning formula} \quad (9)
\end{aligned}
$$

where V is a cross-sectional average velocity of the flow, $R = \dfrac{A}{P}$ is a hydraulic radius, $P = 2y + B$ is a wetted perimeter, k is a conversion factor, n is the Gauckler-Manning coefficient.

For $B \gg y$, we can approximate $R \approx y$, so in terms of Q and y we have $S_f \approx \dfrac{n^2|Q|Q}{k^2B^2y^{10/3}}$.

In terms of y and flow velocity V equations (1) and (2) can be rewritten as

$$\begin{aligned} \frac{\partial y}{\partial t} + V\frac{\partial y}{\partial x} + y\frac{\partial V}{\partial x} &= 0, \quad x \in [a,b], t \in [t_0, t_1] \qquad (10) \\ \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x} + g(S_f - S_0) &= 0, \quad x \in [a,b], t \in [t_0, t_1]. \qquad (11) \end{aligned}$$

The criteria for the subcritical, supercritical or critical flow is the Froude number, $F_r = \dfrac{V}{c}$, where c is the celerity of a gravity wave defined as

$$c = \sqrt{g\frac{A}{B}} = \sqrt{gy}. \tag{12}$$

In terms of Q and y the Froude number can be written as

$$F_r = \frac{V}{c} = \frac{Q}{A\sqrt{gy}} = \frac{Q}{By\sqrt{gy}} = \frac{Q}{B\sqrt{gy^3}}. \tag{13}$$

If $F_r < 1$ we deal with the subcritical flow, if $F_r = 1$ or > 1 we have critical or supercritical flow, respectively.

Numerical scheme

Preissman scheme

In this scheme the partial derivatives and other variables are approximated as follows

$$\begin{aligned} \left(\frac{\partial f}{\partial t}\right)\Big|_{(x_i,y_k)} &= \frac{(f_i^{k+1} + f_{i+1}^{k+1}) - (f_i^k + f_{i+1}^k)}{2\Delta t} \qquad (14) \\ \left(\frac{\partial f}{\partial x}\right)\Big|_{(x_i,y_k)} &= \frac{\theta(f_{i+1}^{k+1} - f_i^{k+1})}{\Delta x} + \frac{(1-\theta)(f_{i+1}^k - f_i^k)}{\Delta x} \qquad (15) \\ \bar{f}|_{(x_i,y_k)} &= \frac{1}{2}\theta(f_{i+1}^{k+1} + f_i^{k+1}) + \frac{1}{2}(1-\theta)(f_{i+1}^k + f_i^k), \qquad (16) \end{aligned}$$

where θ is a weighting coefficient. The scheme is unconditionally stable if $0.55 < \theta \leq 1$.

Then the discretized equations can be written for $i = \overline{1, N-1}$

$$\begin{aligned}
B\frac{(y_i^{k+1} + y_{i+1}^{k+1}) - (y_i^k + y_{i+1}^k)}{2\Delta t} &+ \\
+\frac{\theta(Q_{i+1}^{k+1} - Q_i^{k+1})}{\Delta x} + \frac{(1-\theta)(Q_{i+1}^k - Q_i^k)}{\Delta x} &= 0, \quad (17) \\
\frac{(Q_i^{k+1} + Q_{i+1}^{k+1}) - (Q_i^k + Q_{i+1}^k)}{2\Delta t} &+ \\
+\frac{\theta\left[\left(\frac{Q^2}{By}\right)_{i+1}^{k+1} - \left(\frac{Q^2}{By}\right)_i^{k+1}\right]}{\Delta x} + \frac{(1-\theta)\left[\left(\frac{Q^2}{By}\right)_{i+1}^{k} - \left(\frac{Q^2}{By}\right)_i^{k}\right]}{\Delta x} &+ \\
+gB\bar{y}_i^k\left(\frac{\theta(y_{i+1}^{k+1} - y_i^{k+1})}{\Delta x} + \frac{(1-\theta)(y_{i+1}^k - y_i^k)}{\Delta x} + \bar{S}_{f,i}^k - \bar{S}_{0,i}^k\right) &= 0. \quad (18)
\end{aligned}$$

The simplification leads to

$$\begin{aligned}
(y_i^{k+1} + y_{i+1}^{k+1}) + \frac{2\Delta t\theta}{B\Delta x}(Q_{i+1}^{k+1} - Q_i^{k+1}) &- \\
-(y_i^k + y_{i+1}^k) + \frac{2\Delta t(1-\theta)}{B\Delta x}(Q_{i+1}^k - Q_i^k) &= 0, \quad (19) \\
(Q_i^{k+1} + Q_{i+1}^{k+1}) + \frac{2\Delta t\theta}{B\Delta x}\left[\left(\frac{Q^2}{By}\right)_{i+1}^{k+1} - \left(\frac{Q^2}{By}\right)_i^{k+1}\right] &- \\
-(Q_i^k + Q_{i+1}^k) + \frac{2\Delta t(1-\theta)}{B\Delta x}\left[\left(\frac{Q^2}{By}\right)_{i+1}^{k} - \left(\frac{Q^2}{By}\right)_i^{k}\right] &+ \\
+gB\Delta t\bar{y}_i^k\left(\frac{\theta(y_{i+1}^{k+1} - y_i^{k+1})}{\Delta x} + \frac{(1-\theta)(y_{i+1}^k - y_i^k)}{\Delta x} + \bar{S}_{f,i}^k - \bar{S}_{0,i}^k\right) &= 0. \quad (20)
\end{aligned}$$

Boundary conditions give us 2 additional equations:

$$y_N^{k+1} = y_b(t_{k+1}), \quad (21)$$

$$Q_1^{k+1} = Q_a(t_{k+1}). \quad (22)$$

Numerical simulations