Urysohn's Lemma:

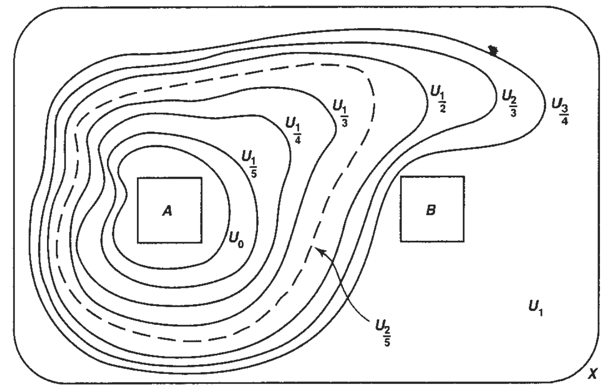
These notes cover parts of sections 33, 34, and 35. Not covered is complete regularity.
Urysohn's Lemma gives a method for constructing a continuous function separating closed sets.

Urysohn's Lemma If A and B are disjoint, closed in a normal space X, there exists a map $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$

Note: In a metric space, the following function does this.

$$f(x) = \frac{1}{2}\left(\frac{d(x,A) - d(x,B)}{d(x,A) + d(x,B)} + 1\right)$$

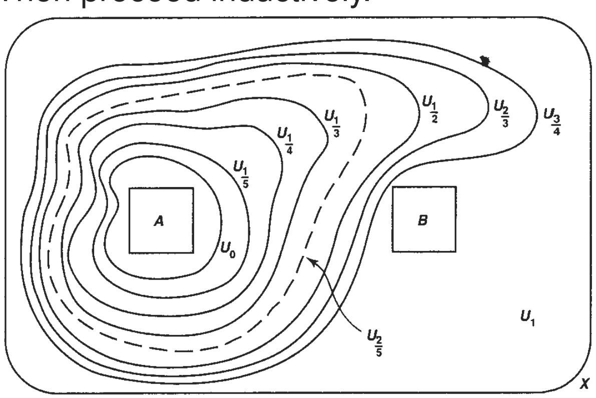
Proof - Step 1



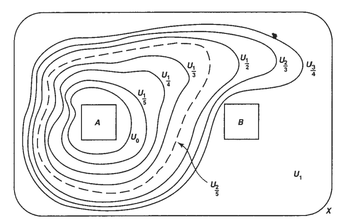
Order the rationals in $[0,1]$: $\{1, 0, r_3, r_4, \ldots\}$. For each $p \in Q$ construct inductively an open set U_p in X so that if $p < r$, then $\overline{U_p} \subset U_r$.

Proof - Continued

To get started, let $U_1 = X \setminus B$, and U_0 (using normality and $A^{\text{closed}} \subset U_1^{\text{open}}$) be an open set containing A with $\overline{U_0} \subset U_1$.
Then proceed inductively.



Proof, Continued



Step 2 For $p \in Q$, define $U_p = \emptyset$ if $p < 0$ and $U_p = X$ if $p > 1$.

Step 3 Define $f(x) = \inf\{p | x \in U_p\}$

Step 4 Check that:

- $x \in \overline{U_r} \Rightarrow f(x) \leq r$
- $x \notin \overline{U_r} \Rightarrow f(x) \geq r$

Proof - Continued

Step 5 Show that f is continuous and has the desired properties using Step 4 and the fact that if

$$c < p < f(x_0) < q < d \text{ and } U = U_q \setminus \overline{U_p}$$

then $f(U) \subset [p,q] \subset (c,d)$

Note: If A and B are G_δ sets (countable intesections of open sets), then the function f from Urysohn's Lemma can be constructed so $f^{-1}(1) = B$ and $f^{-1}(0) = A$.

Tietze's Extension Theorem

We will use Urysohn's Lemma to prove:

Tietze's Extension Theorem: (Extending functions.)
If $A^{closed} \subset X^{normal}$ and $f : A \to Y$ is continuous where Y is either R or an interval in R, then there exists a continuous function $F : X \to Y$ extending f, i.e. $F(a) = f(a)$ for each a in A.

and:

Urysohn's Metrization Theorem: If X is regular and second countable, then X is metrizable. In fact, X is homeomorphic to a subspace of I^ω.