Polynomially Weighted Spaces of Bandlimited Functions: Reproducing Kernels, Complete Interpolating Sequences, and Sampling Theorems

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Abstract

Any function that is bandlimited in the distributional sense grows at most polynomially on the real line. Consequently, any bandlimited function with bandwidth σ belongs to a polynomially weighted Paley-Wiener space $B_N(\sigma)$, for some sufficiently large integer N, with inner product $\langle f, g \rangle_{B_N} = \int_{\mathbb{R}} f(t)\,\overline{g}(t)\,(1+t^2)^{-N}\,dt$. We show that the spaces $B_N(\sigma)$, $N \in \mathbb{Z}$, are reproducing kernel Hilbert spaces and, more specifically, de Branges spaces. We determine their reproducing kernels and complete interpolating sequences, together with the associated sampling expansions (including orthogonal ones). We also present more general sampling theorems allowing finitely many derivative samples; these include, as special cases, classical expansions of Valiron and Tschakaloff.

Keywords: complete interpolating sequence, Shannon sampling theorem, nonuniform sampling, Paley-Wiener space

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1 Introduction

The present work is concerned with the sampling and interpolation of bandlimited functions. The sampling problem involves the stable reconstruction of a bandlimited function f from its values $f(\lambda_k)$ on a discrete set $\Lambda = \{\lambda_k,\, k \in \mathbb{Z}\} \subset \mathbb{C}$. The associated interpolation problem consists in finding a bandlimited function f that interpolates given values c_k at the points of Λ, provided the sequence $(c_k)_{k\in\mathbb{Z}}$ satisfies appropriate conditions. Complete interpolating sequences are sampling sets Λ that provide unique solutions for both the sampling and the interpolation problem. In order to define bandlimited functions we first recall that for functions $f \in L^1(\mathbb{R})$ the Fourier transform is given by $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$. This definition can be extended to larger classes of functions and to distributions (generalized functions), see, e.g., [39].

Definition 1. *A function is called bandlimited with bandwidth σ, or σ-bandlimited, if its distributional Fourier transform has support in the interval $[-\sigma, \sigma]$, where $0 < \sigma < \infty$. Equivalently, a function f is σ-bandlimited if and only if f is an entire function and satisfies the growth condition*

$$|f(z)| \,\leq\, \gamma\,(1+|z|)^n\; e^{\sigma|Im\; z|} \tag{1}$$

for all $z \in \mathbb{C}$, and some non-negative integer n and constant $\gamma > 0$ [39, Theorem 7.23].

It is evident from (1) that a bandlimited function f has at most polynomial growth on the real line. This motivates the following definition of a family of inner product spaces of bandlimited functions.

Definition 2. *For $N \in \mathbb{Z}$ let $B_N(\sigma)$ denote the space of σ-bandlimited functions f such that $\|\, f \,\|_{B_N} = \left(\int_{\mathbb{R}} |f(t)|^2\,(1+t^2)^{-N}\,dt\right)^{1/2} \,<\, \infty$. $B_N(\sigma)$ is equipped with the inner product*

$$\langle f,\, g\rangle_{B_N} = \int_{\mathbb{R}} f(t)\,\overline{g}(t)\,(1+t^2)^{-N}\,dt.$$

The spaces $B_N(\sigma)$ are nested with respect to N, i.e., $B_M(\sigma) \subset B_N(\sigma)$ for $M < N$. In the following we will sometimes write the bandwidth σ in the form $\sigma = \pi\omega$. The fundamental example is the Paley-Wiener space $B_0(\pi\omega)$, where the norm and inner product are those of L_2. The space $B_0(\pi\omega)$ is a reproducing kernel Hilbert space (RKHS). The reproducing kernel is given by $K_u(z) = \omega\,\text{sinc}(\omega(z-\overline{u}))$, where the cardinal sine function $\text{sinc}(z)$ is given by $\text{sinc}(z) = \sin(\pi z)/(\pi z)$. The complete interpolating sequences for $B_0(\pi\omega)$ have been characterized in deep results by Pavlov [36], Hruščev et al. [21], Minkin [32], and Lyubarskii and Seip [28]. The most prominent example is the uniform sampling set $\Lambda = \{k/\omega,\, k \in \mathbb{Z}\}$, which leads to the classical interpolation and sampling theorem that is often traced back to Whittaker [50], Kotelnikov [24], and Shannon [42]:

Theorem A. *For any sequence of complex numbers $(c_k)_{k\in\mathbb{Z}}$ that satisfies $\sum_{k\in\mathbb{Z}}|c_k|^2 < \infty$ the unique solution $f \in B_0(\pi\omega)$ of the interpolation problem $f(k/\omega) = c_k$, $k \in \mathbb{Z}$, is given by*

$$f(z) = \sum_{k\in\mathbb{Z}} c_k \operatorname{sinc}(\omega z - k). \tag{2}$$

Furthermore, every $f \in B_0(\pi\omega)$ satisfies $\sum_{k\in\mathbb{Z}}|f(k/\omega)|^2 = \omega \parallel f \parallel_{B_0}^2 < \infty$, and thus admits the representation (2) with $c_k = f(k/\omega)$, $k \in \mathbb{Z}$. The cardinal series (2) converges absolutely; in the norm of B_0; and uniformly on horizontal strips of finite width in the complex plane.

For some historical background on the cardinal series, see, e.g., [8] and [16]. A comprehensive survey of the early literature is provided in [22]. General introductions to sampling theory, surveys, and monographs from a variety of perspectives include, for example, [9, 12–14, 17, 18, 31, 46, 55, 56]. The goal of the present work is to study the spaces $B_N(\sigma)$ for $N \neq 0$ and to establish analogues of some classical results for $B_0(\sigma)$. Our foundation and starting point is [2]; we generalize its framework and obtain sharper sampling and interpolation results in the reproducing kernel Hilbert space setting.

Naturally, as N increases, additional information about $f \in B_N(\sigma)$ will be required. This information can be obtained in different ways. In the present paper we study two approaches that both require a minimal number of samples. The approach of taking some additional samples of the function f is studied in Sections 4–6. It will be shown that for $N > M$, the sampling theorem for $B_N(\sigma)$ requires $N-M$ additional samples compared to that for $B_M(\sigma)$. Alternatively, the $N-M$ additional samples can be samples of derivatives of f. This case is studied in Section 7. Sampling theorems for these two approaches and $N > 0$ were developed in [2, 9, 19, 20, 26, 37, 43, 45, 47–49] and are discussed further below. Two of our results, Theorem 3 and Theorem 5, generalize and sharpen these prior sampling theorems by providing a stronger sense of convergence, i.e., convergence in the norm of $B_N(\sigma)$ instead of uniform convergence on compact subsets of $\mathbb{C}$, and/or applying to a larger class of sampling sets. Furthermore, they are proved in the unified framework of sampling theory in reproducing kernel Hilbert spaces. Alternative approaches for sampling bandlimited functions of polynomial growth involve oversampling in the sense of sampling at a higher rate; see, e.g., [10, 26, 30, 54]. This can give very fast numerical convergence of the sampling series [23, 38, 40]. Since we are concerned with sampling at the minimal sampling rate, methods that require this kind of oversampling are not studied in this paper.

The paper is organized as follows. The next section reviews some pertinent concepts from sampling theory in reproducing kernel Hilbert spaces. This includes the definition of complete interpolating sequences, the class of sampling sets under consideration, as well as a generalization of Theorem A for this general setting. These concepts and results will then be applied to the spaces $B_N(\sigma)$ in the remaining sections. The main contributions of this paper are contained in Theorems 1–6. Theorem 1 in Section 3 establishes the $B_N(\sigma)$, $N \neq 0$, as reproducing kernel Hilbert spaces. Explicit expressions for the reproducing kernels are obtained for $N > 0$, while for $N < 0$ it is shown that the kernels can be determined recursively. These results are based on a direct verification that the spaces $B_N(\sigma)$, $N \in \mathbb{Z}$, are de Branges spaces [11]. The spaces $B_N(\sigma)$, $N > 0$ have appeared in the prior literature on sampling bandlimited functions of polynomial growth [2, 26, 48, 49, 54]. For example, Zakai [54] suggested the space $B_1(\sigma)$ as natural function space for sampling theory, and Lee [26, Theorem 4] proved that $B_0(\sigma)$ is dense in $B_N(\sigma)$, $N > 0$. Walter [49] considered the subspace $B^0_{2N}(\sigma)$ of $B_{2N}(\sigma)$, $N > 0$, that consists of the functions $f \in B_{2N}(\sigma)$ such that $f(z)/(1+z^2)^N$ is an entire function. He noted that $B^0_{2N}(\sigma)$ is an RKHS and identified its reproducing kernel. On a more abstract level, general connections between de Branges spaces and a large class of weighted Paley-Wiener spaces, including the existence of complete interpolating sequences, have been studied by Lyubarskii and Seip [29]. A systematic RKHS treatment of the spaces $B_N(\sigma)$ themselves for all $N \in \mathbb{Z}$, together with the explicit de Branges space identification and reproducing kernel formulas developed here, does not seem to have been carried out.

Theorem 2 in Section 4 identifies the complete interpolating sequences for $B_N(\sigma)$, $N \neq 0$. It implies that for $N > 0$ a complete interpolating sequence (CIS) for $B_N(\sigma)$ is obtained by adding N points to a CIS for $B_0(\sigma)$, while for $N < 0$ one obtains a CIS for $B_N(\sigma)$ by deleting $|N|$ points from a CIS for $B_0(\sigma)$. It is also shown that a complete interpolating sequence for $B_N(\sigma)$ cannot simultaneously be a CIS for another space $B_{N'}(\sigma')$ with $(N', \sigma') \neq (N, \sigma)$.

Theorem 3 in Section 5 provides the generalization of Theorem A for sampling on complete interpolating sequences in the spaces $B_N(\sigma)$, $N \in \mathbb{Z}$. This extends and sharpens our earlier result for $N > 0$ [2, Theorem 4]. Corollary 2 gives an example of a sampling expansion for $B_1(\sigma)$ that requires one additional sample compared to the cardinal series, but in return provides stability with regard to bounded errors in the samples. This result is also used to obtain an explicit example of a function in $B_1(\pi\omega)$ that has bounded samples on $\frac{1}{\omega}\mathbb{Z}$ but is itself unbounded on the real line (Corollary 4). A variety of further examples and numerical experiments, as well as a discussion of prior results by Walter [48, 49] and Shin et al. [43], can be found in [2].

The explicit expressions for the reproducing kernels yield a complete characterization of the orthogonal expansions for the spaces $B_N(\sigma)$, $N \geq 0$, which are given in Theorem 4 in Section 6. For $N > 0$ the orthogonal sampling sets are nonuniform and the sampling points are obtained as solutions of a nonlinear equation. An example of an orthogonal expansion in $B_1(\sigma)$ is provided.

Section 7 is devoted to the case where finitely many derivative samples are included. Results for this case go as far back as Valiron's 1925 paper on Lagrange interpolation [47]. Subsequent work includes theorems for uniform sampling by Tschakaloff [45], Pfaffelhuber [37], and Lee [26], with the most general theorem due to Hoskins and Sousa Pinto [19, 20]. Schmeisser [9, p. 96] extended Valiron's formula to a class of nonuniform sampling sets. Theorem 5, stated in analogy to the Hoskins and Sousa Pinto theorem, generalizes this prior work. In all of these cases the derivative samples are taken at a single point. Theorem 6, our final result, allows for derivative samples to be taken at more than one point. Corollary 8 provides an example. Finally, Section 8 contains the more technical definitions and proofs.

2 Complete interpolating sequences and sampling expansions in reproducing kernel Hilbert spaces

In this section we review some known results about complete interpolating sequences and the associated sampling expansions in reproducing kernel Hilbert spaces, and state them in the form needed for the remainder of the paper. The included proofs rely on the basic theory of reproducing kernel Hilbert spaces, Riesz bases, and frames, which can be found, for example, in [53, pp. 13–16, 25–31, 154–159]. While being an integral part of the presentation, they do not claim originality. For a broader introduction and context, see, e.g., [3] and [13], as well as [14, 17, 18, 33, 41, 52]. To begin, recall that a reproducing kernel Hilbert space (RKHS) H on a domain X is a separable Hilbert space of functions $f : X \to \mathbb{C}$ such that for every $u \in X$ there exists an element K_u of H (the reproducing kernel at u) such that $f(u) = \langle f, K_u \rangle$ for all $f \in H$. Throughout this paper the domain X will be the set $\mathbb{C}$ of complex numbers. Let $\widetilde{K}_u$ denote the normalized kernel $\widetilde{K}_u = K_u / \parallel K_u \parallel$. It will always be assumed that H is such that K_u does not vanish identically for any $u \in \mathbb{C}$. The following class of sampling sets will be considered.

Definition 3. *A countable set* $\Lambda = \{\lambda_k,\, k \in \mathbb{Z}\} \subset \mathbb{C}$ *is called a* complete interpolating sequence (CIS) *for* H *if the following two conditions hold:*

1. *For any sequence* $\{c_k\}_{k\in\mathbb{Z}}$ *such that* $\sum_{k\in\mathbb{Z}} \frac{|c_k|^2}{\|K_{\lambda_k}\|^2} < \infty$ *the interpolation problem*

$$f(\lambda_k) = c_k, \quad k \in \mathbb{Z}, \tag{3}$$

   has a solution $f \in H$*.*

2. *For all* $f \in H$ *one has*

$$A \parallel f \parallel^2 \;\le\; \sum_{k\in\mathbb{Z}} \frac{|f(\lambda_k)|^2}{\parallel K_{\lambda_k} \parallel^2} \;\le\; B \parallel f \parallel^2 \tag{4}$$

   with constants $0 < A \le B < \infty$ *that are independent of* f*.*

Unless otherwise specified the λ_k *are enumerated with non-decreasing real parts, i.e., Re* $\lambda_k \le$ *Re* λ_{k+1}*, and such that* $\operatorname{Re}\lambda_k \le 0$ *for* $k \le 0$*, and* $\operatorname{Re}\lambda_k \ge 0$ *for* $k \ge 0$*.*

The condition (4) implies that the solution to the interpolation problem (3) is unique. In case of $H = B_0(\sigma)$ the definition of a CIS given above is slightly more general than the definition that we used in [2], which did impose the additional condition $\sup |\operatorname{Im}\lambda_k| < \infty$. The following characterization of complete interpolating sequences will be used throughout this paper.

Theorem B. *A set* $\Lambda = \{\lambda_k,\, k \in \mathbb{Z}\} \subset \mathbb{C}$ *is a CIS for an RKHS* H *if and only if the normalized kernels* $\{\widetilde{K}_{\lambda_k},\, k \in \mathbb{Z}\}$ *form a Riesz basis of* H*.*

Proof Assume Λ is a CIS for H. The condition (4) implies that the set of normalized kernels $\{\widetilde{K}_{\lambda_k},\, k \in \mathbb{Z}\}$ is a frame. Let λ_l be an arbitrary element of Λ. Then the unique solution of the interpolation problem $f(\lambda_k) = \delta_{kl},\, k \in \mathbb{Z}$, is a nontrivial function that vanishes on $\Lambda \backslash \{\lambda_l\}$. That is, it is orthogonal to $\{\widetilde{K}_{\lambda_k},\, k \in \mathbb{Z},\, k \neq l\}$. Hence removing a single element from $\{\widetilde{K}_{\lambda_k},\, k \in \mathbb{Z}\}$ leaves an incomplete set. Therefore $\{\widetilde{K}_{\lambda_k},\, k \in \mathbb{Z}\}$ is an exact frame and thus a Riesz basis, cf. [53, p. 157]. Now assume that $\{\widetilde{K}_{\lambda_k},\, k \in \mathbb{Z}\}$ is a Riesz basis. Since a Riesz basis is also a frame, one has condition (4). Let $\{S_{\lambda_k},\, k \in \mathbb{Z}\}$ denote the Riesz basis that is biorthogonal to $\{\widetilde{K}_{\lambda_k},\, k \in \mathbb{Z}\}$, i.e., $\langle S_{\lambda_k}, \widetilde{K}_{\lambda_l}\rangle = \delta_{kl}$. For $\sum_{k\in\mathbb{Z}} \| K_{\lambda_k} \|^{-2} |c_k|^2 < \infty$, the expansion $f(z) = \sum_{k\in\mathbb{Z}} \frac{c_k}{\|K_{\lambda_k}\|} S_{\lambda_k}(z)$ converges unconditionally to a function $f \in H$ that satisfies

$$f(\lambda_l) \;=\; \langle f, K_{\lambda_l}\rangle \;= \| K_{\lambda_l} \| \sum_{k\in\mathbb{Z}} \frac{c_k}{\| K_{\lambda_k} \|} \langle S_{\lambda_k}, \widetilde{K}_{\lambda_l}\rangle \;=\; c_l, \quad l \in \mathbb{Z}.$$

Hence f solves the interpolation problem (3). □

The concept of a complete interpolating sequence leads directly to an associated sampling and interpolation theorem.

Theorem C. *Let $\Lambda = \{\lambda_k,\, k \in \mathbb{Z}\}$ be a CIS for an RKHS H. For $k \in \mathbb{Z}$ let φ_k be the unique function in H that satisfies $\varphi_k(\lambda_l) = \delta_{kl},\, l \in \mathbb{Z}$. For any sequence of complex numbers $(c_k)_{k\in\mathbb{Z}}$ that satisfies $\sum_{k\in\mathbb{Z}} \| K_{\lambda_k} \|^{-2} |c_k|^2 < \infty$, the unique solution $f \in H$ of the interpolation problem $f(\lambda_k) = c_k,\, k \in \mathbb{Z}$, is given by*

$$f(z) = \sum_{k\in\mathbb{Z}} c_k\, \varphi_k(z). \tag{5}$$

Furthermore, every $f \in H$ satisfies $\sum_{k\in\mathbb{Z}} \| K_{\lambda_k} \|^{-2} |f(\lambda_k)|^2 < \infty$, and thus admits the representation (5) with $c_k = f(\lambda_k),\, k \in \mathbb{Z}$. The series (5) converges unconditionally in the norm of H.

Proof Since Λ is a CIS, every $f \in H$ satisfies $\sum_{k\in\mathbb{Z}} \| K_{\lambda_k} \|^{-2} |f(\lambda_k)|^2 < \infty$ according to Definition 3. Furthermore, $\{\widetilde{K}_{\lambda_k},\, k \in \mathbb{Z}\}$ is a Riesz basis of H according to Theorem B. Let $\{S_{\lambda_k},\, k \in \mathbb{Z}\}$ denote the corresponding biorthogonal Riesz basis. Then each $f \in H$ admits the expansion

$$f = \sum_{k\in\mathbb{Z}} \langle f, \tilde{K}_{\lambda_k}\rangle\, S_{\lambda_k} \;=\; \sum_{k\in\mathbb{Z}} \frac{f(\lambda_k)}{\| K_{\lambda_k} \|}\, S_{\lambda_k}$$

with unconditional convergence in the norm of H. Now let

$$\varphi_k = \frac{S_{\lambda_k}}{\| K_{\lambda_k} \|}. \tag{6}$$

This will yield (5) with $c_k = f(\lambda_k)$, provided that the φ_k satisfy $\varphi_k(\lambda_l) = \delta_{kl}$. This follows from the biorthogonality, i.e., $\varphi_k(\lambda_l) = \langle \varphi_k, K_{\lambda_l}\rangle = \frac{\|K_{\lambda_l}\|}{\|K_{\lambda_k}\|}\langle S_{\lambda_k}, \widetilde{K}_{\lambda_l}\rangle = \delta_{kl}$. Now let $(c_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ satisfy $\sum_{k\in\mathbb{Z}} \| K_{\lambda_k} \|^{-2} |c_k|^2 < \infty$. Since $\{S_{\lambda_k},\, k \in \mathbb{Z}\}$ is a Riesz basis, the series $f = \sum_{k\in\mathbb{Z}} \frac{c_k}{\|K_{\lambda_k}\|} S_{\lambda_k} = \sum_{k\in\mathbb{Z}} c_k\, \varphi_k$ converges unconditionally in the norm of H. Since $\varphi_k(\lambda_l) = \delta_{kl}$, one has $f(\lambda_l) = c_l,\, l \in \mathbb{Z}$. □

Convergence in the norm of H implies uniform convergence on subsets S of $\mathbb{C}$ for which $\{\| K_u \|, u \in S\}$ is bounded. This follows from $|f(u)| = |\langle f, K_u\rangle| \leq \|K_u\| \|f\|$. The expansion (5) is called orthogonal if the sampling functions φ_k, $k \in \mathbb{Z}$, form an orthogonal basis of H. The following lemma provides a necessary and sufficient criterion.

Lemma A. *Let $\Lambda = \{\lambda_k, k \in \mathbb{Z}\}$ be a CIS of H. Then the sampling functions φ_k, $k \in \mathbb{Z}$, in (5) are mutually orthogonal if and only if*

$$K_{\lambda_k}(\lambda_l) = 0 \quad \text{for } k, l \in \mathbb{Z},\ k \neq l. \tag{7}$$

In this case, $\varphi_k = \frac{K_{\lambda_k}}{\|K_{\lambda_k}\|^2}$, $k \in \mathbb{Z}$.

Proof Assume Λ is a CIS of H such that (7) holds. Then $\{\tilde{K}_{\lambda_k}, k \in \mathbb{Z}\}$ is an orthonormal basis of H, so $S_{\lambda_k} = \tilde{K}_{\lambda_k}$, and (6) then gives $\varphi_k = K_{\lambda_k} / \| K_{\lambda_k} \|^2$, $k \in \mathbb{Z}$. To prove the reverse direction, assume that Λ is a CIS such that the φ_k, $k \in \mathbb{Z}$, are mutually orthogonal. Then (6) implies that the S_{λ_k}, $k \in \mathbb{Z}$, are also mutually orthogonal. Since $\{\tilde{K}_{\lambda_k}, k \in \mathbb{Z}\}$ is the unique Riesz basis biorthogonal to $\{S_{\lambda_k}, k \in \mathbb{Z}\}$, one has $S_{\lambda_k} = c_k \tilde{K}_{\lambda_k}$, $c_k \neq 0$, $k \in \mathbb{Z}$. In fact, $c_k = \langle S_{\lambda_k}, \tilde{K}_{\lambda_k}\rangle = 1$. Then (6) yields $\varphi_k = S_{\lambda_k} / \| K_{\lambda_k} \| = K_{\lambda_k} / \| K_{\lambda_k} \|^2$. For $k \neq l$ one has $0 = \langle S_{\lambda_k}, \widetilde{K}_{\lambda_l}\rangle = \langle \tilde{K}_{\lambda_k}, \widetilde{K}_{\lambda_l}\rangle = (\| K_{\lambda_k} \| \| K_{\lambda_l} \|)^{-1} K_{\lambda_k}(\lambda_l)$, i.e., (7) holds. □

3 Reproducing kernels

The following theorem establishes that $B_N(\sigma)$ is a RKHS. It also yields explicit expressions for the reproducing kernels for $B_N(\sigma)$, $N \geq 0$, and a recursion formula for the kernels when $N < 0$.

Theorem 1. *$B_N(\sigma)$ is a reproducing kernel Hilbert space for all $N \in \mathbb{Z}$. The reproducing kernel K_u^N of $B_N(\sigma)$ has the form*

$$K_u^N(z) = \frac{i}{2} \frac{E_N(z)\overline{E_N(u)} - E_N^*(z)\overline{E_N^*(u)}}{\pi(z - \overline{u})} \tag{8}$$

where $E_N^*(z) = \overline{E_N(\overline{z})}$. For $N \geq 0$, the function E_N is given by

$$E_N(z) = (1 - iz)^N e^{-i\sigma z}, \quad N \geq 0. \tag{9}$$

For $N < 0$, E_N is determined by the recursion

$$E_N(z) = \sqrt{\frac{\pi}{K_i^{N+1}(i)}}\ K_i^{N+1}(z), \quad N \in \mathbb{Z}. \tag{10}$$

The theorem will be proved in Section 8.2 by showing that $B_N(\sigma)$ is a de Branges space $H(E_N)$ as defined and studied in [11]. The definition of a de Branges space is given in Section 8.1. With $\sigma = \pi\omega$, equations (8) and (9) give the following kernels for $B_N(\pi\omega)$, $N = 0, 1, 2$.

$$K_u^0(z) = \omega\,\mathrm{sinc}(\omega(z-\overline{u})) \tag{11}$$

$$K_u^1(z) = \omega\,(1+z\overline{u})\,\mathrm{sinc}(\omega(z-\overline{u})) \;+\; \frac{1}{\pi}\,\cos(\pi\omega(z-\overline{u})) \tag{12}$$

$$\begin{aligned} K_u^2(z) &= \omega\left((z^2-1)(\overline{u}^2-1)+4z\overline{u}\right)\,\mathrm{sinc}(\omega(z-\overline{u})) \\ &\quad + \;\frac{2}{\pi}\,(1+z\overline{u})\cos(\pi\omega(z-\overline{u})) \end{aligned} \tag{13}$$

For $N = -1$ one obtains from (8), (10), and (11)

$$\begin{aligned} &K_u^{-1}(z) \\ &= \frac{i}{2}\,\frac{\pi\omega^2}{\omega\,\mathrm{sinc}(2\omega i)}\,\frac{\mathrm{sinc}(\omega(z+i))\,\overline{\mathrm{sinc}(\omega(u+i))} \;-\; \overline{\mathrm{sinc}(\omega(\overline{z}+i))}\,\mathrm{sinc}(\omega(\overline{u}+i))}{\pi(z-\overline{u})} \\ &= \frac{\pi\omega\,(1+z\overline{u})\,\mathrm{sinc}(\omega(z-\overline{u})) - \coth(2\pi\omega)\,\cos(\pi\omega(z-\overline{u})) + \frac{\cos(\pi\omega(z+\overline{u}))}{\sinh(2\pi\omega)}}{\pi(1+z^2)\,(1+\overline{u}^2)} \end{aligned} \tag{14}$$

4 Complete interpolating sequences

The complete interpolating sequences for the Paley-Wiener space $B_0(\sigma)$ have been completely characterized [21, 28, 32, 36]. For an introduction, examples, tools for constructing new CISs from known ones, numerical experiments, and further references see [2]. We will now show that for $N > 0$, the complete interpolating sequences for $B_N(\sigma)$ are obtained by adding N points to a complete interpolating sequence for $B_0(\sigma)$, while for $N < 0$ one obtains a CIS for $B_N(\sigma)$ by deleting N points from a CIS for $B_0(\sigma)$.

Theorem 2. *Let $N \in \mathbb{Z}$ and $\Lambda = \{\lambda_k,\ k \in \mathbb{Z}\}$ be a CIS for $B_N(\sigma)$.*

1. *If $\mu \notin \Lambda$, then $\Lambda \cup \{\mu\}$ is a CIS for $B_{N+1}(\sigma)$.*
2. *If $\tilde{\lambda} \in \Lambda$, then $\Lambda \backslash \{\tilde{\lambda}\}$ is a CIS for $B_{N-1}(\sigma)$.*

Proof We first introduce some notation. Let $N \in \mathbb{Z}$ and let K_u^N denote the reproducing kernel for $B_N(\sigma)$ at the point $u \in \mathbb{C}$. Let $\{S_{\lambda_k}, k \in \mathbb{Z}\}$ be the Riesz basis of $B_N(\sigma)$ that is biorthogonal to $\{\tilde{K}_{\lambda_k}^N,\ k \in \mathbb{Z}\}$. In particular, one has $S_{\lambda_k}(\lambda_k) \neq 0$, and $S_{\lambda_k}(\lambda_l) = 0$ for $k \neq l$. For $\mu \in \mathbb{C}$ let $B_{N,\,\mu}(\sigma)$ denote the subspace of $B_N(\sigma)$ given by

$$B_{N,\,\mu}(\sigma) = \{f \in B_N(\sigma) :\ f(\mu) = 0\}. \tag{15}$$

$B_{N,\,\mu}(\sigma)$ is a closed subspace of $B_N(\sigma)$ and thus a RKHS in its own right. Furthermore, consider the linear operator

$$T_\mu^N :\ B_N(\sigma) \to B_{N+1,\,\mu}(\sigma), \quad T_\mu^N f(z) \;=\; (z-\mu)\,f(z).$$

Since $|t-\mu|^2/(1+t^2)$ is bounded on $\mathbb{R}$, T_μ^N is a bounded operator. If $T_\mu^N f = T_\mu^N g$, then $f(z)$ and $g(z)$ are two entire functions that must agree everywhere except possibly at the point $z = \mu$. Hence $f = g$, and it follows that T_μ^N is one-to-one. Let $g \in B_{N+1,\,\mu}(\sigma)$ and consider $f(z) = g(z)/(z-\mu)$. Since g satisfies (1) and $g(\mu) = 0$, f is an entire function that also satisfies (1), and is therefore σ-bandlimited. Since $\lim_{t\to\pm\infty} |t-\mu|^2/(1+t^2) = 1$, one has that $f \in B_N(\sigma)$, and thus $T_\mu^N f = g$. Hence T_μ^N is also onto, and thus both bounded and bijective. By the Bounded Inverse Theorem [5, p. 188], T_μ^N has a continuous inverse, and is thus a topological isomorphism.

To prove part 1, let $\mu \notin \Lambda$. Since $\{S_{\lambda_k}, k \in \mathbb{Z}\}$ is a Riesz basis of $B_N(\sigma)$, and T_μ^N is a topological isomorphism, it follows that $\{T_\mu^N S_{\lambda_k},\, k \in \mathbb{Z}\}$ is a Riesz basis of $B_{N+1,\,\mu}(\sigma)$ [15, p. 196, Lemma 7.10]. Furthermore, it is clear that K_μ^{N+1} lies in the orthogonal complement $(B_{N+1,\,\mu}(\sigma))^\perp$ of $B_{N+1,\,\mu}(\sigma)$. If g is an element of $(B_{N+1,\,\mu}(\sigma))^\perp$ that is orthogonal to K_μ^{N+1}, then $g(\mu) = \left\langle g,\, K_\mu^{N+1} \right\rangle_{B_{N+1}} = 0$, so g lies also in $B_{N+1,\,\mu}(\sigma)$. Hence $g \equiv 0$ and it follows that $(B_{N+1,\,\mu}(\sigma))^\perp$ is one-dimensional and thus equals the span of K_μ^{N+1}. Since $B_N(\sigma)$ contains functions $f(z)$ that do not vanish at $z = \mu$, there is $k_0 \in \mathbb{Z}$ such that $S_{\lambda_{k_0}}(\mu) \neq 0$. Let $\psi(z) = (z - \lambda_{k_0}) S_{\lambda_{k_0}}(z)$. Applying Lemma 8 in Section 8.3 with $H = B_{N+1}(\sigma)$, $V = B_{N+1,\,\mu}(\sigma)$, and $B = \{\psi\}$ yields that $\{\psi\} \cup \{T_\mu^N S_{\lambda_k},\, k \in \mathbb{Z}\}$ is a Riesz basis of $B_{N+1}(\sigma)$. Its biorthogonal Riesz basis has the form $\{\left(\overline{\psi(\mu)}\right)^{-1} K_\mu^{N+1}\} \cup \{\left(\overline{(\lambda_k - \mu) S_{\lambda_k}(\lambda_k)}\right)^{-1} K_{\lambda_k}^{N+1},\, k \in \mathbb{Z}\}$. Hence the unit vectors $\{\widetilde{K}_\mu^{N+1}\} \cup \{\widetilde{K}_{\lambda_k}^{N+1},\, k \in \mathbb{Z}\}$ are a Riesz basis as well [53, p. 26]. It now follows from Theorem B that $\Lambda \cup \{\mu\}$ is a CIS of $B_{N+1}(\sigma)$.

To prove part 2, consider the expansion $f = \sum_{\lambda\in\Lambda} \frac{f(\lambda)}{\|K_\lambda^N\|_{B_N}} S_\lambda$ of a function $f \in B_{N,\tilde{\lambda}}(\sigma)$ with respect to the Riesz basis $\{S_\lambda,\, \lambda \in \Lambda\}$ defined above. Since $f(\tilde{\lambda}) = 0$, the term for $\lambda = \tilde{\lambda}$ vanishes. Furthermore, for $\lambda \neq \tilde{\lambda}$ one has $S_\lambda(\tilde{\lambda}) = 0$, i.e., $S_\lambda \in B_{N,\tilde{\lambda}}(\sigma)$. It follows that $\{S_\lambda,\, \lambda \in \Lambda \backslash \{\tilde{\lambda}\}\}$ is a Riesz basis of $B_{N,\,\tilde{\lambda}}(\sigma)$. By the Bounded Inverse Theorem, the operator $T_{\tilde{\lambda}}^{N-1} : B_{N-1}(\sigma) \to B_{N,\,\tilde{\lambda}}(\sigma)$ has a bounded inverse $U = (T_{\tilde{\lambda}}^{N-1})^{-1}$, given by $Uf(z) = (z - \tilde{\lambda})^{-1} f(z)$, $f \in B_{N,\,\tilde{\lambda}}(\sigma)$. Since $U : B_{N,\,\tilde{\lambda}}(\sigma) \to B_{N-1}(\sigma)$ is bounded and bijective, it follows that $\{US_\lambda, \lambda \in \Lambda\backslash\{\tilde{\lambda}\}\}$ is a Riesz basis of $B_{N-1}(\sigma)$. The biorthogonal Riesz basis is given by $\{\overline{\left(\frac{\lambda - \tilde{\lambda}}{S_\lambda(\lambda)}\right)} K_\lambda^{N-1},\, \lambda \in \Lambda\backslash\{\tilde{\lambda}\}\}$. As before, it follows that the unit vectors $\{\widetilde{K}_\lambda^{N-1},\, \lambda \in \Lambda\backslash\{\tilde{\lambda}\}\}$ are also a Riesz basis of $B_{N-1}(\sigma)$, i.e., $\Lambda\backslash\{\tilde{\lambda}\}$ is a CIS of $B_{N-1}(\sigma)$. □

Lemma 1. *Let $\Lambda = \{\lambda_k,\, k \in \mathbb{Z}\}$ be a CIS for $B_N(\sigma)$, $N \in \mathbb{Z}$, where the λ_k are ordered as in Definition 3. A function*

$$\varphi(z) = c\,(z - \lambda_0) \prod_{k=1}^{\infty} \left(1 - \frac{z}{\lambda_k}\right) \left(1 - \frac{z}{\lambda_{-k}}\right), \quad c \neq 0, \tag{16}$$

is called a generating function for Λ. The infinite product converges conditionally, and $\varphi \in B_{N+1}(\sigma)$. The sampling functions $\varphi_k \in B_N(\sigma)$ in the expansion (5), i.e., the unique functions in $B_N(\sigma)$ satisfying $\varphi_k(\lambda_l) = \delta_{kl}$, $l \in \mathbb{Z}$, are given by

$$\varphi_k(z) = \frac{\varphi(z)}{\varphi'(\lambda_k)\,(z - \lambda_k)}, \quad k \in \mathbb{Z}. \tag{17}$$

Proof For $N = 0$ this result is well-known; see, e.g., [2, 21, 28]. For $N \neq 0$ the assertions for $\varphi(z)$ then follow from repeated application of Theorem 2. Since $\varphi \in B_{N+1}(\sigma)$, the functions φ_k given in (17) lie in $B_N(\sigma)$. It follows from (16) and (17) that for $k \in \mathbb{Z}$ the function φ_k satisfies $\varphi_k(\lambda_l) = \delta_{kl}$, $l \in \mathbb{Z}$. Since Λ is a CIS, φ_k is the only such function in $B_N(\sigma)$. □

Corollary 1. *If $\Lambda \subset \mathbb{C}$ is a CIS for both $B_N(\sigma)$ and $B_{N'}(\sigma')$, then $N = N'$ and $\sigma = \sigma'$.*

Proof Assume Λ is a CIS for $B_N(\sigma)$. According to Theorem 2, deleting N points from Λ if $N \geq 0$, or augmenting Λ with $|N|$ additional points if $N < 0$, yields a CIS Λ_0 of $B_0(\sigma)$. It follows that the generating function φ of Λ and the generating function φ_0 of Λ_0 differ only by a polynomial factor and therefore have the same exponential type. On the other hand, the exponential type of φ_0 is equal to σ [28, Theorem 1]. Hence the exponential type of φ is equal to σ. Repeating this reasoning with $B_{N'}(\sigma')$ establishes that $\sigma = \sigma'$. Now assume that Λ is a CIS of both $B_N(\sigma)$ and $B_{N'}(\sigma)$ with $N > N'$. Since Λ is a CIS of $B_N(\sigma)$, according to Theorem 2, deleting $N - N'$ points from Λ yields a CIS Λ' of $B_{N'}(\sigma)$. Hence both Λ and Λ' are complete interpolating sequences for $B_{N'}(\sigma)$. This, however, is not possible since it follows immediately from Definition 3 that deleting points from a CIS will destroy the property of being a CIS. It follows that one must have $N = N'$. □

As an example, consider a uniform sampling set of the form $\Lambda_{h,\alpha} = \alpha + h\mathbb{Z}$, with spacing $h > 0$ and shift $\alpha \in \mathbb{R}$. It is well-known that $\Lambda_{h,\alpha}$ is a CIS for $B_0(\sigma)$ with $\sigma = \pi/h$. It then follows from Corollary 1 that for $N \neq 0$ the space $B_N(\sigma)$ cannot have a CIS that is a uniform sampling set of the form $\Lambda_{h,\alpha}$.

5 Sampling expansions

Application of Theorem C to the spaces $B_N(\sigma)$ requires suitable upper and lower bounds for the norms $\| K^N_{\lambda_k} \|_{B_N}$. These are furnished by the following Lemma.

Lemma 2. *Let $N \in \mathbb{Z}$ and let K^N_u denote the reproducing kernel of $B_N(\sigma)$. Then there exist constants $0 < c_1 \leq c_2$ such that for all $u \in \mathbb{C}$*

$$c_1\,(1+|u|^2)^N\,\frac{e^{2\sigma|\,\mathrm{Im}\,u|}}{1+|\,\mathrm{Im}\,u|}\ \leq\ \|\,K^N_u\,\|^2_{B_N}\ \leq\ c_2\,(1+|u|^2)^N\,\frac{e^{2\sigma|\,\mathrm{Im}\,u|}}{1+|\,\mathrm{Im}\,u|}. \tag{18}$$

Proof For $N = 0$ this result is known (cf. [28]), and can be directly verified from $K^0_u(z) = \frac{\sin(\sigma(z-\overline{u}))}{\pi(z-\overline{u})}$. For $N \neq 0$ the assertion then follows from $|N|$ successive applications of Lemma 9 in Section 8.3. □

Lemma 2 implies in particular that $\| K^N_u \|_{B_N}$ is bounded on compact subsets of $\mathbb{C}$ for $N > 0$, and on horizontal strips of finite width for $N \leq 0$. Usually the sampling points λ_k are either all real or lie in a horizontal strip such that $\sup_{k\in\mathbb{Z}} |\,\mathrm{Im}\,\lambda_k| < \infty$. In this case the quantities $(1 + |\,\mathrm{Im}\,(\lambda_k)|)\,e^{-2\sigma|\,\mathrm{Im}\,\lambda_k|}$ are bounded away from both zero and infinity. One therefore has together with Lemma 2:

Lemma 3. *Let* $\Lambda = \{\lambda_k,\, k \in \mathbb{Z}\}$ *be a CIS of* $B_N(\sigma)$*. If* $\sup_{k\in\mathbb{Z}} |\operatorname{Im} \lambda_k| < \infty$*, then define* $\eta_k = 1$ *for all* $k \in \mathbb{Z}$*. Otherwise let* $\eta_k = (1 + |\operatorname{Im}(\lambda_k)|)\, e^{-2\sigma |\operatorname{Im} \lambda_k|}$*,* $k \in \mathbb{Z}$*. Then there are constants* $0 < c_1 \le c_2$ *such that*

$$c_1 \frac{\eta_k}{(1+|\lambda_k|^2)^N} \;\le\; \frac{1}{\| K^N_{\lambda_k} \|^2_{B_N}} \;\le\; c_2 \frac{\eta_k}{(1+|\lambda_k|^2)^N}.$$

We can now apply Theorem C with $H = B_N(\sigma)$ and obtain with the help of Lemmas 1–3 the following general sampling theorem.

Theorem 3. *Let* $N \in \mathbb{Z}$*, and* $\Lambda = \{\lambda_k,\, k \in \mathbb{Z}\}$ *be a CIS of* $B_N(\sigma)$ *with generating function* φ*. For* $k \in \mathbb{Z}$ *let* φ_k *be as in (17) and* η_k *as in Lemma 3. For any sequence of complex numbers* $(c_k)_{k\in\mathbb{Z}}$ *that satisfies* $\sum_{k\in\mathbb{Z}} \frac{|c_k|^2\, \eta_k}{(1+|\lambda_k|^2)^N} < \infty$*, the unique solution* $f \in B_N(\sigma)$ *of the interpolation problem* $f(\lambda_k) = c_k$*,* $k \in \mathbb{Z}$*, is given by*

$$f(z) = \sum_{k\in\mathbb{Z}} c_k\, \varphi_k(z). \tag{19}$$

Furthermore, every $f \in B_N(\sigma)$ *satisfies* $\sum_{k\in\mathbb{Z}} \frac{|f(\lambda_k)|^2\, \eta_k}{(1+|\lambda_k|^2)^N} < \infty$*, and thus admits the representation (19) with* $c_k = f(\lambda_k)$*. The series (19) converges unconditionally in the norm of* $B_N(\sigma)$*, uniformly on compact subsets of* $\mathbb{C}$ *for* $N > 0$*, and uniformly on horizontal strips of finite width for* $N \le 0$*.*

For $N = 0$ and $\sup_{k\in\mathbb{Z}} |\operatorname{Im} \lambda_k| < \infty$, Theorem 3 yields [2, Theorem 3], i.e., a generalized Paley-Wiener-Levinson theorem. For $N > 0$, Theorem 3 provides a sharpened version of [2, Theorem 4], with convergence in the norm instead of uniform convergence on compacts. The notation can be converted to that of [2] as follows. Let Λ be a CIS of $B_0(\pi)$ with generating function φ, and $\mu_1, \dots, \mu_M$ be distinct points in $\mathbb{C}\backslash\Lambda$. Then according to Theorem 2, $\tilde{\Lambda} = \Lambda \cup \{\mu_1, \dots, \mu_M\}$ is a CIS of $B_M(\pi)$ with generating function $\tilde{\varphi}(z) = P_\mu(z)\, \varphi(z)$, where $P_\mu(z) = \prod_{l=1}^M (z - \mu_l)$. Now apply Theorem 3 with $N = M$, $\Lambda = \tilde{\Lambda}$. The numerical experiments reported in [2] can therefore also serve as examples for the application of Theorem 3.

Apart from the classical Paley-Wiener space $B_0(\sigma)$, the space $B_1(\sigma)$ may have received the most attention in the applied literature. In a well-known paper, Zakai [54] recommended $B_1(\sigma)$ as the natural space for sampling theory and showed that for $f \in B_1(\sigma)$ the cardinal series (2) with $c_k = f(k/\omega)$ converges to f uniformly on compact subsets of $\mathbb{C}$ in case of oversampling, i.e., for $\omega > \sigma/\pi$ [54, Theorem 3]. Theorem 3 provides sampling expansions for $B_1(\sigma)$ that do not require such oversampling. For example, since $\cos(\sigma z)$ is a σ-sine-type function, its zeros are a CIS of $B_0(\sigma)$, cf. [2]. According to Theorem 2, adding the point $\lambda_0 = 0$ then gives a CIS Λ_1 of $B_1(\sigma)$. The elements of Λ_1 are the zeros of $\varphi(z) = z\, \cos(\sigma z)$. With $\sigma = \pi\omega$ this gives

$$\lambda_0 = 0, \text{ and } \lambda_k = \frac{1}{\omega}\left(k - \frac{1}{2}\right), \quad \lambda_{-k} = -\lambda_k, \quad k \in \mathbb{N}. \tag{20}$$

The functions φ_k can be determined via (17) or by observing that φ_k is the unique function in $B_1(\sigma)$ that satisfies $\varphi_k(\lambda_l) = \delta_{kl}$, $l \in \mathbb{Z}$. Since for $k, l \neq 0$ one has $\lambda_k - \lambda_l \in \frac{1}{\omega}\mathbb{Z}$ it is easily verified that

$$\varphi_0(z) = \cos(\pi\omega z)$$
$$\varphi_k(z) = \frac{z}{\lambda_k}\,\mathrm{sinc}(\omega(z - \lambda_k)), \quad k \neq 0.$$

Applying Theorem 3 now gives the following result.

Corollary 2. *Let $f \in B_1(\pi\omega)$, and λ_k, $k \in \mathbb{Z}$, as in (20). Then*

$$f(z) = f(0)\,\cos(\pi\omega z) \;+\; z \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{f(\lambda_k)}{\lambda_k}\,\mathrm{sinc}(\omega(z - \lambda_k)) \tag{21}$$

with unconditional convergence in the norm of $B_1(\pi\omega)$ and uniform convergence on compact subsets of $\mathbb{C}$.

How does the sampling set Λ_1 compare to the classical uniform sampling set $\Lambda_0 = \{k/\omega,\ k \in \mathbb{Z}\}$, which is a CIS for $B_0(\pi\omega)$ and leads to Theorem A and the cardinal series (2)? Both Λ_0 and Λ_1 are symmetric real sampling sets. Λ_1 requires one additional sample, and (21) is not an orthogonal expansion. On the other hand, for functions $f \in B_0(\sigma)$ the expansion (21) can be expected to converge faster than the cardinal series due to the additional factor $1/\lambda_k$; see [2] for numerical experiments in similar cases. In addition, (21) can be used to reconstruct functions from the larger space $B_1(\sigma)$. This addresses a well-known drawback of the cardinal series, namely, its instability with regard to bounded perturbations of the samples of the form ϵ_k, $k \in \mathbb{Z}$, with $|\epsilon_k| < \epsilon < \infty$ for all k. The cardinal series may diverge in such cases; see, e.g., [6, 13, 23, 25, 34, 54]. Consider, for example, a perturbation of the form

$$\epsilon_k = \begin{cases} 0 & \text{for } k \leq 0 \\ \epsilon\,(-1)^k & \text{for } k > 0, \end{cases} \qquad k \in \mathbb{Z},\ \epsilon > 0. \tag{22}$$

Then for every $\epsilon > 0$ the cardinal series $\sum_{k\in\mathbb{Z}} \epsilon_k\,\mathrm{sinc}(\omega z - k) = \epsilon\,\sin(\pi\omega z)\sum_{k=1}^{\infty} \frac{1}{\pi(\omega z - k)}$ diverges for every $z \notin \frac{1}{\omega}\mathbb{Z}$. On the other hand, when using a CIS for $B_N(\sigma)$ with $N > 0$, one has the following stability result.

Corollary 3. *Let $N \geq 1$ and $\Lambda = \{\lambda_k,\ k \in \mathbb{Z}\}$ be a CIS for $B_N(\sigma)$ with generating function φ. Let c_k be such that $|c_k| \leq \epsilon < \infty$ for all $k \in \mathbb{Z}$. Then $g = \sum_{k\in\mathbb{Z}} c_k\,\varphi_k$ is the unique function in $B_N(\sigma)$ such that $g(\lambda_k) = c_k$, $k \in \mathbb{Z}$. One has $\|\, g \,\|_{B_N} \leq C\,\epsilon$, and therefore also $|g(u)| \leq C_S\,\epsilon$ for $u \in S$, for any compact subset S of $\mathbb{C}$.*

Proof Note that the constant function $f(z) = 1$ is in $B_N(\sigma)$ for all $N \geq 1$. It then follows from (4) that $\sum_{k\in\mathbb{Z}} \frac{1}{\|K^N_{\lambda_k}\|^2_{B_N}} < \infty$, which in turn implies $\sum_{k\in\mathbb{Z}} \frac{|c_k|^2}{\|K^N_{\lambda_k}\|^2_{B_N}} < \infty$. Theorem C then gives that $g(z) = \sum_{k\in\mathbb{Z}} c_k\, \varphi_k(z)$ is the unique function in $B_N(\sigma)$ that satisfies $g(\lambda_k) = c_k$, $k \in \mathbb{Z}$. Now (4) implies that $\| \, g \, \|_{B_N}^2 \leq \, C_1 \sum_{k\in\mathbb{Z}} \frac{|c_k|^2}{\|K_{\lambda_k}^N\|_{B_N}^2} \leq \left(C_1 \sum_{k\in\mathbb{Z}} \frac{1}{\|K_{\lambda_k}^N\|_{B_N}^2} \right) \epsilon^2 = C\epsilon^2$. According to Lemma 2, $\| \, K_u^N \, \|_{B_N}$ is bounded for $u \in S$, S compact. Hence $|g(u)| \leq \| \, K_u^N \, \|_{B_N} \, \| \, g \, \|_{B_N} \leq C_S \, \epsilon$ for $u \in S$. □

One can also use (21) to explicitly find the functions $g \in B_1(\pi\omega)$ that satisfy $g(k/\omega) = \epsilon_k$ for ϵ_k as in (22). This provides a concrete example of a bandlimited function that has bounded samples but is unbounded on the real line.

Corollary 4. *Let $\omega > 0$ and $g \in B_1(\pi\omega)$ such that $g(k/\omega) = \epsilon_k$, $k \in \mathbb{Z}$, with ϵ_k as in (22). Then g has the form*

$$g(z) \; = \; \frac{\epsilon}{\pi} \, \sin(\pi\omega z) \, \psi(1-\omega z) \; + \; C \, \sin(\pi\omega z), \tag{23}$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function, and $C = g\left(\frac{1}{2\omega}\right) - \frac{\epsilon}{\pi}\, \psi\left(\frac{1}{2}\right)$. In particular, g is unbounded on the real line.

Proof The representation (23) follows from applying (21) to the function $f(z) = g(z + \frac{1}{2\omega})$, using the identity

$$\sum_{k=1}^{\infty} \frac{1}{(k-t)(u-k)} = \frac{\psi(1-u) - \psi(1-t)}{u-t}, \quad u, t \not\in \mathbb{N},$$

with $t = \frac{1}{2}$ and $u = \omega z + \frac{1}{2}$, and then setting $g(z) = f(z - \frac{1}{2\omega})$. The reflection formula

$$\psi(1-x) = \psi(x) + \pi \cot(\pi x)$$

and the inequalities

$$\log(x) - \frac{1}{x} \; \leq \; \psi(x) \; \leq \; \log(x) - \frac{1}{2x}, \quad x > 0,$$

then yield that g is unbounded on the real axis, with logarithmic growth of its local maxima as $x \to \infty$. □

Levin [27, p. 157] obtained necessary and sufficient conditions for a bounded sequence of samples $(c_k)_{k\in\mathbb{Z}}$ such that the functions $g \in B_1(\pi)$ that satisfy $g(k) = c_k$, $k \in \mathbb{Z}$, do remain bounded on the real line.

6 Orthogonal sampling expansions

We will now determine the orthogonal sampling expansions for $B_N(\sigma)$, $N \geq 0$. The following theorem establishes that for $N \in \mathbb{N} \cup \{0\}$ and a given point $x \in \mathbb{R}$, there exists a real-valued CIS of $B_N(\sigma)$ that contains the point x and leads to an orthogonal sampling expansion, and that all orthogonal expansions are of this form.

Theorem 4. *Let $N \geq 0$, $\sigma > 0$, and $x \in \mathbb{R}$. For $k \in \mathbb{Z}$ let $\lambda_k \in \mathbb{R}$ be the solution of the equation*

$$\sigma\,\lambda_k \;+\; N\,\arctan(\lambda_k) = \alpha_x \;+\; k\pi, \tag{24}$$

where $\alpha_x \in [0,\pi)$ is given by $\alpha_x = (\sigma x + N \arctan x) \mod \pi$. Then $\Lambda = \{\lambda_k,\, k \in \mathbb{Z}\}$ is a CIS of $B_N(\sigma)$ that contains x and gives the orthogonal sampling expansion

$$f(z) = \sum_{k\in\mathbb{Z}} \frac{f(\lambda_k)}{\|\, K^N_{\lambda_k} \|_{B_N}}\, \widetilde{K}^N_{\lambda_k}(z) \;=\; \sum_{k\in\mathbb{Z}} \frac{f(\lambda_k)}{K^N_{\lambda_k}(\lambda_k)}\, K^N_{\lambda_k}(z), \quad f \in B_N(\sigma), \tag{25}$$

with the reproducing kernel K^N_u of $B_N(\sigma)$ as in (8), and thus

$$K^N_t(t) \;= \|\, K^N_t \,\|^2_{B_N} = (1+t^2)^N \left(\frac{\sigma}{\pi} \,+\, \frac{N}{\pi\,(1+t^2)}\right), \quad t \in \mathbb{R}. \tag{26}$$

The series converges in the norm of $B_N(\sigma)$ as well as uniformly on compact subsets of $\mathbb{C}$. On the other hand, no CIS of $B_N(\sigma)$ that contains a point $\lambda_k \notin \mathbb{R}$ yields an orthogonal sampling expansion.

The proof of the theorem will be presented in Section 8.4. The orthonormality and completeness of the $\widetilde{K}^N_{\lambda_k}$ imply the following immediate corollary.

Corollary 5. *Let $N \geq 0$, $x \in \mathbb{R}$, and λ_k, $k \in \mathbb{Z}$, as in (24). Then for any $f \in B_N(\sigma)$,*

$$\sum_{k\in\mathbb{Z}} \frac{|f(\lambda_k)|^2}{\|\, K^N_{\lambda_k} \|^2_{B_N}} = \sum_{k\in\mathbb{Z}} \left| \left\langle f,\, \widetilde{K}^N_{\lambda_k} \right\rangle_{B_N} \right|^2 = \|\, f \,\|^2_{B_N} \;=\; \int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)^N}\, dt. \tag{27}$$

For $N \geq 1$ and $f(t) = 1$, (26) and (27) imply the identity

$$\sum_{k\in\mathbb{Z}} \frac{1}{\|\, K^N_{\lambda_k} \|^2_{B_N}} = \sum_{k\in\mathbb{Z}} \frac{\pi}{(1+\lambda_k^2)^{N-1}(N + \sigma\,(1+\lambda_k^2))} \;=\; \frac{\sqrt{\pi}\;\Gamma(N-\frac{1}{2})}{\Gamma(N)}. \tag{28}$$

We now consider examples for $N = 0, 1$. For $N = 0$ and $\sigma = \pi\omega$, equation (26) yields $K^0_t(t) = \frac{\sigma}{\pi} = \omega$ for all $t \in \mathbb{R}$. Choosing $x = 0$, equation (24) then gives $\lambda_k = \frac{\pi}{\sigma}\, k = \frac{k}{\omega}$, $k \in \mathbb{Z}$, so the sampling set is $\Lambda = \frac{1}{\omega}\,\mathbb{Z}$. Then the expansion (25) reads

$$f(z) = \sum_{k\in\mathbb{Z}} \frac{f(k/\omega)}{\omega}\, K^0_{k/\omega}(z) = \sum_{k\in\mathbb{Z}} f(k/\omega)\,\mathrm{sinc}(\omega z - k), \quad f \in B_0(\pi\omega).$$

That is, one obtains the cardinal series (2) with $c_k = f(k/\omega)$, as expected. For $x \neq 0$ one has $\lambda_k = \frac{\pi k + \alpha_x}{\pi\omega}$ with $\alpha_x = (\pi\omega x) \mod \pi$. Hence the sampling set containing x is simply a shifted copy of $\frac{1}{\omega}\,\mathbb{Z}$.

Now consider $N = 1$, $\sigma = \pi\omega$, and let $x = 0$. Then the sampling points λ_k are determined by $\lambda_k = \frac{k}{\omega} - \frac{\arctan(\lambda_k)}{\pi\omega}$. One has $\lambda_0 = 0$ and $\lambda_{-k} = -\lambda_k$, that is, the sequence is symmetric around the origin. For large positive k, $\arctan(\lambda_k) \simeq \frac{\pi}{2}$, so λ_k will be very close to $\frac{1}{\omega}\left(k - \frac{1}{2}\right)$. Equation (26) now yields $K_t^1(t) = \omega(1+t^2) + \frac{1}{\pi}$, $t \in \mathbb{R}$. Inserting this and the kernel (12) into the expansion (25) gives the following orthogonal sampling expansion:

Corollary 6. *Let $\omega > 0$ and $f \in B_1(\pi\omega)$. Let $\lambda_k \in \mathbb{R}$ be given by*

$$\pi\omega\lambda_k \ + \arctan(\lambda_k) = \pi k, \quad k \in \mathbb{Z}.$$

Then f admits the orthogonal sampling expansion

$$f(z) = \sum_{k\in\mathbb{Z}} \frac{\pi f(\lambda_k)}{1+\pi\omega\,(1+\lambda_k^2)} \left(\omega(1+\lambda_k z)\,\mathrm{sinc}(\omega(z-\lambda_k)) \ + \ \frac{\cos(\pi\omega(z-\lambda_k))}{\pi}\right) \tag{29}$$

The expansion converges in the norm of $B_1(\pi\omega)$ and uniformly on compact subsets of $\mathbb{C}$.

7 Sampling expansions with finitely many derivative samples

In this section we consider sampling theorems that include a finite number of samples of derivatives of f. The set-up is as follows. Let $f \in B_N(\sigma)$ with $N = M + L$, $L \geq 0$. The set of sampling points is a CIS Λ of $B_M(\sigma)$. At some of the sampling points one also takes samples of derivatives of f such that the total number of derivative samples is equal to L. We begin with the case where the derivative samples are all taken at the same point $\lambda_m \in \Lambda$. The following theorem generalizes a theorem for uniform sampling by Hoskins and Sousa Pinto [19, 20], which in turn generalized earlier results by Valiron [47], Pfaffelhuber [37], and Lee [26, Theorem 1]. It also generalizes a result by Schmeisser [9, p. 96] for nonuniform sampling. The theorem includes an auxiliary function $\eta \in B_{M+1}(\sigma)$ that can be freely chosen subject to the condition $\eta(\lambda_m) \neq 0$. The polynomial P occurring in the theorem is the order $L-1$ Taylor polynomial about λ_m of the function $f(z)/\eta(z)$.

Theorem 5. *Let $M \in \mathbb{Z}$, $\Lambda = \{\lambda_k,\, k \in \mathbb{Z}\}$ a CIS of $B_M(\sigma)$ with generating function φ, $m \in \mathbb{Z}$, and $\eta \in B_{M+1}(\sigma)$ such that $\eta(\lambda_m) \neq 0$. Let $f \in B_{M+L}(\sigma)$, and $L \in \mathbb{N} \cup \{0\}$. For $l = 0, \ldots, L$ let*

$$c_l \;=\; \frac{1}{l!}\,\frac{d^l}{dz^l}\,\frac{f(z)}{\eta(z)}\bigg|_{z=\lambda_m}\,, \quad \text{and let} \quad P(z) \;=\; \sum_{l=0}^{L-1} c_l\,(z-\lambda_m)^l.$$

Then

$$f(z) = P(z)\eta(z) \;+\; (z-\lambda_m)^L g(z) \quad \text{with } g \in B_M(\sigma). \tag{30}$$

Furthermore,

$$\begin{aligned} f(z) &= P(z)\,\eta(z) \;+\; c_L\,\eta(\lambda_m)\,(z-\lambda_m)^L \varphi_m(z) \\ &\quad + \;(z-\lambda_m)^L \sum_{k \in \mathbb{Z}\setminus\{m\}} \frac{f(\lambda_k) - P(\lambda_k)\eta(\lambda_k)}{(\lambda_k-\lambda_m)^L}\,\varphi_k(z), \end{aligned} \tag{31}$$

with φ_k as in (17). The series converges in the norm of $B_M(\sigma)$, and therefore also in the norm of $B_{M+L}(\sigma)$ as well as uniformly on compact subsets of $\mathbb{C}$.

Proof Assume that $f \in B_{M+L}(\sigma)$. Let

$$\begin{aligned} g(z) &= (z-\lambda_m)^{-L}\,(f(z) - P(z)\eta(z)) \\ &= (z-\lambda_m)^{-L}\left(\frac{f(z)}{\eta(z)} - P(z)\right)\eta(z) \end{aligned} \tag{32}$$

It is apparent from the second equation that g does not have a singularity at λ_m, since P is the order $L-1$ Taylor polynomial of $f(z)/\eta(z)$ about λ_m. The first equation then implies that $g \in B_M(\sigma)$. This establishes (30). Now (31) follows from applying Theorem 3 to g and observing that $g(\lambda_m) = \lim_{z\to\lambda_m} g(z) = c_L\,\eta(\lambda_m)$. □

It is worthwhile to note that since (30) obviously implies that $f \in B_{M+L}(\sigma)$, it follows that $f \in B_{M+L}(\sigma)$ if and only if (30) holds. For $M = 0$ and $\eta(z) = 1$, (30) yields a characterization of $B_L(\sigma)$ obtained by Lee [26, Theorem 1], and, for the special case $L = 1$, already by Zakai [54, Theorem 1].

Most prior results considered the case $M = 0$ and uniform sampling, that is, $\Lambda = \frac{1}{\omega}\mathbb{Z}$, $\omega = \sigma/\pi$, $\varphi(z) = \sin(\pi\omega z)$, $m = \lambda_m = 0$. Hoskins and Sousa Pinto [19], [20, Theorem 3.21] obtained (30) and (31) for this case using the theory of distributions, showing uniform convergence on compacts, with $\eta \in B_1(\sigma)$ such that $\widehat{\eta}$ is a distribution of order zero. Choosing $\eta = \cos(\pi\omega t)$ gives an expansion found by Pfaffelhuber [37]. Choosing $\eta = \operatorname{sinc}(\omega t)$ and $\omega = 1$ yields an expansion found by Valiron [47, p. 204]; cf. [9, p. 96] . With regard to nonuniform sampling, for $M = m = 0$, $\sigma = \pi$, φ a π-sine-type function with a zero at $\lambda_0 = 0$, and $\eta = \varphi_0$, (31) yields an interpolation formula derived by Schmeisser [9, p. 96] with a contour integral method.

For $M = 0$, $L = 1$, and $\eta = \varphi_m$, Theorem 5 yields the following generalized Valiron-Tschakaloff formula, which extends and sharpens our earlier result [1, Theorem 8]. Note that in this case $\eta(\lambda_k) = \delta_{km}$, $k \in \mathbb{Z}$, and $\eta'(\lambda_m) = \varphi''(\lambda_m)/(2\varphi'(\lambda_m))$.

Corollary 7. *Let $f \in B_1(\sigma)$, $\Lambda = \{\lambda_k,\, k \in \mathbb{Z}\}$ a CIS of $B_0(\sigma)$ with generating function φ, and $m \in \mathbb{Z}$. Then*

$$\begin{aligned} f(z) \;&=\; f'(\lambda_m)\frac{\varphi(z)}{\varphi'(\lambda_m)} \;+\; f(\lambda_m)\left(\varphi_m(z) - \frac{\varphi''(\lambda_m)}{2(\varphi'(\lambda_m))^2}\,\varphi(z)\right) \\ &\quad+\; (z-\lambda_m)\sum_{k\in\mathbb{Z}\setminus\{m\}} \frac{f(\lambda_k)}{\lambda_k-\lambda_m}\,\varphi_k(z). \end{aligned} \tag{33}$$

The series converges in the norm of $B_1(\sigma)$ and uniformly on compact subsets of $\mathbb{C}$.

We now consider the case where derivatives may be taken at more than one point. Let $(\mu_k)_{k\in\mathbb{Z}}$ be a sequence of nonnegative integers such that all but finitely many of the μ_k are zero. Let $\Lambda = \{\lambda_k,\, k \in \mathbb{Z}\}$ be a CIS of $B_M(\sigma)$ with generating function φ. Assume the available samples are

$$f^{(l)}(\lambda_k), \quad k \in \mathbb{Z},\ l = 0,\dots,\mu_k.$$

We seek to reconstruct a function $f \in B_N(\sigma)$, $N = M + \sum_{k\in\mathbb{Z}} \mu_k$ from these samples. We begin by constructing the sampling functions for these data. That is, for $k \in \mathbb{Z}$, $l = 0,\dots,\mu_k$ we seek functions $\phi_{kl} \in B_N(\sigma)$ such that

$$\phi_{kl}^{(j)}(\lambda_n) = \delta_{kn}\delta_{jl} \qquad \text{for } n \in \mathbb{Z},\ j = 0,\dots,\mu_n. \tag{34}$$

To this end, we define

$$\phi(z) = c\,(z-\lambda_0)^{1+\mu_0}\prod_{k=1}^{\infty}\left(1-\frac{z}{\lambda_k}\right)^{1+\mu_k}\left(1-\frac{z}{\lambda_{-k}}\right)^{1+\mu_{-k}}, \quad c \neq 0, \tag{35}$$

$$\psi_k(z) = (z-\lambda_k)^{-1-\mu_k}\,\phi(z), \quad k \in \mathbb{Z} \tag{36}$$

If all μ_k are equal to zero, then (35) reduces to a generating function (16) for the CIS Λ. This also implies that $\phi \in B_{N+1}(\sigma)$. It is then straightforward to verify that for each $k \in \mathbb{Z}$ the functions $\phi_{kl} \in B_N(\sigma)$, $l = 0,\dots,\mu_k$ can be successively computed by means of

$$\phi_{k\mu_k}(z) = \frac{(z-\lambda_k)^{\mu_k}}{\mu_k!\,\psi_k(\lambda_k)}\,\psi_k(z), \tag{37}$$

$$\begin{aligned} \phi_{kl}(z) &= \frac{(z-\lambda_k)^{l}}{l!\,\psi_k(\lambda_k)}\,\psi_k(z) \;-\; \sum_{m=l+1}^{\mu_k}\binom{m}{l}\frac{\psi_k^{(m-l)}(\lambda_k)}{\psi_k(\lambda_k)}\,\phi_{km}(z), \\ &\qquad l = \mu_k-1, \mu_k-2, \dots, 0. \end{aligned} \tag{38}$$

Theorem 6. *Let $M \in \mathbb{Z}$, $\Lambda = \{\lambda_k,\, k \in \mathbb{Z}\}$ a CIS of $B_M(\sigma)$, $(\mu_k)_{k\in\mathbb{Z}}$ a sequence of nonnegative integers with $\sum_{k\in\mathbb{Z}} \mu_k < \infty$, $N = M + \sum_{k\in\mathbb{Z}} \mu_k$, and $f \in B_N(\sigma)$. Then*

$$f(z) = \sum_{k\in\mathbb{Z}} \sum_{l=0}^{\mu_k} f^{(l)}(\lambda_k)\, \phi_{kl}(z), \tag{39}$$

with ϕ_{kl} as in (34), (37), (38). The expansion converges unconditionally in the norm of $B_N(\sigma)$ and uniformly on compact subsets of $\mathbb{C}$.

The theorem will be proved in Section 8.5. As an example we consider uniform sampling on a set $\Lambda = d\,\mathbb{Z}$, $d = \pi/\sigma$, with additional samples of the first derivatives at the points $\lambda_{-1} = -d$ and $\lambda_1 = d$. This yields

Corollary 8. *Let $\sigma > 0$, $d = \pi/\sigma$ and $f \in B_2(\sigma)$. Then*

$$\begin{aligned} f(z) = {} & \frac{z^2-d^2}{d^2} \sum_{\substack{k\in\mathbb{Z}\\ |k|\neq 1}} \frac{f(kd)}{k^2-1} \operatorname{sinc}\left(\frac{z-kd}{d}\right) \\ & + \; f(d)\, \frac{z+d}{2d} \left(\frac{\sin(\pi z/d)}{2\pi} + \operatorname{sinc}\left(\frac{z-d}{d}\right)\right) \\ & + \; f(-d)\, \frac{z-d}{2d} \left(\frac{\sin(\pi z/d)}{2\pi} - \operatorname{sinc}\left(\frac{z+d}{d}\right)\right) \\ & + \; \frac{f'(-d)(z-d) \; - \; f'(d)(z+d)}{2\pi} \sin(\pi z/d). \end{aligned} \tag{40}$$

The expansion converges in the norm of $B_2(\sigma)$ as well as uniformly on compact subsets of $\mathbb{C}$.

Proof We have $N = 2$, $M = 0$, $\Lambda = d\,\mathbb{Z}$, $d = \pi/\sigma$, $\mu_{-1} = \mu_1 = 1$, and $\mu_k = 0$ for $|k| \neq 1$. This gives

$$\phi(z) = (z^2 - d^2)\, \sin(\pi z/d).$$

For $|k| \neq 1$ one has $\mu_k = 0$ and thus

$$\psi_k(z) = \frac{\phi(z)}{z-dk} = (-1)^k \frac{\pi}{d} (z^2-d^2) \operatorname{sinc}((z-dk)/d).$$

Hence

$$\phi_{k,0}(z) = \frac{\psi_k(z)}{\psi_k(dk)} \;=\; \frac{z^2-d^2}{d^2(k^2-1)} \operatorname{sinc}((z-dk)/d), \quad |k| \neq 1.$$

For $k = 1$ one has $\lambda_1 = d$ and $\mu_1 = 1$ which gives

$$\psi_1(z) = \frac{\phi(z)}{(z-d)^2} \;=\; (z+d)\, \frac{\sin(\pi z/d)}{z-d} \;=\; -\frac{\pi}{d}\,(z+d) \operatorname{sinc}((z-d)/d)$$

with $\psi_1(d) = -2\pi$ and $\psi_1'(d) = -\pi/d$. Then (37) gives

$$\phi_{1,1}(z) = \frac{z-d}{\psi_1(d)}\, \psi_1(z) \;=\; -\frac{z+d}{2\pi} \sin(\pi z/d).$$

From (38) one then obtains

$$\phi_{1,0}(z) \;=\; \frac{\psi_1(z)}{\psi_1(d)} - \frac{\psi_1'(d)}{\psi_1(d)}\phi_{1,1}(z) \;=\; \frac{z+d}{2d}\left(\frac{\sin(\pi z/d)}{2\pi} + \mathrm{sinc}((z-d)/d)\right).$$

Similarly, for $k = -1$ one has $\lambda_{-1} = -d$, and

$$\phi_{-1,1}(z) \;=\; \frac{z-d}{2\pi}\sin(\pi z/d), \quad \phi_{-1,0}(z) \;=\; \frac{z-d}{2d}\left(\frac{\sin(\pi z/d)}{2\pi} - \mathrm{sinc}((z+d)/d)\right).$$

The result now follows from applying Theorem 6. □

To conclude this section, we note that a sampling theorem for $B_{2N}(\pi)$, $N > 0$, found by Walter [49, Corollary 3.3] can be obtained from Theorem 6 by choosing $M = 2$, $\Lambda = \Lambda_0 \cup \{i, -i\}$, $\Lambda_0 = \{t_k,\, k \in \mathbb{Z}\} \subset \mathbb{R}$ such that $\sup_k |t_k - k| < 1/4$ and $t_k = -t_{-k}$, $k \geq 0$, (the first condition implies that Λ_0 is a CIS of $B_0(\pi)$), and including the derivative samples $f^{(j)}(\pm i)$, $j = 1, \ldots, N-1$.

8 Proofs

8.1 de Branges spaces

In order to prove Theorem 1 we need the following concepts, definitions, and results. In [11] de Branges introduced and deeply analyzed a class of reproducing kernel Hilbert spaces of entire functions, now called de Branges spaces. The definition will use the following concepts from the theory of analytic functions:

Definition 4. *For f an entire function let f^* be the entire function given by*

$$f^*(z) = \overline{f(\overline{z})}. \tag{41}$$

A function f is of bounded type in the upper half-plane if it is the ratio of two analytic functions that are bounded in the upper half-plane.

The mean type of a function g in the upper half-plane ([11, p. 26]) is given by

$$h = \limsup_{y\to\infty} \frac{\log|g(iy)|}{y}.$$

Definition 5. *([11, p. 50]) Let E be an entire function which satisfies the inequality $|E(z)| > |E(\overline{z})|$ in the upper half-plane $\mathrm{Im}\, z > 0$. The space $H(E)$ associated with E consists of entire functions f such that*

$$\|f\|_E^2 \;=\; \int_{\mathbb{R}} |f(t)/E(t)|^2\, dt \;<\; \infty,$$

and such that both ratios $f(z)/E(z)$ and $f^*(z)/E(z)$ are of bounded type and of nonpositive mean type in the upper half-plane. $H(E)$ is equipped with the inner product

$$\langle f,\, g\rangle_E = \int_{\mathbb{R}} f(t)\overline{g(t)}/|E(t)|^2\, dt.$$

De Branges showed [11, p. 50–51, Theorem 19] that $H(E)$ is a RKHS with reproducing kernel

$$K_u(z) = \frac{i}{2}\frac{E(z)\overline{E(u)} - E^*(z)\overline{E^*(u)}}{\pi(z-\overline{u})}. \tag{42}$$

From equation (42) one directly obtains that

$$K_{\overline{u}} = K_u^*, \tag{43}$$
$$K_z(u) = \overline{K_u(z)}, \quad \text{and} \tag{44}$$
$$K_z(z) = \frac{|E(z)|^2 - |E(\overline{z})|^2}{4\pi \ \mathrm{Im}\ z} \ = K_{\overline{z}}(\overline{z}). \tag{45}$$

Different functions $E(z)$ may generate the same space $H(E)$. For example, $E(z)$ can be replaced by $e^{i\alpha}E(z)$ for some $\alpha \in \mathbb{R}$ without change of the corresponding space ([11, p. 55]). For a full classification of the functions $E(z)$ that give rise to the same space $H(E)$ see [51, p. 496].

8.2 Proof of Theorem 1.

The proof of Theorem 1 will be presented in several steps. We first establish that the spaces $B_N(\sigma)$, $N \in \mathbb{Z}$ are Hilbert spaces and then prove the theorem for $N \geq 0$. We then show that the $B_N(\sigma)$ for $N < 0$ are reproducing kernel Hilbert spaces (Lemma 6), and establish recursion relations that link the reproducing kernels of $B_N(\sigma)$ and $B_{N+1}(\sigma)$ (Lemma 7). Finally we prove Theorem 1 for $N < 0$. The following fundamental result from the theory of reproducing kernel Hilbert spaces will also be needed. It states that an RKHS is determined by its reproducing kernel.

Lemma 4. *([35, Proposition 2.3], cf. [4, p.344, item (4)]) Let H_r, $r = 1, 2$ be RKHSs on X with kernels K^r, $r = 1, 2$. Let $\| \cdot \|_r$ denote the norm of H_r. If $K_u^1(z) = K_u^2(z)$ for all $u, z \in X$, then $H_1 = H_2$ and $\| f \|_1 = \| f \|_2$ for every f.*

Lemma 5. *The spaces $B_N(\sigma)$, $N \in \mathbb{Z}$, are Hilbert spaces.*

Proof For $N \in \mathbb{Z}$ let $\widetilde{H}^{-N}(\sigma)$ denote the Sobolev space of tempered distributions f with support contained in $[-\sigma, \sigma]$ such that $\| f \|^2_{\widetilde{H}^{-N}} = \int_R (1+t^2)^{-N} |\hat{f}(t)|^2 \, dt < \infty$; cf. Triebel [44, p. 317]. It follows that $f \in \widetilde{H}^{-N}(\sigma)$ if and only if $\hat{f} \in B_N(\sigma)$, and $\| f \|_{\widetilde{H}^{-N}} = \| \hat{f} \|_{B_N}$. That is, the Fourier transform is an isometric isomorphism between $\widetilde{H}^{-N}(\sigma)$ and $B_N(\sigma)$. Since $\widetilde{H}^{-N}(\sigma)$ is a Hilbert space, so is $B_N(\sigma)$. □

Proof of Theorem 1 for $N \geq 0$. We will show that $B_N(\sigma)$, $N \geq 0$, is a de Branges space $H(E)$ with $E(z) = E_N(z) = (1-iz)^N e^{-i\sigma z} = i^{-N}(z+i)^N e^{-i\sigma z}$. Clearly, E_N is an entire function that has no zeros in the upper half-plane. For $z = x+iy$ in the upper half-plane, i.e., $y > 0$, one has

$$\begin{aligned}|E_N(z)| &= (x^2 + (1+y)^2)^{N/2}\, e^{\sigma\, y} \\ &> (x^2 + (1-y)^2)^{N/2}\, e^{-\sigma y} \;=\; |E_N(\overline{z})|.\end{aligned}$$

For $t \in \mathbb{R}$ one has $|E_N(t)|^2 = (1+t^2)^N$. Hence $B_N(\sigma)$ and $H(E_N)$ have the same inner product and norm. Let $f \in B_N(\sigma)$. It follows from [26, Theorem 1] that there is $C > 0$ such that for all $z \in \mathbb{C}$

$$|f(z)| \le C\,(1+|z|)^N\, e^{\sigma\,|\,\mathrm{Im}\; z|}, \tag{46}$$

and $|f^*(z)| = |f(\overline{z})|$ then obviously satisfies the same inequality. Since $|1 - iz|^2 = 1 + |z|^2 + 2\ \mathrm{Im}\ z$, one has for z in the upper half-plane

$$\left|\frac{f(z)}{E_N(z)}\right| \le C \left(\frac{1+|z|}{\sqrt{1+|z|^2}}\right)^N .$$

Hence $f(z)/E_N(z)$ is bounded in the upper half-plane and therefore both of bounded type and of nonpositive mean type in the upper half-plane; cf. [11, p. 51]. The same is true for $f^*(z)/E_N(z)$. It follows that $f \in H(E_N)$. Hence $B_N(\sigma)$ is a subspace of $H(E_N)$. The reproducing kernel of $H(E_N)$ is given by

$$\begin{aligned}K_u^N(z) &= \frac{i}{2}\,\frac{E_N(z)\overline{E_N(u)} - E_N^*(z)\overline{E_N^*(u)}}{\pi(z-\overline{u})} \\ &= \frac{i}{2}\,\frac{(z+i)^N\,(\overline{u}-i)^N\,e^{-i\sigma(z-\overline{u})} \;-\; (z-i)^N\,(\overline{u}+i)^N\,e^{i\sigma(z-\overline{u})}}{\pi\,(z-\overline{u})}\end{aligned} \tag{47}$$

For fixed u the function in the numerator of (47) is a function in $B_{N+1}(\sigma)$ that vanishes at $z = \overline{u}$. Hence $K_u^N \in B_N(\sigma)$ (cf. the proof of Theorem 2). Thus $B_N(\sigma)$ is a RKHS with reproducing kernel K_u^N. It now follows from Lemma 4 that $B_N(\sigma) = H(E_N)$. Furthermore, it is straightforward to verify from (8) and (9) that E_N satisfies the recursion (10) for $N \ge 0$. This proves Theorem 1 for the case $N \ge 0$. □

Lemma 6. *$B_N(\sigma)$ is a reproducing kernel Hilbert space for $N < 0$.*

Proof We proceed by induction. The claim holds for $N = 0$. Assume that $B_N(\sigma)$ is a RKHS for some $N \le 0$. Consider the operator $T_i : B_{N-1}(\sigma) \to B_N(\sigma)$, $T_i g(z) = (z-i)\,g(z)$. It is straightforward to check that $\|\, T_i g \,\|_{B_N} = \|\, g \,\|_{B_{N-1}}$. Let $g \in B_{N-1}(\sigma)$, $z \in \mathbb{C}$, $z \ne i$, and let K_z^N denote the reproducing kernel in $B_N(\sigma)$. Then

$$\begin{aligned}|g(z)| = \frac{|T_i\, g(z)|}{|z-i|} \;&=\; \frac{\left|\left\langle T_i g,\, K_z^N \right\rangle_{B_N}\right|}{|z-i|} \\ &\le \frac{\|\, T_i\, g \,\|_{B_N}\, \|\, K_z^N \,\|_{B_N}}{|z-i|} \;=\; \frac{\|\, K_z^N \,\|_{B_N}}{|z-i|}\;\; \|\, g \,\|_{B_{N-1}}\,.\end{aligned}$$

Hence evaluation at points $z \neq i$ is a continuous functional on $B_{N-1}(\sigma)$. For $z = i$ one uses an analogous argument with i replaced by $-i$, and T_i replaced by T_{-i} with $T_{-i}g(z) = (z+i)g(z)$. Hence $B_{N-1}(\sigma)$ is a Hilbert space of functions for which point evaluation at any $z \in \mathbb{C}$ is a continuous functional. Therefore $B_{N-1}(\sigma)$ is a RKHS. □

Lemma 7. *For $N \in \mathbb{Z}$ let $K_u^N(z)$ denote the reproducing kernel of $B_N(\sigma)$ corresponding to a point $u \in \mathbb{C}$, and let $\widetilde{K}_u^N = K_u^N / \parallel K_u^N \parallel_{B_N}$. Then*

$$K_u^{N+1}(z) = \left\langle K_u^{N+1}, \widetilde{K}_i^{N+1} \right\rangle_{B_{N+1}} \widetilde{K}_i^{N+1}(z) \; + \; (z-i)(\overline{u}+i)K_u^N(z) \tag{48}$$

$$= \left\langle K_u^{N+1}, \widetilde{K}_{-i}^{N+1} \right\rangle_{B_{N+1}} \widetilde{K}_{-i}^{N+1}(z) \; + \; (z+i)(\overline{u}-i)K_u^N(z). \tag{49}$$

Proof Recall that $B_{N+1,i}(\sigma)$ denotes the closed subspace of $B_{N+1}(\sigma)$ given by $B_{N+1,i}(\sigma) = \{f \in B_{N+1}(\sigma) : f(i) = 0\}$. Let g_u be given by $g_u(z) = (z-i)(\overline{u}+i)K_u^N(z)$. Then $g_u \in B_{N+1,i}(\sigma)$. Let $f \in B_{N+1,i}(\sigma)$. Then the function $h(z) = f(z)/(z-i)$ is in $B_N(\sigma)$, and, for $u \neq i$,

$$\begin{aligned} \langle f, g_u \rangle_{B_{N+1}} &= (u-i) \int_{\mathbb{R}} \frac{f(x)(x+i)\overline{K_u^N(x)}}{(1+x^2)^N(x+i)(x-i)} dx \\ &= (u-i) \left\langle h, K_u^N \right\rangle_{B_N} \; = \; (u-i)h(u) \; = \; f(u). \end{aligned}$$

Since $g_i(z)$ vanishes identically, one also has $\langle f, g_i \rangle_{B_{N+1}} = 0 = f(i)$. Hence g_u is the reproducing kernel for $B_{N+1,i}(\sigma)$. It follows that g_u is the orthogonal projection of K_u^{N+1} onto $B_{N+1,i}(\sigma)$; cf. [4, p. 345, item (7)]. As shown in the proof of Theorem 2, the orthogonal complement of $B_{N+1,i}(\sigma)$ in $B_{N+1}(\sigma)$ is the one-dimensional subspace generated by K_i^{N+1}. The orthogonal projection of K_u^{N+1} onto this subspace is given by $\left\langle K_u^{N+1}, \widetilde{K}_i^{N+1} \right\rangle_{B_{N+1}} \widetilde{K}_i^{N+1}(z)$. Hence $K_u^{N+1}(z) = \left\langle K_u^{N+1}, \widetilde{K}_i^{N+1} \right\rangle_{B_{N+1}} \widetilde{K}_i^{N+1}(z) + g_u(z)$, which is (48). An analogous argument with functions vanishing at the point $-i$ instead of at i yields (49). □

Proof of Theorem 1 for $N < 0$. We proceed by induction. For $N = 0$ one has that $B_0(\sigma)$ is a de Branges space $H(E_0)$ with $E_0(z) = e^{-i\sigma z}$ [11, p. 50]. Now assume that $B_{N+1}(\sigma)$ is a de Branges space with reproducing kernel K_u^{N+1}. Subtracting (49) from (48) gives

$$2i(z-\overline{u})K_u^N(z) = \frac{K_u^{N+1}(-i)\, K_{-i}^{N+1}(z)}{K_{-i}^{N+1}(-i)} \; - \; \frac{K_u^{N+1}(i)\, K_i^{N+1}(z)}{K_i^{N+1}(i)}.$$

With the help of (43), (44), and (45) this yields

$$\begin{aligned} K_u^N(z) &= \frac{i}{2} \frac{\pi}{K_i^{N+1}(i)} \frac{K_i^{N+1}(z)\, \overline{K_i^{N+1}(u)} \; - \; (K_i^{N+1})^*(z)\, \overline{(K_i^{N+1})^*(u)}}{\pi\,(z-\overline{u})} \\ &= \frac{i}{2} \frac{E_N(z)\, \overline{E_N(u)} \; - \; E_N^*(z)\, \overline{E_N^*(u)}}{\pi\,(z-\overline{u})}, \end{aligned} \tag{50}$$

with E_N as in (10), i.e., $E_N = \sqrt{\frac{\pi}{K_i^{N+1}(i)}}\, K_i^{N+1}$. Clearly, E_N is an entire function. Since for every $z \in \mathbb{C}$ there are functions $f \in B_N(\sigma)$ such that $f(z) \neq 0$, one has $K_z^N(z) > 0$ for all $z \in \mathbb{C}$. Let $z \in \mathbb{C}$ be in the upper half-plane, i.e., $\text{Im } z > 0$. Then (50) gives

$$0 < K_z^N(z) \;=\; \frac{|E_N(z)|^2 \;-\; |E_N(\overline{z})|^2}{4\pi \;\text{Im } z},$$

which implies $|E_N(z)|^2 > |E_N(\overline{z})|^2$. Hence E_N generates a de Branges space $H(E_N)$. From (42) and (50) it is evident that $H(E_N)$ and $B_N(\sigma)$ are two reproducing kernel Hilbert spaces of entire functions that share the same reproducing kernel. Lemma 4 then implies that $B_N(\sigma) = H(E_N)$. □

8.3 Auxiliary results

This subsection contains two technical lemmas. Lemma 8 is used in the proofs of Theorem 2 and Theorem 6. Lemma 9 is needed for Theorem 3.

Lemma 8. *Let V be a closed subspace of a Hilbert space H such that its orthogonal complement $V^\perp$ has dimension $n < \infty$. Let $P_{V^\perp}$ denote the orthogonal projection onto $V^\perp$. Let R be a Riesz basis of V and $B = \{b_1, \ldots, b_n\} \subset H$ such that $P_{V^\perp} B$ is a basis of $V^\perp$. Then $R \cup B$ is a Riesz basis of H.*

Proof We show that $R \cup B$ is an unconditional basis of H where the norms of the basis vectors are bounded away from both zero and infinity. The latter property is true for R since it is a Riesz basis and for B because the hypothesis implies that B is a finite and linearly independent set. Hence it also holds for $R \cup B$. Let $R = \{r_k, k \in I\}$, where I is a countable index set, and let $\tilde{b}_j = P_{V^\perp} b_j$, $j = 1, \ldots, n$. Let $f \in H$ be arbitrary. It needs to be shown that there is a unique expansion $f = \sum_{k \in I} c_k r_k + \sum_{j=1}^n d_j b_j$ with unconditional convergence. Since $\sum_{k \in I} c_k\, r_k \in V$, the coefficients d_j must satisfy $P_{V^\perp} f = P_{V^\perp} \sum_{j=1}^n d_j b_j = \sum_{j=1}^n d_j\, \tilde{b}_j$. Since $\{\tilde{b}_1, \ldots, \tilde{b}_n\}$ is a basis of $V^\perp$, there exist uniquely determined coefficients d_j that satisfy this condition. Let $\tilde{f} = f - \sum_{j=1}^n d_j\, b_j$. Then $P_{V^\perp} \tilde{f} = 0$, i.e., $\tilde{f} \in V$. Since R is a Riesz basis of V, there exist uniquely determined coefficients c_k, $k \in I$, such that $\tilde{f} = \sum_{k \in I} c_k\, r_k$, and the convergence of this expansion is unconditional. □

Lemma 9. *For $N \in \mathbb{Z}$ let $K_u^N(z)$ denote the reproducing kernel of $B_N(\sigma)$ at a point $u \in \mathbb{C}$. Then there is a constant b_N such that for all $u \in \mathbb{C}$*

$$(1 + |u|^2)\; \| \, K_u^N \, \|_{B_N}^2 \;\leq\; \| \, K_u^{N+1} \, \|_{B_{N+1}}^2 \;\leq\; b_N \,(1 + |u|^2)\; \| \, K_u^N \, \|_{B_N}^2 \,. \tag{51}$$

Proof Throughout this proof, $\langle \cdot, \cdot \rangle$ will denote the inner product in $B_{N+1}(\sigma)$. We begin by observing that for $u \in \mathbb{C}$ one has

$$|u \pm i|^2 = 1 + |u|^2 \pm 2 \;\text{Im } u \;=\; (1 + |u|^2) \left(1 \pm \frac{2 \;\text{Im } u}{1 + |u|^2} \right),$$

and therefore

$$\max\{|u + i|^2, |u - i|^2\} = (1 + |u|^2) \left(1 + \frac{2 \,|\; \text{Im } u|}{1 + |u|^2} \right).$$

It follows that for all $u \in \mathbb{C}$

$$1+|u|^2 \le \max\{|u+i|^2, |u-i|^2\} \ \le\ 2(1+|u|^2). \tag{52}$$

The two terms on the right-hand side of (48) are mutually orthogonal in $B_{N+1}(\sigma)$, and the same is true for the two terms on the right-hand side of (49). Furthermore, one has $\| (z \pm i)K_u^N \|_{B_{N+1}} = \| K_u^N \|_{B_N}$. One then obtains from (48) and (49) that

$$\| K_u^N \|_{B_N}^2 = \| K_u^{N+1} \|_{B_{N+1}}^2 \ \frac{1-\left|\langle \widetilde{K}_u^{N+1},\, \widetilde{K}_i^{N+1}\rangle\right|^2}{|u-i|^2}, \quad u \neq i, \text{ and} \tag{53}$$

$$\| K_u^N \|_{B_N}^2 = \| K_u^{N+1} \|_{B_{N+1}}^2 \ \frac{1-\left|\langle \widetilde{K}_u^{N+1},\, \widetilde{K}_{-i}^{N+1}\rangle\right|^2}{|u+i|^2}, \quad u \neq -i. \tag{54}$$

It follows from (53) that $\| K_u^N \|_{B_N}^2 \le \frac{\|K_u^{N+1}\|_{B_{N+1}}^2}{|u-i|^2}$ for $u \neq i$, and from (54) that $\| K_u^N \|_{B_N}^2 \le \frac{\|K_u^{N+1}\|_{B_{N+1}}^2}{|u+i|^2}$ for $u \neq -i$. Combining these two estimates and using (52) yields

$$\| K_u^N \|_{B_N}^2 \le \frac{\| K_u^{N+1} \|_{B_{N+1}}^2}{\max\{|u+i|^2, |u-i|^2\}} \ \le\ \frac{\| K_u^{N+1} \|_{B_{N+1}}^2}{(1+|u|^2)},$$

which establishes the left inequality in (51). On the other hand, (53) implies that

$$\| K_u^N \|_{B_N}^2 \ge \| K_u^{N+1} \|_{B_{N+1}}^2 \ \frac{1-\left|\langle \widetilde{K}_u^{N+1},\, \widetilde{K}_i^{N+1}\rangle\right|^2}{\max\{|u+i|^2, |u-i|^2\}},$$

while (54) yields that

$$\| K_u^N \|_{B_N}^2 \ge \| K_u^{N+1} \|_{B_{N+1}}^2 \ \frac{1-\left|\langle \widetilde{K}_u^{N+1},\, \widetilde{K}_{-i}^{N+1}\rangle\right|^2}{\max\{|u+i|^2, |u-i|^2\}}.$$

Combining these estimates and using (52) gives

$$\| K_u^N \|_{B_N}^2 \ge \| K_u^{N+1} \|_{B_{N+1}}^2 \ \frac{1-\min\left\{\left|\langle \widetilde{K}_u^{N+1},\, \widetilde{K}_i^{N+1}\rangle\right|^2,\ \left|\langle \widetilde{K}_u^{N+1},\, \widetilde{K}_{-i}^{N+1}\rangle\right|^2\right\}}{2\,(1+|u|^2)}. \tag{55}$$

Let V denote the two-dimensional subspace of $B_{N+1}(\sigma)$ that is spanned by K_i^{N+1} and K_{-i}^{N+1}, and let $D = \{x \in V : \| x \|_{B_{N+1}} \le 1\}$. The function $f : D \to \mathbb{R}$, $f(x) = \min\left\{\left|\langle x,\, \widetilde{K}_i^{N+1}\rangle\right|^2,\ \left|\langle x,\, \widetilde{K}_{-i}^{N+1}\rangle\right|^2\right\}$ is continuous. Clearly, $f(x) \le 1$ for all $x \in D$. If $f(x) = 1$ for some $x \in D$, then x must be a unimodular multiple of both $\widetilde{K}_i^{N+1}$ and $\widetilde{K}_{-i}^{N+1}$. But $\widetilde{K}_i^{N+1}$ and $\widetilde{K}_{-i}^{N+1}$ are not collinear, since otherwise there would be a constant c such that $g(i) = c\, g(-i)$ for all $g \in B_{N+1}(\sigma)$, which is not the case. It follows that $f(x) < 1$ for all $x \in D$. Since V is finite-dimensional, D is compact. Hence $m = \max_{x \in D} f(x) < 1$. Let $P_V : B_{N+1}(\sigma) \to V$ denote the orthogonal projection onto V. Then $P_V \widetilde{K}_u^{N+1} \in D$ and $\langle \widetilde{K}_u^{N+1},\, \widetilde{K}_{\pm i}^{N+1}\rangle = \langle P_V \widetilde{K}_u^{N+1},\, \widetilde{K}_{\pm i}^{N+1}\rangle$. Hence $\min\left\{\left|\langle \widetilde{K}_u^{N+1},\, \widetilde{K}_i^{N+1}\rangle\right|^2,\ \left|\langle \widetilde{K}_u^{N+1},\, \widetilde{K}_{-i}^{N+1}\rangle\right|^2\right\} = f(P_V \widetilde{K}_u^{N+1}) \le m < 1$ for all $u \in \mathbb{C}$. It now follows from (55) that

$$\| K_u^N \|_{B_N}^2 \ \ge\ \| K_u^{N+1} \|_{B_{N+1}}^2 \ \frac{1-m}{2\,(1+|u|^2)},$$

which establishes the right inequality in (51). $\square$

8.4 Proof of Theorem 4.

Proof We will first show that (24) gives a CIS of $B_N(\sigma)$. For $N = 0$ this is obvious. Let $N > 0$. Since $\arctan(t)$ is bounded on $\mathbb{R}$, it follows from (24) that $\lim_{k\to\pm\infty}\lambda_k = \pm\infty$. Consequently, $\lim_{k\to\pm\infty}\arctan(\lambda_k) = \pm\frac{\pi}{2}$. Therefore,

$$\lim_{k\to+\infty}\left(\lambda_k - \frac{1}{\sigma}\left[\alpha_x + \left(k - \frac{N}{2}\right)\pi\right]\right) = 0, \tag{56}$$

$$\lim_{k\to-\infty}\left(\lambda_k - \frac{1}{\sigma}\left[\alpha_x + \left(k + \frac{N}{2}\right)\pi\right]\right) = 0. \tag{57}$$

First assume that N is even. Define γ_k, $k \in \mathbb{Z}$ by

$$\gamma_k = \begin{cases} \lambda_{k+\frac{N}{2}} & \text{for} \quad k > 0 \\ \lambda_0 & \text{for} \quad k = 0 \\ \lambda_{k-\frac{N}{2}} & \text{for} \quad k < 0 \end{cases}.$$

Let ρ_k, $k \in \mathbb{Z}$ be given by $\rho_k = \frac{1}{\sigma}(\alpha_x + k\pi)$. Clearly, the equidistant set $\{\rho_k,\ k \in \mathbb{Z}\}$ is a CIS for $B_0(\sigma)$. Since $\lim_{k\to\pm\infty}(\gamma_k - \rho_k) = 0$, it follows in particular that for all but finitely many k one has $|\gamma_k - \rho_k| \le d$ for some $d < \frac{\pi}{4\sigma}$. It then follows from Lemma 1 and Lemma 4 in [2] that $\{\gamma_k,\ k \in \mathbb{Z}\}$ is also a CIS of $B_0(\sigma)$. Compared to $\{\gamma_k,\ k \in \mathbb{Z}\}$, Λ has the N additional points $\lambda_{-N/2}, \ldots, \lambda_{-1}, \lambda_1, \ldots, \lambda_{N/2}$. It now follows from Theorem 2 that Λ is a CIS of $B_N(\sigma)$. If N is odd, we use an analogous argument by rewriting (56) and (57) as

$$\lim_{k\to+\infty}\left(\lambda_k - \frac{1}{\sigma}\left[\alpha_x + \frac{\pi}{2} + \left(k - \frac{N+1}{2}\right)\pi\right]\right) = 0,$$

$$\lim_{k\to-\infty}\left(\lambda_k - \frac{1}{\sigma}\left[\alpha_x + \frac{\pi}{2} + \left(k + \frac{N-1}{2}\right)\pi\right]\right) = 0,$$

defining $\rho_k = \frac{1}{\sigma}\left(\alpha_x + \frac{\pi}{2} + k\pi\right)$, $k \in \mathbb{Z}$ and

$$\gamma_k = \begin{cases} \lambda_{k+\frac{N+1}{2}} & \text{for} \quad k > 0 \\ \lambda_0 & \text{for} \quad k = 0 \\ \lambda_{k-\frac{N-1}{2}} & \text{for} \quad k < 0 \end{cases}. \tag{58}$$

We now argue as in the case of even N to conclude that Λ is a CIS for $B_N(\sigma)$: By construction, the sequence $\{\rho_k,\ k \in \mathbb{Z}\}$ is a CIS for $B_0(\sigma)$, and the points γ_k satisfy $\lim_{k\to\pm\infty}(\gamma_k - \rho_k) = 0$. As before, it follows that $\{\gamma_k,\ k \in \mathbb{Z}\}$ is a CIS of $B_0(\sigma)$. Λ is obtained from $\{\gamma_k,\ k \in \mathbb{Z}\}$ by adding the N points $\lambda_{\frac{1-N}{2}}, \ldots, \lambda_{-1}, \lambda_1, \ldots \lambda_{\frac{N+1}{2}}$. Theorem 2 then implies that Λ is a CIS of $B_N(\sigma)$.

The assertion that Λ satisfies the condition (7) and thus yields an orthogonal expansion will be proved by applying a general construction due to de Branges [11, p. 54] to the specific situation at hand. According to Theorem 1, $B_N(\sigma)$ is the de Branges space $H(E_N)$ with $E_N(z) = (1 - iz)^N e^{-i\sigma z}$. Note that $E_N(0) = 1$. For $t \in \mathbb{R}$ we now write $E_N(t) = |E_N(t)|\, e^{-i\psi(t)}$, with $\psi : \mathbb{R} \to \mathbb{R}$ a continuous function such that $\psi(0) = 0$. Inserting this into (42) gives for $s, t \in \mathbb{R}$

$$K_s^N(t) = |E_N(s)|\ |E_N(t)|\ \frac{\sin(\psi(s) - \psi(t))}{\pi(s - t)}. \tag{59}$$

Since E_N does not have any real zeros, it follows for $s, t \in \mathbb{R}$, $s \ne t$, that

$$K_s^N(t) = 0 \quad \text{if and only if} \quad \psi(s) - \psi(t) \ = \ k\pi \text{ for some } k \in \mathbb{Z}. \tag{60}$$

Furthermore, letting $s \to t$ in (59) yields

$$\psi'(t) = \frac{\pi\, K_t^N(t)}{|E_N(t)|^2}. \tag{61}$$

The expression (26) for $K_t^N(t)$ follows from (8), (9), and from observing that

$$K_t^N(t) = \lim_{\epsilon \to 0} K_{t+i\epsilon}^N(t+i\epsilon) \;=\; \frac{1}{2\pi}\,\frac{d}{d\epsilon}\,|E_N(t+i\epsilon)|^2\Big|_{\epsilon=0}.$$

It then follows from (61), (9), and (26) that ψ satisfies

$$\psi'(t) = \sigma \;+\; \frac{N}{1+t^2}, \quad \psi(0) = 0,$$

which gives

$$\psi(t) = \sigma t \;+\; N\,\arctan(t), \quad t \in \mathbb{R}. \tag{62}$$

Hence the equations (24) can be written as $\psi(\lambda_k) = \alpha_x + k\pi$, $k \in \mathbb{Z}$, and (60) then shows that (7) holds. Hence Λ gives an orthogonal expansion. Since $\alpha_x = \psi(x) \mod \pi$, it follows from (24) that $x \in \Lambda$.

Now assume that $\Lambda = \{\lambda_j,\, j \in \mathbb{Z}\}$ is a CIS of $B_N(\sigma)$ such that $\lambda_k \notin \mathbb{R}$ for some $k \in \mathbb{Z}$. We will show that then the condition (7) does not hold. First we consider the case that λ_k is a zero of $E_N(z)$. For $N = 0$ one has $E_N(z) = e^{-i\sigma z}$ which has no zeros. For $N > 0$, $z = -i$ is the only zero of $E_N(z) = (1-iz)^N\, e^{-i\sigma z}$. In this case, equation (8) gives $K_{-i}^N(z) = -\frac{i}{2}\, E_N(i)\, \frac{E_N^*(z)}{\pi(z-i)} = \frac{E_N(i)}{2\pi}\,(1+iz)^{N-1}\, e^{i\sigma z}$ which has one root for $N > 1$ and no roots for $N = 1$. Since any CIS of $B_N(\sigma)$ contains infinitely many points, the condition (7) requires $K_{\lambda_k}^N$ to have infinitely many roots. Hence it does not hold. Therefore we can assume that $E_N(\lambda_k) \neq 0$. Let $l \in \mathbb{Z}$ be such that $\lambda_l \notin \{\lambda_k, \overline{\lambda_k}\}$, and $E_N(\lambda_l)\, E_N(\overline{\lambda_l}) \neq 0$. Such l exists since E_N has at most one root, while Λ has infinitely many elements. If $K_{\lambda_k}^N(\lambda_l) = 0$, then (8) gives $E_N(\lambda_l)\, \overline{E_N(\lambda_k)} = E_N^*(\lambda_l)\, E_N(\overline{\lambda_k})$, which in turn implies

$$\left|\frac{E_N(\overline{\lambda_k})}{E_N(\lambda_k)}\right| = \left|\frac{E_N(\lambda_l)}{E_N^*(\lambda_l)}\right| \;=\; \left|\frac{E_N(\lambda_l)}{E_N\,(\overline{\lambda_l})}\right|. \tag{63}$$

Recall that for $\operatorname{Im} z > 0$ one has $|E_N(z)| > |E_N(\overline{z})|$. Hence, it follows from (63) that if $\operatorname{Im} \lambda_k > 0$, then $\operatorname{Im} \lambda_l < 0$, and vice versa. That is, one has $(\operatorname{Im} \lambda_k)(\operatorname{Im} \lambda_l) < 0$. Now let m be such that $\lambda_m \notin \{\lambda_k, \overline{\lambda_k}, \lambda_l, \overline{\lambda_l}\}$ and $E_N(\lambda_m)\, E_N(\overline{\lambda_m}) \neq 0$.Then (7) requires that both $K_{\lambda_k}^N(\lambda_m) = 0$ and $K_{\lambda_l}^N(\lambda_m) = 0$, which implies that $(\operatorname{Im} \lambda_k)(\operatorname{Im} \lambda_m) < 0$ as well as $(\operatorname{Im} \lambda_l)(\operatorname{Im} \lambda_m) < 0$, which contradicts $(\operatorname{Im} \lambda_k)(\operatorname{Im} \lambda_l) < 0$. □

8.5 Proof of Theorem 6

Before proving the theorem, we establish that the derivative is a continuous operator on $B_N(\sigma)$.

Lemma 10. *The derivative operator $Df = f'$ is a continuous operator on $B_N(\sigma)$ for any $N \in \mathbb{Z}$ and $\sigma > 0$.*

Proof The derivative rule for the distributional Fourier transform implies that f' is σ-bandlimited whenever f is. It remains to be shown that for $N \in \mathbb{Z}$ and $f \in B_N(\sigma)$

$$\parallel f' \parallel_{B_N} \;\leq\; C_N \parallel f \parallel_{B_N}. \tag{64}$$

It is well-known that (64) holds for $N = 0$; see, e.g., [53, p. 91]. For $N > 0$ one can directly verify from (9) that $E'_N(z)/E_N(z)$ is a bounded analytic function in the upper half-plane. Baranov [7, Theorem 3.2] showed that this is sufficient for (64) to hold in a de Branges space. For $N < 0$ we proceed by induction. Assume that $N \in \mathbb{Z}$ is such that (64) holds for $B_N(\sigma)$. Let $f \in B_{N-1}(\sigma)$. Recall that $\| f \|_{B_N} \leq \| f \|_{B_{N-1}}$, and that the operator T_i given by $T_i f(z) = (z - i)f(z)$ satisfies $\| T_i f \|_{B_N} = \| f \|_{B_{N-1}}$, as well as $(T_i f)' = f + T_i f'$. It follows that

$$\begin{aligned} \| f' \|_{B_{N-1}} &= \| T_i f' \|_{B_N} = \| (T_i f)' - f \|_{B_N} \leq \| (T_i f)' \|_{B_N} + \| f \|_{B_N} \\ &\leq C_N \| T_i f \|_{B_N} + \| f \|_{B_N} \leq (C_N + 1) \| f \|_{B_{N-1}}, \end{aligned}$$

so (64) holds also for $B_{N-1}(\sigma)$. □

Proof of Theorem 6 It follows from Theorem 1 and Lemma 10 that point evaluation of the j-th derivative $f^{(j)}$ is a continuous linear functional on $B_N(\sigma)$. For $u \in \mathbb{C}$ and $j \in \mathbb{N} \cup \{0\}$, let $K^N_{u,j}$ denote the element of $B_N(\sigma)$ such that $\langle f, K^N_{u,j} \rangle_{B_N} = f^{(j)}(u)$ for all $f \in B_N(\sigma)$. Note that $K^N_{u,0} = K^N_u$. By hypothesis, $\Lambda = \{\lambda_k, k \in \mathbb{Z}\}$ is a CIS of $B_M(\sigma)$. Let $\phi(z)$ be as in (35). Let J denote the set $J = \{k \in \mathbb{Z}, \mu_k > 0\}$, and $|J|$ the number of elements in J. Let $S = \sum_{k \in \mathbb{Z}} \mu_k$. Since $S < \infty$, one has $|J| < \infty$. Removing from Λ the points λ_k with $k \in J$ gives the set $\tilde{\Lambda} = \{\lambda_k, k \in \mathbb{Z} \backslash J\}$. By Theorem 2, $\tilde{\Lambda}$ is a CIS for $B_L(\sigma)$ with $L = M - |J|$. Let

$$P(z) = \prod_{k \in J} (z - \lambda_k)^{1 + \mu_k}.$$

By Lemma 1, $\tilde{\varphi}(z) = \frac{\phi(z)}{P(z)}$ is a generating function for the CIS $\tilde{\Lambda}$ of $B_L(\sigma)$. For $k \in \mathbb{Z} \backslash J$ let $\tilde{\varphi}_k(z) = \frac{\tilde{\varphi}(z)}{\tilde{\varphi}'(\lambda_k)(z - \lambda_k)}$. It then follows from equation (6) in the proof of Theorem C that

$$R_{B_L} = \{\| K^L_{\lambda_k} \|_{B_L} \ \tilde{\varphi}_k(z), \ k \in \mathbb{Z} \backslash J\}$$

is a Riesz basis for $B_L(\sigma)$. Here K^L_u denotes the reproducing kernel for $B_L(\sigma)$. For any $u \in \mathbb{C}$ and $Q \in \mathbb{Z}$, the mapping $f(z) \to (z - u)f(z)$ is an injective continuous linear mapping from $B_Q(\sigma)$ into $B_{Q+1}(\sigma)$. It follows that the mapping $f \to P(z)f(z)$ is a continuous linear mapping that maps $B_L(\sigma)$ bijectively onto the subspace $V \subset B_N(\sigma)$ given by

$$V = \{f \in B_N(\sigma) : \ f^{(j)}(\lambda_k) = 0, \ k \in J, \ j = 0, \ldots, \mu_k\}.$$

V is the intersection of the null spaces of the $S + |J|$ linearly independent continuous linear functionals represented by $K^N_{\lambda_k, j}$, $j = 0, \ldots, \mu_k$, $k \in J$. Hence V is a closed subspace of $B_N(\sigma)$. Since

$$R_V = P(z) \, R_{B_L} \ = \ \{\| K^L_{\lambda_k} \|_{B_L} \ P(z) \, \tilde{\varphi}_k(z), \ k \in \mathbb{Z} \backslash J\}$$

is the image of the Riesz basis R_{B_L} under a continuous bijective mapping, it is a Riesz basis of V [15, Lemma 7.10]. Note that $B_{V^\perp} = \{K^N_{\lambda_k, j}, k \in J, j = 0, \ldots, \mu_k\}$ is a basis of $V^\perp$, the orthogonal complement of V, and let $P_{V^\perp}$ denote the orthogonal projection onto $V^\perp$. We wish to apply Lemma 8 with

$$B = \{\phi_{kl}, \ k \in J, \ l = 0, \ldots, \mu_k\},$$

where $\phi_{kl} \in B_N(\sigma)$ as in (34), (37), (38). For $k, n \in J$, $l \in \{0 \ldots, \mu_k\}$, and $j \in \{0, \ldots, \mu_n\}$ one has $\left\langle P_{V^\perp} \phi_{kl},\, K^N_{\lambda_n, j} \right\rangle_{B_N} = \left\langle \phi_{kl},\, K^N_{\lambda_n, j} \right\rangle_{B_N} = \phi^{(j)}_{kl}(\lambda_n) = \delta_{kn}\, \delta_{jl}$. Hence $P_{V^\perp} B$ is biorthogonal to $B_{V^\perp}$ and thus a basis of $V^\perp$. It now follows from Lemma 8 that $R = B \cup R_V$ is a Riesz basis of $B_N(\sigma)$. The biorthogonal Riesz basis is given by

$$R^* = \left\{ K^N_{\lambda_k, l},\, k \in J,\, l = 0, \ldots, \mu_k \right\} \,\cup\, \left\{ \frac{K^N_{\lambda_k}}{\overline{P(\lambda_k)}\, \| K^L_{\lambda_k} \|_{B_L}},\, k \in \mathbb{Z} \backslash J \right\}.$$

The expansion of $f \in B_N(\sigma)$ with respect to the Riesz basis R is then given by

$$f(z) = \sum_{k \in J} \sum_{l=0}^{\mu_k} f^{(l)}(\lambda_k)\, \phi_{kl}(z) \;+ \sum_{k \in \mathbb{Z} \backslash J} f(\lambda_k) \frac{P(z)\, \tilde{\varphi}_k(z)}{P(\lambda_k)}. \tag{65}$$

For $k \in \mathbb{Z} \backslash J$ one has $\mu_k = 0$. In order to establish (39) it remains to be shown that for $k \in \mathbb{Z} \backslash J$ one has $\frac{P(z)\, \tilde{\varphi}_k(z)}{P(\lambda_k)} = \phi_{k0}$. Both functions take the value 1 at $z = \lambda_k$. Since $\frac{P(z)\, \tilde{\varphi}_k(z)}{P(\lambda_k)} = \frac{\phi(z)}{P(\lambda_k) \tilde{\varphi}'(\lambda_k)\, (z - \lambda_k)}$, and by (37), $\phi_{k0}(z) = \frac{\phi(z)}{\psi_k(\lambda_k)\, (z - \lambda_k)}$, the two functions differ at most by a multiplicative constant. Since they agree at the point $z = \lambda_k$, they must be equal. This establishes (39). The convergence properties follow from (65) being an expansion with respect to a Riesz basis. □

References

[1] Al-Hammali H, Faridani A (2020) The zeros of a sine-type function and the peak value problem. Signal Processing 167:107274

[2] Al-Hammali H, Faridani A (2023) Uniform and non-uniform sampling of bandlimited functions at minimal density with a few additional samples. Sampl Theory Signal Process Data Anal 21:2. https://doi.org/10.1007/s43670-022-00041-7

[3] Al-Sa'di S (2012) Sampling and interpolation in Hilbert spaces of entire functions. PhD dissertation. Iowa State University Digital Repository https://doi.org/10.31274/etd-180810-586

[4] Aronszajn N (1950) Theory of reproducing kernels. Trans Amer Math Soc 68:337–404

[5] Axler S (2020) Measure Theory, Integration & Real Analysis. Springer, Cham, https://doi.org/10.1007/978-3-030-33143-6

[6] Bailey B, Madych W (2014) Cardinal sine series: Convergence and uniqueness. Sampling Theory in Signal and Image Processing 13(1):21–33

[7] Baranov A (2000) Differentiation in the Branges spaces and embedding theorems. Journal of Mathematical Sciences 101(2):2881–2913

[8] Butzer P, Ferreira P, Higgins J, et al (2011) Interpolation and sampling: E.T. Whittaker, K. Ogura and their followers. J Fourier Anal Appl 17:320–354. https://doi.org/10.1007/s00041-010-9131-8

[9] Butzer PL, Schmeisser G, Stens RL (2001) An introduction to sampling analysis. In: Marvasti F (ed) Nonuniform Sampling: Theory and Practice. Kluwer Academic / Plenum Publishers, New York, p 17–121

[10] Campbell LL (1968) Sampling theorem for the Fourier transform of a distribution with bounded support. SIAM Journal on Applied Mathematics 16(3):626–636

[11] De Branges L (1968) Hilbert Spaces of Entire Functions. Prentice-Hall, Englewood Cliffs

[12] Feichtinger HG (2023) Sampling via the Banach Gelfand triple. In: Casey SD, Dodson MM, Ferreira PJSG, et al (eds) Sampling, Approximation, and Signal Analysis: Harmonic Analysis in the Spirit of J. Rowland Higgins. Springer International Publishing, Cham, pp 211–242, https://doi.org/10.1007/978-3-031-41130-4_10

[13] García AG (2015) Sampling theory and reproducing kernel Hilbert spaces. In: Alpay D (ed) Operator Theory. Springer, Basel, pp 97–110

[14] García AG (2024) The Use of Frames in Sampling Theory. Springer, Cham

[15] Heil C (2011) A Basis Theory Primer: Expanded Edition. Birkhäuser, Boston, https://doi.org/10.1007/978-0-8176-4687-5

[16] Higgins JR (1985) Five short stories about the cardinal series. Bulletin of the American Mathematical Society 12(1):45–89

[17] Higgins JR (1996) Sampling Theory in Fourier and Signal Analysis: Foundations. Clarendon Press, Oxford

[18] Higgins JR (2014) Sampling in reproducing kernel Hilbert space. In: Zayed A, Schmeisser G (eds) New Perspectives on Approximation and Sampling Theory. Birkhäuser, Cham

[19] Hoskins R, de Sousa Pinto J (1984) Sampling expansions for functions bandlimited in the distributional sense. SIAM Journal on Applied Mathematics 44(3):605–610

[20] Hoskins RF, Sousa Pinto J (2011) Theories of generalised Functions. Distributions, ultradistributions and other generalised functions. Woodhead Publishing Limited, Oxford

[21] Hruščev SV, Nikol'skii NK, Pavlov BS (1981) Unconditional bases of exponentials and of reproducing kernels. In: Complex Analysis and Spectral Theory, Lecture Notes in Mathematics, vol 864. Springer Verlag, Berlin, pp 214–335

[22] Jerri AJ (1977) The Shannon sampling theorem–its various extensions and applications: A tutorial review. Proceedings of the IEEE 65(11):1565–1596

[23] Kircheis M, Potts D, Tasche M (2024) On numerical realizations of Shannon's sampling theorem. Sampl Theory Signal Process Data Anal 22:13. https://doi.org/10.1007/s43670-024-00087-9

[24] Kotelnikov VA (1933) On the carrying capacity of the ether and wire in telecommunications. In: Material for the First All-Union Conference on Questions of Communication, Izd. Red. Upr. Svyazi RKKA, Moscow

[25] Landau H (1967) Sampling, data transmission, and the Nyquist rate. Proceedings of the IEEE 55(10):1701–1706

[26] Lee AJ (1976) Characterization of bandlimited functions and processes. Information and Control 31(3):258–271

[27] Levin BY (1996) Lectures on entire functions, Translations of Mathematical Monographs, vol 150. American Mathematical Society, Providence

[28] Lyubarskii YI, Seip K (1997) Complete interpolating sequences for Paley–Wiener spaces and Muckenhoupt's (A_p) condition. Revista Matematica Iberoamericana 13(2):361–376

[29] Lyubarskii YI, Seip K (2002) Weighted Paley-Wiener spaces. J Amer Math Soc 15:979–1006

[30] Madych WR (2020) Sampling series, refinable sampling kernels, and frequency band limited functions. In: Casey SD, Okoudjou KA, Robinson M, et al (eds) Sampling: Theory and Applications: A Centennial Celebration of Claude Shannon. Birkhäuser, Cham, pp 93–140

[31] Marks RJ (2009) Handbook of Fourier Analysis and its Applications. Oxford University Press, Oxford

[32] Minkin AM (1992) Reflection of exponents, and unconditional bases of exponentials. St Petersburg Math J 3(5):1043–1068

[33] Nashed MZ, Walter GG (1991) General sampling theorems for functions in reproducing kernel Hilbert spaces. Math Control Signals Systems 4:363–390

[34] Partington J (1996) Recovery of functions by interpolation and sampling. J Math Anal Appl 189:301–309

[35] Paulsen VI, Raghupati M (2016) An Introduction to the Theory of reproducing kernel Hilbert Spaces, Cambridge studies in advanced mathematics, vol 152. Cambridge University Press, Cambridge

[36] Pavlov BS (1979) Basicity of an exponential system and Muckenhoupt's condition. Dokl Akad Nauk SSSR 247(1):37–40

[37] Pfaffelhuber E (1971) Sampling series for band-limited generalized functions. IEEE Transactions on Information Theory 17(6):650–654

[38] Qian L (2003) On the regularized Whittaker-Kotel'nikov-Shannon sampling formula. Proc Amer Math Soc 131(4):1169–1176

[39] Rudin W (1991) Functional Analysis. McGraw-Hill, New York

[40] Schmeisser G, Stenger F (2007) Sinc approximation with a Gaussian multiplier. Sampling Theory in Signal and Image Processing 6(2):199–221

[41] Seip K (1998) Developments from nonharmonic Fourier series. Documenta Mathematica, Extra Volume ICM 1998 II pp 713–722

[42] Shannon CE (1949) Communication in the presence of noise. Proceedings of the IRE 37(1):10–21

[43] Shin CE, Chung SY, Kim D (2004) General sampling theorem using contour integral. J Math Anal Appl 291:50–65

[44] Triebel H (1978) Interpolation Theory, Function Spaces, Differential Operators. North-Holland Publishing Company, Amsterdam

[45] Tschakaloff L (1934) Zweite Lösung der Aufgabe 105. Jahresber Deutsch Math–Verein 43:11–12

[46] Unser M (2000) Sampling–50 years after Shannon. Proceedings of the IEEE 88(4):569–587

[47] Valiron MG (1925) Sur la formule d'interpolation de Lagrange. Bulletin des Sciences Mathématiques 49(2). Pages 181–192 and 203–224.

[48] Walter GG (1988) Sampling bandlimited functions of polynomial growth. SIAM Journal on Mathematical Analysis 19(5):1198–1203

[49] Walter GG (1992) Nonuniform sampling of bandlimited functions of polynomial growth. SIAM Journal on Mathematical Analysis 23(4):995–1003

[50] Whittaker ET (1915) On the functions which are represented by the expansion of interpolating theory. Proc Roy Soc Edinburgh 35:181–194

[51] Woracek H (2015) De Branges spaces and growth aspects. In: Alpay D (ed) Operator Theory. Springer, Basel, pp 489–523

[52] Yao K (1967) Applications of reproducing kernel Hilbert spaces–bandlimited signal models. Information and Control 11:429–444

[53] Young RM (2001) An Introduction to Nonharmonic Fourier Series, Revised First Edition. Academic Press, San Diego

[54] Zakai M (1965) Band-limited functions and the sampling theorem. Information and Control 8:143–158

[55] Zayed AI (1993) Advances in Shannon's Sampling Theory. CRC Press, Boca Raton

[56] Zayed AI, Butzer PL (2001) Lagrange interpolation and sampling theorems. In: Marvasti F (ed) Nonuniform Sampling: Theory and Practice. Kluwer Academic/-Plenum Publishers, New York, pp 123–168