A Sampling Theorem by Perturbing the Zeros of a Sine-Type Function

Hussain Al-Hammali[a] and Adel Faridani[b]

[a]Department of Mathematics, College of Natural Sciences, Kyungpook National University, Daegu, The Republic of Korea; [b]Department of Mathematics, Oregon State University, Corvallis, OR, 97331, USA

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ABSTRACT
In this paper, we present a generalisation of the classical Shannon sampling theorem that allows for sampling sets that are perturbations of the set of zeros of a sine-type function. Such sampling sets may be non-equidistant and non-periodic.

KEYWORDS
Shannon sampling; Paley-Wiener spaces; zeros of a sine-type function; Riesz basis; non-equidistant sampling.

1. Introduction and preliminaries

The Shannon sampling theorem, which is also known as the Whittaker-Kotelnikov-Shannon (WKS) theorem, allows the reconstruction of a band-limited function from its sampled values. It reads as follows: *if a function f is band-limited to $[-\pi, \pi]$, i.e., it is represented as*

$$f(t) = \int_{-\pi}^{\pi} g(x)e^{-ixt}dx, \quad t \in \mathbb{R}$$

for some function $g \in L^2(-\pi, \pi)$, then f can be reconstructed from its samples, $f(k)$, $k \in \mathbb{Z}$. The reconstruction formula is

$$f(t) = \sum_{k=-\infty}^{\infty} f(k)\frac{\sin \pi (t-k)}{\pi (t-k)}, \quad t \in \mathbb{R}. \tag{1}$$

The series converges absolutely, in the L^2-sense, and uniformly on $\mathbb{R}$.

Some early works in this field include those by Whittaker [1], Ogura [2], Kotelnikov [3], and Shannon [4]. For a historical overview on sampling theorems, see [5–8].

We require some definitions and results that will explain the theme of the generalisation we are interested in. An entire function f is of *exponential type* at most σ

CONTACT Hussain Al-Hammali. Email: hussain@knu.ac.kr

($\sigma > 0$), and we write f is a function of exponential type $\leq \sigma$, if for any $\epsilon > 0$ there exists an A_ϵ such that

$$|f(z)| \leq A_\epsilon e^{(\sigma+\epsilon)|z|}$$

for all $z \in \mathbb{C}$. Let E denote the space of all entire functions and let E_σ denote the class of all entire functions of exponential type $\leq \sigma$. The Paley- Wiener spaces $\mathcal{PW}_\sigma^p$ are defined as follows.

Definition 1.1. A function f is in the *Paley- Wiener space* $\mathcal{PW}_\sigma^p$, $1 \leq p \leq \infty$, if $f(z) = \int_{-\sigma}^{\sigma} g(w)e^{izw}dw,\ z \in \mathbb{C}$ for some $g \in L^p[-\sigma, \sigma]$, where the norm is given by

$$\|f\|_{\mathcal{PW}_\sigma^p} := \left(\frac{1}{2\pi}\int_{-\sigma}^{\sigma} |g\,(w)|^p\,dw\right)^{1/p}$$

for $1 \leq p < \infty$.

A *band-limited function* is a tempered distribution whose Fourier transform has compact support. The functions in the spaces $\mathcal{PW}_\sigma^2$ and $\mathcal{PW}_\sigma^1$ are examples of band-limited functions. A closely related function space is the *Bernstein space* $\mathcal{B}_\sigma^p$ which consists of all functions in E_σ whose restrictions to the real line are in $L^p(\mathbb{R})$. The norm for $\mathcal{B}_\sigma^p$, $1 \leq p \leq \infty$ is given by $\|f\|_{\mathcal{B}_\sigma^p} := \|f\|_p$. Again the functions in this space are band-limited functions. This can be seen by the theorems of Phragmén-Lindelöf [9, p. 39] and Paley-Wiener-Schwartz [10, p. 198].

The two normed spaces $\mathcal{B}_\sigma^2$ and $\mathcal{PW}_\sigma^2$ are identical and this can be seen by the theorems of Plancherel [11, Theorem 2.13], and Paley-Wiener [11, p. 67]. The Bernstein and Paley-Wiener spaces can be ordered as follows:

$$\mathcal{B}_\sigma^1 \subset \mathcal{B}_\sigma^2 \subset \ldots \subset \mathcal{B}_\sigma^\infty$$

and

$$\ldots \subset \mathcal{PW}_\sigma^2 \subset \mathcal{PW}_\sigma^1.$$

In addition, we obtain the following ordered inclusions:

$$\mathcal{B}_\sigma^2 = \mathcal{PW}_\sigma^2 \subset \mathcal{PW}_\sigma^1 \subset \mathcal{B}_{\sigma,0}^\infty \subset \mathcal{B}_\sigma^\infty,$$

where the elements f in $\mathcal{B}_{\sigma,0}^\infty$ are those in $\mathcal{B}_\sigma^\infty$ that satisfy $\lim_{|t|\to\infty} f(t) = 0$. A function of exponential type $\leq \sigma$ is said to be a *σ-sine-type* function if

(i) the zeros of f are simple and separated (that is, uniformly discrete, $\inf_{j\neq k} |\lambda_j - \lambda_k| \geq \underline{\delta}$ for all $k \in \mathbb{Z}$ and some $\underline{\delta} > 0$) and
(ii) there exist A, B, and η such that

$$Ae^{\sigma|y|} \leq |f(x+iy)| \leq Be^{\sigma|y|} \tag{2}$$

for all $x, y \in \mathbb{R}$ and $|y| \geq \eta$.

By the horizontal strip of finite width, we shall mean the following:

$$L = \{x + iy \in \mathbb{C} \mid x \in \mathbb{R}, |y| \leq M \text{ for some positive number } M\}.$$

Looking at condition (ii) we see that the zeros of a π-sine-type function lie on a horizontal strip of finite width.

The function $\sin \pi z$ is an example of a π-sine-type function. An example of a π-sine-type function with non-equidistant zeros is

$$\varphi_{\alpha,\beta}(z) = \cos \pi z - \beta \sin \alpha \pi z$$

for some suitable choices of $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. For example, one may choose $\alpha = 1/\sqrt{3}$ and $\beta = 0.5$.

We now define the generating function that will be used to create the sampling basis. The generating function is defined by the following canonical product

$$\varphi(z) = \lim_{N \to \infty} \prod_{|k| \leq N} \left(1 - \frac{z}{\lambda_k}\right), \tag{3}$$

where $(1 - z/\lambda_k)$ is replaced by z if $\lambda_k = 0$. If the λ_k's are the integers, then the canonical product (3) yields $\frac{1}{\pi} \sin \pi z$. A result by Levin-Ostrovskii [12, p. 85] shows that the canonical product (3) forms a π-sine-type function if the λ_k's differ from the zeros of a π-sine-type function by a sequence $\{\delta_k\}_{k\in\mathbb{Z}} \in l^p$, $1 < p < \infty$.

Boche and Mönich [13, Theorem 2] deduced a sampling series for the class of functions $\mathcal{B}^\infty_{\sigma,0} \supset \mathcal{B}^2_\sigma = \mathcal{PW}^2_\sigma$ that generalizes (1) where the sampling set is the zeros of a π-sine-type function instead of the integers. In their sampling series, one can obtain (1) when the sampling set is chosen to be the integers. Thus, the WKS sampling series can be used for the class $\mathcal{B}^\infty_{\sigma,0}$ where the convergence is uniform over compact subsets of $\mathbb{R}$. Another type of a generalisation of the WKS sampling series is the following result by Higgins. It is a generalisation in a sense that it allows a reconstruction using a perturbed sampling set from the integers within a quarter. It reads as follows.

Theorem 1.2 (Higgins [14]; see also Seip [15])**.** *Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers such that*

$$|\lambda_k - k| \leq D < \frac{1}{4},$$

and let $\varphi(t)$ be defined as in (3). Then for all $f \in \mathcal{PW}^2_\pi$, we obtain

$$f(t) = \sum_{k=-\infty}^{\infty} f(\lambda_k) \frac{\varphi(t)}{\varphi'(\lambda_k)(t - \lambda_k)}. \tag{4}$$

The convergence is uniform over $\mathbb{R}$.

The series (4) is of Lagrange-type form, i.e., the sampling basis elements satisfy the condition

$$\varphi_k(\lambda_l) = \frac{\varphi(\lambda_l)}{\varphi'(\lambda_k)(\lambda_l - \lambda_k)} = \delta_{kl}, \quad k, l \in \mathbb{Z}.$$

We can point out that the sampling series (4) with the restriction $\lambda_{-k} = -\lambda_k$ is called the Paley-Wiener-Levinson theorem, see [16, p. 115] and [17]. In [17] A. G. García deduced the sampling series (4) using the fact that the Fourier transform defines an isometric isomorphism from $L^2[-\pi,\pi]$ onto the space $\mathcal{PW}_\pi^2$ and had the series extended to a horizontal strip of finite width. A sequence of vectors $\{\varphi_k\}_{k\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is called a *Riesz basis* if $\{\varphi_k\}_{k\in\mathbb{Z}}$ is complete in $\mathcal{H}$ and there exist constants A and B such that for all M, $N \in \mathbb{N}$ and arbitrary scalars c_k's we have

$$A\sum_{k=-M}^{N}|c_k|^2 \leq \left\|\sum_{k=-M}^{N} c_k\varphi_k\right\|^2 \leq B\sum_{k=-M}^{N}|c_k|^2. \tag{5}$$

When we have to deal with perturbations in the sampling set, we use *Kadec's 1/4-theorem*, see [18, p. 36].

Theorem 1.3 (Kadec)**.** *If $\{\lambda_k\}_{k\in\mathbb{Z}}$ is a sequence of real numbers for which*

$$|\lambda_k - k| \leq D < \frac{1}{4}, \quad k = 0, \pm 1, \pm 2, \ldots$$

then $\left\{e^{i\lambda_k t}\right\}_{k\in\mathbb{Z}}$ forms a Riesz basis for $L^2[-\pi,\pi]$.

Later we present a result that generalizes the above result by Kadec.

2. Derivation of the Main Result

Let us now consider the perturbation as $\lambda_k^* = \lambda_k + \delta_k$ where $\{\delta_k\}_{k\in\mathbb{Z}} \in l^\infty$ and define φ^* as

$$\varphi^*(z) = \lim_{N\to\infty}\prod_{|k|\leq N}\left(1 - \frac{z}{\lambda_k + \delta_k}\right),$$

where $(1 - z/(\lambda_k + \delta_k))$ is replaced by z if $\lambda_k + \delta_k = 0$. In addition, we define $\varphi_k^*(z)$ as

$$\varphi_k^*(z) = \frac{\varphi^*(z)}{\varphi *'\left(\lambda_k^*\right)\left(z - \lambda_k^*\right)}.$$

Two sequences $\{x_n\}$ and $\{y_n\}$ in a Hilbert space $\mathcal{H}$ are said to be *biorthogonal* if

$$\langle x_n, y_m\rangle = \delta_{nm}$$

for every n and m. In his proof of the Paley- Wiener- Levinson theorem (Theorem 1.2 with t chosen to be complex), A. G. García [17] used a theorem by Titchmarsh. The theorem can be stated as follows, cf. [19, Theorem VI].

Theorem 2.1 (Titchmarsh)**.** *Let $g \in L^1[-\pi,\pi]$ and define the entire function f as*

$$f(z) = \int_{-\pi}^{\pi} g(w)e^{zw}dw.$$

Then, f has infinitely many zeros, $\{z_n\}_{n\in\mathbb{N}}$, with nondecreasing absolute values, such that

$$f(z) = Az^m e^{\left(\frac{a+b}{2}\right)z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)$$

for some $m \in \mathbb{N} \cup \{0\}$, where $[a,b] \subseteq [-\pi,\pi]$ is the smallest interval that contains the support of g. The infinite product is conditionally convergent.

For our application of Titchmarsh's theorem, we also need the following lemma. Recall that a sequence of vectors $\{f_k\}_{k\in\mathbb{Z}}$ in a normed space X is said to be *complete* if its linear span is dense in X. For the completeness in Hilbert spaces, it is equivalent to saying that the only vector that is orthogonal to all f_k's is the zero vector. Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ be a sequence of real or complex numbers. The *completeness radius* of Λ is defined to be the number

$$R(\Lambda) = \sup \left\{ r \mid \left\{ e^{i\lambda_k t} \right\}_{k\in\mathbb{Z}} \text{is complete in } C[-r,r] \right\}.$$

The completeness radius does not change if we replace $C[-r,r]$ by the spaces $L^p[-r,r]$ with $1 \le p < \infty$, see [18, p. 120].

Lemma 2.2. *Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ be such that $\{f_k(t) = e^{i\lambda_k t},\ k \in \mathbb{Z}\}$ is a Riesz basis of $L^2(-\pi,\pi)$. Let $\{g_k\}_{k\in\mathbb{Z}}$ denote the corresponding biorthogonal basis of $\{f_k\}_{k\in\mathbb{Z}}$. Let $[\alpha_k,\beta_k] \subseteq [-\pi,\pi]$ be the smallest interval that contains the support of g_k. Then $\alpha_k = -\pi$ and $\beta_k = \pi$.*

Proof. Assume that $[\alpha_k,\beta_k]$ is a proper subset of $[-\pi,\pi]$ for some $k \in \mathbb{Z}$, let $c = (\alpha_k + \beta_k)/2$, and consider the shifted function $h(t) = \overline{g_k}(t+c)$. Then h is supported in $[-(\beta_k - \alpha_k)/2,\ (\beta_k - \alpha_k)/2] = [-\beta\pi, \beta\pi]$ for some $\beta \in (0,1)$. The inverse Fourier transform of h now satisfies $\tilde{h}(\lambda_n) = \int h(t)e^{i\lambda_n t}dt = \int \overline{g_k}(t+c)e^{i\lambda_n t}\,dt = e^{-i\lambda_n c}\int \overline{g_k}(t)e^{i\lambda_n t}\,dt = e^{-i\lambda_n c}\langle f_n, g_k\rangle = e^{-i\lambda_n c}\delta_{kn}$, i.e., $\tilde{h}$ vanishes on $\Lambda' = \Lambda - \{\lambda_k\}$. According to Beurling and Malliavin [20], the closure radii (the completeness radii) of Λ and Λ' are equal. Therefore, because $\beta < 1$, $\tilde{h} \in \mathcal{PW}^2_{\beta\pi}$ vanishes on Λ', so it must vanish identically, which is a contradiction. $\square$

We restate the following theorem by Katsnel'son, cf. [21].

Theorem 2.3 (Katsnel'son)**.** *Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be the set of zeros of a σ-sine-type function and let $\{\delta_k\}_{k\in\mathbb{Z}}$ be a sequence of complex numbers satisfying the conditions*

$$|\operatorname{Re}\delta_k| \le dp, \sup_{k\in\mathbb{Z}} |\operatorname{Im}\delta_k| < \infty$$

where $p = \inf_k |\mathrm{Re}\lambda_k - \mathrm{Re}\lambda_{k+1}|$ and $d < \frac{1}{4}$ is a constant. Then, the sequence $\left\{e^{i(\lambda_k+\delta_k)t}\right\}_{k\in\mathbb{Z}}$ is a Riesz basis in $L^2(-\sigma,\sigma)$.

The following is the main result of this paper. It is a generalisation in the sense that the perturbed sampling points in Higgins's result are replaced by complex numbers with a certain maximum distance from the zeros of a π-sine-type function.

Theorem 2.4. *Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be the set of zeros of a π-sine-type function and let $\{\lambda_k^*\}_{k\in\mathbb{Z}}$ be a sequence of complex numbers satisfying*

$$|\mathrm{Re}\lambda_k^* - \mathrm{Re}\lambda_k| \le dp, \sup_{k\in\mathbb{Z}} |\mathrm{Im}\lambda_k^* - \mathrm{Im}\lambda_k| < \infty, \tag{6}$$

where $d < \frac{1}{4}$ and $p = \inf_k |\mathrm{Re}\lambda_k - \mathrm{Re}\lambda_{k+1}|$. Then, for all $f \in \mathcal{PW}_\pi^2$, we obtain

$$f(z) = \sum_{k\in\mathbb{Z}} f(\lambda_k^*) \frac{\varphi^*(z)}{\varphi^{*\prime}(\lambda_k^*)(z-\lambda_k^*)}, \tag{7}$$

where the convergence is uniform on any horizontal strip in $\mathbb{C}$ of finite width.

Proof. By Theorem 2.3, we have that the sequence $\{e^{i\lambda_k^*(.)}\}_{k\in\mathbb{Z}}$ forms a Riesz basis over $L^2(-\pi,\pi)$. The sequence $\{e^{i\lambda_k^*(.)}\}_{k\in\mathbb{Z}}$ possesses a complete biorthogonal sequence $\{g_k\}_{k\in\mathbb{Z}}$ in $L^2(-\pi,\pi)$, see [18, Theorem 9, p. 27]. The sequence $\{g_k\}_{k\in\mathbb{Z}}$ is a Riesz basis being biorthogonal to $\{e^{i\lambda_k^*(.)}\}_{k\in\mathbb{Z}}$, see [18, p. 30]. Let $h_k = \overline{g_k}$. It follows that

$$\int_{-\pi}^{\pi} h_n(x) e^{i\lambda_m^* x} dx = \left\langle e^{i\lambda_m^*(.)}, g_n \right\rangle = \delta_{mn}. \tag{8}$$

We define the function G_n as

$$G_n(z) = \int_{-\pi}^{\pi} h_n(x) e^{izx} dx. \tag{9}$$

By using the biorthogonality condition (8) we obtain that

$$G_n(z) = \frac{\varphi^*(z)}{(z-\lambda_n^*)} K(z),$$

with $K(\lambda_n^*) \neq 0$. We claim that the function K has no zeros. The claim will be proved by way of contradiction. Assume that $K(\mu) = 0$ for $\mu \neq \lambda_n^*$ and define the function H as

$$H(z) = G_n(z) \frac{(z-\lambda_n^*)}{(z-\mu)}.$$

Then, the function H belongs to $\mathcal{PW}_\pi^2 = \mathcal{B}_\pi^2$ because G_n does, and the factor $\frac{(z-\lambda_n^*)}{(z-\mu)}$ is asymptotically equal to 1. The function H vanishes on the complete interpolating set $\Lambda = \{\lambda_k^*\}_{k\in\mathbb{Z}}$ and thus $H \equiv 0$. This implies that G_n is identically equal to zero, a contradiction. Therefore, the function K is different from zero everywhere. Now, by virtue of Theorem 2.1 and Lemma 2.2, we obtain

$$G_n(z) = \frac{\varphi^*(z)}{z-\lambda_n^*} K(z) = A z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right).$$

As φ^* has only simple zeros and $K(z)$ has no zeros, we have that $z_k \neq z_l$ for $k \neq l$. Furthermore, one has either $m = 0$ and $\lambda_0 \neq 0$, or $m = 1$ and $\lambda_0 = 0$. In either case

$\{z_k \mid k=1,2,\ldots\} = \{\lambda_k^* \mid k\in\mathbb{Z}, k\neq n\}$. It follows that $K(z)$ is a constant. By the biorthogonality condition, we obtain that

$$1 = G_n\left(\lambda_n^*\right) = \lim_{z\to\lambda_n^*} G_n\left(z\right) = \varphi^{*\prime}\left(\lambda_n^*\right) K\left(\lambda_n^*\right)$$

and, thus, we rewrite (9) as

$$\int_{-\pi}^{\pi} h_n(x)e^{izx}dx = \frac{\varphi^*(z)}{\varphi^{*\prime}\left(\lambda_n^*\right)\left(z-\lambda_n^*\right)} = \varphi_n^*(z). \tag{10}$$

Now, if $f\in\mathcal{PW}_\pi^2$, then

$$f(z) = \int_{-\pi}^{\pi} g(w)e^{izw}dw$$

for some g in $L^2\left(-\pi,\pi\right)$. We have that $\{h_k\}_{k\in\mathbb{Z}}$ forms a Riesz basis for $L^2\left(-\pi,\pi\right)$ and so

$$g(w) = \lim_{N\to\infty}\sum_{m=-N}^{N} c_m h_m(w)$$

converges in the L^2 sense. It follows that $f(\lambda_m^*) = \left\langle e^{i\lambda_m^*(\cdot)}, \overline{g}\right\rangle = \left\langle e^{i\lambda_m^*(\cdot)}, \sum_{k\in\mathbb{Z}}\overline{c_k}g_k\right\rangle = c_m$ by biorthogonality. Now,

$$\begin{aligned}\left|f(z) - \sum_{m=-N}^{N} f(\lambda_m^*)\frac{\varphi^*(z)}{\varphi^{*\prime}(\lambda_m^*)(z-\lambda_m^*)}\right| &= \left|\int_{-\pi}^{\pi} g(w)e^{izw}dw - \sum_{m=-N}^{N} c_m\int_{-\pi}^{\pi} h_m(w)e^{izw}dw\right| \\ &= \left|\int_{-\pi}^{\pi}\left[g(w) - \sum_{m=-N}^{N} c_m h_m(w)\right]e^{izw}dw\right| \\ &\leq \sqrt{2\pi}\left\|g - \sum_{m=-N}^{N} c_m h_m\right\|_2 e^{\pi|\mathrm{Im}z|}.\end{aligned}$$

Thus, for $|\mathrm{Im}z|\leq M$ for some $M\in\mathbb{R}$, we have

$$\lim_{N\to\infty}\left|f(z) - \sum_{m=-N}^{N} f(\lambda_m^*)\frac{\varphi^*(z)}{\varphi^{*\prime}(\lambda_m^*)(z-\lambda_m^*)}\right| \leq \lim_{N\to\infty}\sqrt{2\pi}\left\|g - \sum_{m=-N}^{N} c_m h_m\right\|_2 e^{\pi|\mathrm{Im}z|} = 0,$$

which shows that the convergence is uniform over any horizontal strip of finite width. $\square$

In the following, we give some examples that boil down to specific known results.

Example 2.5. If $\{\lambda_k^*\}_{k\in\mathbb{Z}}$ is a sequence of real numbers and $\{\lambda_k\}_{k\in\mathbb{Z}} = \mathbb{Z}$, which are zeros of a π-sine-type function, then $p = \inf_k|\mathrm{Re}\lambda_k - \mathrm{Re}\lambda_{k+1}| = 1$ and

$|\mathrm{Re}\lambda_k^* - \mathrm{Re}\lambda_k| \le d < \frac{1}{4}$, and thus for all $f \in \mathcal{B}_\pi^2 = \mathcal{PW}_\pi^2$ we obtain that

$$f(t) = \sum_{k\in\mathbb{Z}} f\left(\lambda_k^*\right) \frac{\varphi^*(t)}{\varphi^{*\prime}(\lambda_k^*)(t-\lambda_k^*)}, \quad t \in \mathbb{R},$$

where the convergence is uniform over $\mathbb{R}$. This is the sampling series by Higgins, see Theorem 1.2. Furthermore, if we additionally set $d = 0$, then we obtain the WKS sampling series (1).

3. Stability

The question of stability of the function reconstruction comes into play when we deal with perturbations. We state the following theorem to establish the stability in the sense of Riesz, cf. [11, Theorem 3.12].

Theorem 3.1. *Under Fourier transformation, the pre-image of a Riesz basis for $L^2(-\pi,\pi)$ is a Riesz basis for $\mathcal{PW}_\pi^2$.*

With this result, we see that the sequence $\{\varphi_k^*\}_{k\in\mathbb{Z}}$, the sampling basis, forms a Riesz basis for $\mathcal{PW}_\pi^2$ because it is a pre-image of the Riesz basis $\{h_k\}_{k\in\mathbb{Z}} \in L^2(-\pi,\pi)$; see the first part of the proof of Theorem 2.4 together with (10). Hence, the stability in the sense of (5) follows.

Another type of stability is defined from the sampling set point of view. The set $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is called a *set of stable sampling (or sampling)* for the space $\mathcal{PW}_\pi^2$, if there exists a constant K such that

$$\|f\|_{L^2}^2 \le K \sum_{k\in\mathbb{Z}} |f(\lambda_k)|^2$$

for all $f \in \mathcal{PW}_\pi^2$. Now, if $f \in \mathcal{PW}_\pi^2$, then the sampling series (7) holds. It is shown above that the sequence

$$\varphi_k^*(z) = \frac{\varphi^*(z)}{\varphi^{*\prime}\left(\lambda_k^*\right)\left(z-\lambda_k^*\right)}$$

forms a Riesz basis for $\mathcal{PW}_\pi^2$ and so (5) holds. Therefore, the stability of $\{\lambda_k^*\}_{k\in\mathbb{Z}}$, the set that satisfies (6), in the sense of stable sampling for the space $\mathcal{PW}_\pi^2$ follows from the right-hand side of (5).

4. An Extension to the Function Space $\mathcal{PW}_\pi^1$

In this section, we extend the sampling series to larger function spaces. In practice, the perturbation occurs on a finite number of points. More generally, if we assume that the sequence of perturbation $\{\delta_k\}_{k\in\mathbb{Z}} \in l^2$, then the sampling series (7) can be used for the space $\mathcal{PW}_\pi^1 \supset \mathcal{PW}_\pi^2$. We state the Phragmén-Lindelöf theorem for use in the next result.

Theorem 4.1 (Phragmén-Lindelöf)**.** *If $f(z)$ is an entire function of exponential type σ, and $|f(z)| \leq M$, $-\infty < x < \infty$, then*

$$|f(x+iy)| \leq Me^{\sigma|y|}$$

in the whole plane $\mathbb{C}$.

A special case of a result by Levin-Ostrovskii [12, p. 85] is proved below.

Theorem 4.2. *Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be the zeros of a π-sine-type function and let $\{\delta_k\}_{k\in\mathbb{Z}} \in l^2$. Then, the function*

$$\varphi^*(z) = \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{\lambda_k + \delta_k}\right)$$

is a π-sine-type function.

Proof. First, if $\{\delta_k\}_{k\in\mathbb{Z}}$ is the zero sequence, then there is nothing to prove. Next, we show that φ^* is a π-sine-type function. We define Φ as

$$\Phi(z) = \left|\frac{\varphi^*(z)}{\varphi(z)}\right|,$$

and we let $z \in S = \{x + iy \mid y = H\}$ with

$$H = \sup_{k\in\mathbb{Z}} \{|\mathrm{Im}\lambda_k|\} + 2. \sup_{k\in\mathbb{Z}} \{|\delta_k|\}.$$

We reorder the terms as

$$\begin{aligned}
\Phi(z) &= \left|\lim_{N\to\infty} \prod_{|k|\leq N} \left(1 - \frac{z}{\lambda_k+\delta_k}\right) / \lim_{N\to\infty} \prod_{|k|\leq N} \left(1 - \frac{z}{\lambda_k}\right)\right| \\
&= \lim_{N\to\infty} \left|\prod_{|k|\leq N} \left(1 - \frac{z}{\lambda_k+\delta_k}\right) / \prod_{|k|\leq N} \left(1 - \frac{z}{\lambda_k}\right)\right| \\
&= \lim_{N\to\infty} \left|\prod_{|k|\leq N} \left\{\left(\frac{\lambda_k - z + \delta_k}{\lambda_k+\delta_k}\right) \cdot \left(\frac{\lambda_k}{\lambda_k - z}\right)\right\}\right| \\
&= \lim_{N\to\infty} \prod_{|k|\leq N} \left|\left\{\left(\frac{\lambda_k - z + \delta_k}{\lambda_k - z}\right) \cdot \left(\frac{\lambda_k + \delta_k - \delta_k}{\lambda_k+\delta_k}\right)\right\}\right| \\
&= \lim_{N\to\infty} \prod_{|k|\leq N} \left|\left\{\left(1 + \frac{\delta_k}{\lambda_k - z}\right) \cdot \left(1 - \frac{\delta_k}{\lambda_k+\delta_k}\right)\right\}\right| \qquad (11) \\
&\leq \lim_{N\to\infty} \prod_{|k|\leq N} \left(1 + \frac{|\delta_k|}{|\lambda_k - z|}\right) \cdot \lim_{N\to\infty} \prod_{|k|\leq N} \left(1 + \frac{|\delta_k|}{|\lambda_k+\delta_k|}\right). \qquad (12)
\end{aligned}$$

The limit in (11) splits into the two limits in (12) because each limit exists and is different from zero. Now, we show the convergence of the limits in (12) and that will be done by showing the convergence of the series

$$\sum_{k\in\mathbb{Z}}\frac{|\delta_k|}{|\lambda_k - z|}.$$

We consider the following partitioning of the zeros

$$J_k = \{\lambda_n \mid \lambda_n \in [k, k+1) \times [-iH, iH]\}.$$

Let $N = \max_{n\in\mathbb{Z}}\mathrm{Card}(J_n)$. The set $\{\lambda_k\}_{k\in\mathbb{Z}}$ is uniformly discrete and so there exists $\lambda_{n_k} \in J_k$ such that $|z - \lambda_{n_k}| \le |z - \lambda_n|$ for all $\lambda_n \in J_k$. For an arbitrarily fixed z, we have $\mathrm{Re}z \in [m, m+1)$ for some $m \in \mathbb{Z}$. Thus, there exists $\varepsilon > 0$ such that

$$|z - \lambda_{n_k}| > \varepsilon\left(|k - m| + 1\right) \quad \text{for all } k \in \mathbb{Z}.$$

Hence,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\frac{1}{|z-\lambda_k|^2} &= \sum_{k\in\mathbb{Z}}\sum_{\lambda_n\in J_k}\frac{1}{|z-\lambda_k|^2} \\
&\le \sum_{k\in\mathbb{Z}}\sum_{\lambda_n\in J_k}\frac{1}{|z-\lambda_{n_k}|^2} \\
&\le \sum_{k\in\mathbb{Z}}\frac{N}{|z-\lambda_{n_k}|^2} \\
&< \frac{N}{\varepsilon^2}\sum_{k\in\mathbb{Z}}\frac{1}{\left(|k-m|+1\right)^2} = \gamma < \infty.
\end{aligned}$$

Thus,

$$\sum_{k\in\mathbb{Z}}\frac{|\delta_k|}{|\lambda_k - z|} \le \left(\sum_{k\in\mathbb{Z}}|\delta_k|^2\right)^{1/2}(\gamma)^{1/2} < \infty \tag{13}$$

by using the Cauchy- Schwarz inequality. Therefore, $|\varphi^*(z)| \le C|\varphi(z)| \le \widetilde{C}e^{\pi|H|}$ for all $z \in S$ which means that $|\varphi^*(z)|$ is bounded on the horizontal line $y = H$. Now, $\psi(z) = \varphi^*(x + i(y + H))$ is an entire function of exponential type $\le \pi$ since φ^* has the zeros $\{\lambda_k + \delta_k\}_{k\in\mathbb{Z}}$ where $\{\lambda_k\}_{k\in\mathbb{Z}}$ is the set of zeros of a π-sine-type function and $\{\delta_k\}_{k\in\mathbb{Z}}$ is a bounded sequence of complex numbers; see [12, p. 80]. Thus, by Theorem 4.1, we obtain

$$|\psi(x + iy)| \le Me^{\pi|y|}.$$

Now, shifting up the function ψ by replacing $y \mapsto y - H$, we have

$$|\varphi^*(x + iy)| \le Me^{\pi|y-H|} \le \widetilde{M}e^{\pi|y|} \tag{14}$$

for some $\widetilde{M} > 0$ and all $y \in \mathbb{R}$. For the lower bound, we use the reverse triangle inequality in (11) to obtain

$$\Phi(z) \geq \lim_{N\to\infty} \prod_{|k|\leq N} \left(1 - \frac{|\delta_k|}{|\lambda_k - z|}\right) \cdot \lim_{N\to\infty} \prod_{|k|\leq N} \left(1 - \frac{|\delta_k|}{|\lambda_k + \delta_k|}\right) = \Delta(|y|), \tag{15}$$

where $|y| \geq H$. The right-hand side of the inequality (15) converges to a non-zero limit by using (13). Thus,

$$|\varphi^*(x+iy)| \geq \Delta(|y|)\, |\varphi(x+iy)| \geq \Delta(|y|)\, m e^{\pi|y|}$$

for all $|y| \geq H$. Note that the function $\Delta(|y|)$ is increasing, which makes $\Delta(|y|) \geq \Delta(H)$. Therefore,

$$|\varphi^*(x+iy)| \geq \widetilde{m} e^{\pi|y|} \tag{16}$$

for all $|y| \geq H$. By obtaining the inequalities (16) and (14), the proof is complete. $\square$

Theorem 4.3. *Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be the set of zeros of a π-sine-type function, $\varphi(z)$ be the function defined as in (3) and let*

$$\varphi_n(z) = \frac{\varphi(z)}{\varphi'(\lambda_n)(z-\lambda_n)}.$$

Then, for all $f \in \mathcal{B}_\pi^\infty$ and all z in a compact subset $\mathcal{C} \subset \mathbb{C}$ we obtain

$$\left| f(z) - \sum_{n=-N}^{N} f(\lambda_n)\, \varphi_n(z) \right| \leq C_{\mathcal{C}} \|f\|_\infty$$

for a sufficiently large N.

Proof. Let $z \in \mathcal{C}$ and $H = \sup\{|y| \mid x+iy \in \mathcal{C}\}$. Then, by using the Cauchy integral formula, we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{f(w)}{(w-z)} dw \\ &= \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{[\varphi(w) - \varphi(z) + \varphi(z)]}{(w-z)} \frac{f(w)}{\varphi(w)} dw \\ &= \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{[\varphi(w) - \varphi(z)]}{(w-z)} \frac{f(w)}{\varphi(w)} dw + \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{\varphi(z)}{(w-z)} \frac{f(w)}{\varphi(w)} dw \end{aligned} \tag{17}$$

For the first integral in (17), we use the residue theorem with the rectangular contour Γ_n. The contour Γ_n is defined as $\Gamma_n = \eta_1 \cup \eta_2 \cup \gamma_1 \cup \gamma_2$ where

$$\eta_1 = \{x + iy_n \mid x_{-n} = \mathrm{Re}(\lambda_{-n} + \lambda_{-n-1})/2 \leq x \leq x_n = \mathrm{Re}(\lambda_n + \lambda_{n+1})/2,\ y_n = x_n\},$$

$$\eta_2 = \{x - iy_n \mid x_{-n} = \mathrm{Re}(\lambda_{-n} + \lambda_{-n-1})/2 \leq x \leq x_n = \mathrm{Re}(\lambda_n + \lambda_{n+1})/2,\ y_n = x_n\},$$

$$\gamma_1 = \{x_n + iy \mid x_n = \mathrm{Re}(\lambda_n + \lambda_{n+1})/2,\ |y| < x_n\},$$

and

$$\gamma_2 = \{x_{-n} + iy \mid x_{-n} = \mathrm{Re}(\lambda_{-n} + \lambda_{-n-1})/2,\ |y| < x_n\}.$$

It follows that

$$\mathrm{Res}\left(\frac{[\varphi(w)-\varphi(z)]}{(w-z)}\frac{f(w)}{\varphi(w)}, \lambda_k\right) = \frac{\varphi(z) f(\lambda_k)}{(z-\lambda_k)\,\varphi'(\lambda_k)},$$

for all λ_k's that are inside the contour Γ_n. Thus,

$$f(z) = \sum_{k=-n}^{n} f(\lambda_k)\frac{\varphi(z)}{(z-\lambda_k)\,\varphi'(\lambda_k)} + E_n(z),$$

where

$$E_n(z) = \frac{1}{2\pi i}\oint_{\Gamma_n} \frac{\varphi(z)}{(w-z)}\frac{f(w)}{\varphi(w)}dw.$$

For the contour integral $E_n(z)$, we only carry out the computation over the first quadrant because the computation over the other quadrants follows in the same way. We perform the computation over the vertical segment and the horizontal segment as follows:

$$\begin{aligned} I_{\gamma_1} = \left|\oint_{\gamma_1} \frac{f(w)\,\varphi(z)}{(w-z)\,\varphi(w)}dw\right| &\leq C_1 \|\varphi\|_\infty e^{\pi|H|}\int_{[0,x_n]} \frac{\|f\|_\infty e^{\pi|y|}}{e^{\pi|y|}\,|x_n - \mathrm{Re}z|}dy \\ &\leq \frac{C_1 \|\varphi\|_\infty \|f\|_\infty e^{\pi|H|}\,|x_n|}{|x_n - \mathrm{Re}z|} \end{aligned}$$

by using Theorem 4.1 and (2) for the functions f and φ. For a sufficiently large n, we have $x_n \neq \mathrm{Re}z$. Then,

$$I_{\gamma_1} = O\left(\|f\|_\infty\right) \text{ as } n \to \infty.$$

Similarly,

$$\begin{aligned} I_{\eta_1} = \left|\oint_{\eta_1} \frac{f(w)\,\varphi(z)}{(w-z)\,\varphi(w)}dw\right| &\leq C_1 \|\varphi\|_\infty e^{\pi|H|}\int_{[0,x_n]} \frac{\|f\|_\infty e^{\pi|y|}}{e^{\pi|y|}\,|i\,(x_n - \mathrm{Im}z)|}dx \\ &\leq \frac{C_1 \|\varphi\|_\infty \|f\|_\infty e^{\pi|H|}\,|x_n|}{|i\,(x_n - \mathrm{Im}z)|} \end{aligned}$$

and so

$$I_{\eta_1} = O\left(\|f\|_\infty\right) \quad \text{as } n \to \infty.$$

Therefore,

$$\left|f\left(z\right)-\sum_{n=-N}^{N}f\left(\lambda_{n}\right)\varphi_{n}\left(z\right)\right|\leq C_{\mathcal{C}}\left\|f\right\|_{\infty},$$

which completes the proof. □

In the next result, the density property of $\mathcal{PW}_{\pi}^{2}$ in $\mathcal{PW}_{\pi}^{1}$ is used to obtain an approximate reconstruction for the functions in the space $\mathcal{PW}_{\pi}^{1}$.

Theorem 4.4. *Let φ be a function of π-sine-type with zeros $\{\lambda_k\}_{k\in\mathbb{Z}}$ and let $\lambda_k^*=\lambda_k+\delta_k$ with $\{\delta_k\}_{k\in\mathbb{Z}}\in l^2$ satisfy*

$$|\mathrm{Re}\lambda_k^*-\mathrm{Re}\lambda_k|\leq dp,\ \sup_{k\in\mathbb{Z}}|\mathrm{Im}\lambda_k^*-\mathrm{Im}\lambda_k|<\infty,$$

where $d<\frac{1}{4}$ and $p=\inf_k|\mathrm{Re}\lambda_k-\mathrm{Re}\lambda_{k+1}|$. Then, for all $f\in\mathcal{PW}_{\pi}^{1}$, we have

$$f(z)=\sum_{k\in\mathbb{Z}}f\left(\lambda_k^*\right)\varphi_k^*\left(z\right),$$

where the convergence is uniform over any compact subset $\mathcal{C}$ of $\mathbb{C}$.

Proof. Let $z\in\mathcal{C}$, $H=\sup\{|y|\mid x+iy\in\mathcal{C}\}$, $f\in\mathcal{PW}_{\pi}^{1}$, and $\epsilon>0$. Then, there exist $g_\epsilon\in\mathcal{PW}_{\pi}^{2}$ such that $\|f-g_\epsilon\|_{\mathcal{PW}_{\pi}^{1}}<\epsilon$. Now,

$$\begin{aligned}
&\left|f\left(z\right)-\sum_{k=-N}^{N}f\left(\lambda_k^*\right)\varphi_k^*\left(z\right)\right|=\\
&\left|f\left(z\right)-g_\epsilon\left(z\right)+g_\epsilon\left(z\right)-\sum_{k=-N}^{N}g_\epsilon\left(\lambda_k^*\right)\varphi_k^*\left(z\right)+\sum_{k=-N}^{N}g_\epsilon\left(\lambda_k^*\right)\varphi_k^*\left(z\right)-\sum_{k=-N}^{N}f\left(\lambda_k^*\right)\varphi_k^*\left(z\right)\right|\\
&=\left|\left(f-g_\epsilon\right)\left(z\right)-\left(\sum_{k=-N}^{N}f\left(\lambda_k^*\right)\varphi_k^*\left(z\right)-\sum_{k=-N}^{N}g_\epsilon\left(\lambda_k^*\right)\varphi_k^*\left(z\right)\right)+g_\epsilon\left(z\right)-\sum_{k=-N}^{N}g_\epsilon\left(\lambda_k^*\right)\varphi_k^*\left(z\right)\right|\\
&\leq\left|\left(f-g_\epsilon\right)\left(z\right)-\left(\sum_{k=-N}^{N}\left(f-g_\epsilon\right)\left(\lambda_k^*\right)\varphi_k^*\left(z\right)\right)\right|+\left|g_\epsilon\left(z\right)-\sum_{k=-N}^{N}g_\epsilon\left(\lambda_k^*\right)\varphi_k^*\left(z\right)\right|\\
&\leq 2\pi C_{\mathcal{C}}\left\|f-g_\epsilon\right\|_{\mathcal{PW}_{\pi}^{1}}+(2\pi)^{3/2}e^{\pi|H|}\left\|g_\epsilon\left(.\right)-\sum_{k=-N}^{N}g_\epsilon\left(\lambda_k^*\right)\varphi_k^*\left(.\right)\right\|_{\mathcal{PW}_{\pi}^{2}}\\
&<2\pi C_{\mathcal{C}}\epsilon+\left(2\pi\right)^{3/2}e^{\pi|H|}\epsilon=\left(2\pi C_{\mathcal{C}}+\left(2\pi\right)^{3/2}e^{\pi|H|}\right)\epsilon
\end{aligned}$$

by using Theorem 4.3 for the first part of the inequality and Theorem 2.4 and the fact $|.|\leq 2\pi e^{\pi|y|}\left\|.\right\|_{\mathcal{PW}_{\pi}^{1}}\leq(2\pi)^{3/2}e^{\pi|y|}\left\|.\right\|_{\mathcal{PW}_{\pi}^{2}}$ for the second part. □

5. Numerical Example

In this section, we give an example for Theorem 2.4. In the following, we have the plot of the function

$$f(x) = \frac{\sin \pi \left(x - \frac{2}{10}\right)}{\left(x - \frac{2}{10}\right)} + \frac{\sin \frac{\pi}{3} \left(x - \frac{23}{10}\right)}{\left(x - \frac{23}{10}\right)}$$

that is in the space $\mathcal{PW}_\pi^2$. We consider sampling points that are the zeros of a π-sine-type function. The π-sine-type function we consider here is $g(x) = \cos \pi x - 0.5 \sin \frac{\pi}{\sqrt{3}} x$. It has zeros with $p = 0.7448$. The zeros form a set of non-equidistant and non-periodic sampling points. We perturb the first ten positive zeros. The perturbation achieved by a random function that maintains the quarter condition in Theorem 2.4. The number of terms used for the reconstruction is $2N + 1 = 801$. The truncation error is $\|e\|_{L^\infty[-10,10]} = 0.0025$.

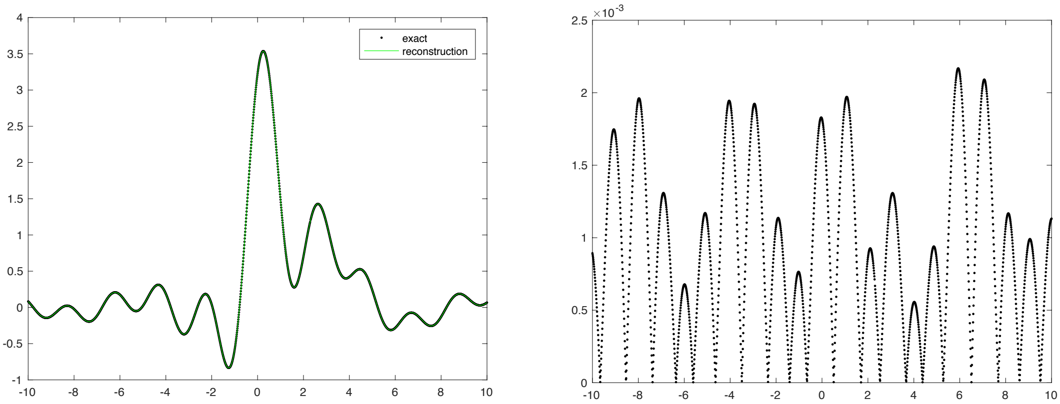

Figure 1. The reconstruction with sampling points that are the zeros of a sine-type function. The reconstruction is on the left and the error is on the right.

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