The Zeros of a Sine-Type Function and the Peak Value Problem

Hussain Al-Hammali [1]
hussain@knu.ac.kr
Kyungpook National University, Department of Mathematics, Daegu, 41566, Republic of Korea

Adel Faridani
faridani@oregonstate.edu
Oregon State University, Department of Mathematics, Corvallis, OR, 97331, USA

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Abstract

The problem of finding an upper bound for the infinity norm of signals from their sample values is called the peak value problem. The peak value problem is a significant problem related to orthogonal frequency division multiplexing (OFDM) which has applications in wireless networks, audio broadcasting, and mobile communications. In this article, we will first answer the question of whether the zeros of a π-sine-type function form a stable sampling set for $\mathcal{B}^{\infty}_{\beta\pi}$, $0 < \beta < 1$. Then, we will address a related question of bounding the infinity norm of a given signal. The sampling set we use only requires a bound on the maximum distance between two consecutive sampling points. Our result for estimating the bound works for a nonuniform sampling set.

Keywords: π-sine-type function; Stable sampling; Bernstein space.

1. Motivation

The primary use of sampling theorems such as the Shannon sampling theorem is to reconstruct signals from their sample values. Sampling theorems may differ from each other in terms of sampling basis. Some bases are constructed using uniform sampling points such as the scaled integers that follow the Nyquist rate, whereas others use nonuniform sampling points. One example of the nonuniform sampling points is the sampling set that is formed by the union of a shifted group (lattice) or, more generally, by unions of shifted lattices [1]. Obviously, sampling theorems show a direct use of the sample values in the signal reconstruction. Following the same acquisition, we pose an interesting question regards to whether the peak value that may occur during signal transmission can be estimated by knowing the signal sample values. The problem of finding an upper bound for the infinity norm of signals from their samples is called the *peak value problem.* For a background on the problem, see [2] and [3].

The peak value problem is a significant problem related to orthogonal frequency division multiplexing (OFDM) which has applications in wireless networks and mobile communications. OFDM has been used for wireless networks for its advantage in converting a wide-band frequency channel into a series of narrow band frequency using a parallel multicarrier transmission scheme [4]. One drawback of OFDM is the high peak-to-average power ratio (PAPR) [5]. Several methods have been developed to overcome this obstacle, which may occur in some circumstances. Some of those methods suggest computing the magnitude of the transmitted signal in their schemes [3], [6] and [7]. Not only computing the magnitude of the transmitted signal does make the problem interesting but also finding such a bound is important for showing the stability of the sampling set over a certain function space.

In this paper, we address two questions related to the peak value problem. First, we provide an affirmative answer to a question proposed by Boche and Mönich of whether the zero set of a π-sine-type function is a set of stable sampling for the space $\mathcal{B}^{\infty}_{\beta\pi}$, $0 < \beta < 1$. We will prove this by making use of a general result by A. Beurling and some specific properties of sine-type functions. The second and main result is an estimate for the peak value of a function in $\mathcal{B}^{\infty}_{\beta\pi}$, $0 < \beta < 1$, based on its samples on a sampling set that has a sufficiently small maximum

[1] The corresponding author. Alternative e-mails are: ssuhyh@yahoo.com & alhammah@lifetime.oregonstate.edu.

gap between consecutive points, but is otherwise arbitrary. An important tool in the proof is the sampling theorem of Valiron-Tschakaloff, for which we prove a new generalized version.

This paper is organized as follows: Section 1 provides a motivation for the problem. Section 2 defines and presents preliminary results. Section 3 discusses the solvability of stable sampling for certain function space. Section 4 discusses a generalization on Valiron-Tschakaloff sampling series. Section 5 presents the estimation of the infinity norm where the sampling set is a perturbed set from the integers. Section 6 provides a numerical example of the main result. Section 7 presents the conclusion of this paper followed by an appendix in Section 8. The results of Sections 4 and 5 are based on the first author's Ph.D. thesis [8].

2. Definitions and Preliminary Results

The norm that we consider here is the infinity norm unless otherwise specified. An entire function f is of *exponential type* at most σ ($\sigma > 0$), and we write f is a function of exponential type $\leq \sigma$, if for any $\epsilon > 0$ there exists an A_ϵ such that

$$|f(z)| \leq A_\epsilon e^{(\sigma+\epsilon)|z|}$$

for all $z \in \mathbb{C}$. Let E denote the space of all entire functions and E_σ denote the class of all entire functions of exponential type $\leq \sigma$. A function of exponential type $\leq \sigma$ is said to be a *σ-sine-type* function if

(i) the zeros $\{\lambda_k\}_{k\in\mathbb{Z}}$ of f are simple and separated (that is, uniformly discrete, $\inf_{j\neq k} |\lambda_j - \lambda_k| \geq \underline{\delta}$ for all $k \in \mathbb{Z}$ and some $\underline{\delta} > 0$) and

(ii) there exist positive constants A, B, and η such that

$$Ae^{\sigma|y|} \leq |f(x+iy)| \leq Be^{\sigma|y|} \tag{1}$$

for all $x, y \in \mathbb{R}$ and $|y| \geq \eta$.

We define the generating function that will be used to create the sampling basis by the following canonical product

$$\varphi(z) = \lim_{N\to\infty} \prod_{|k|\leq N} \left(1 - \frac{z}{\lambda_k}\right), \tag{2}$$

where $(1 - z/\lambda_k)$ is replaced by z if $\lambda_k = 0$. It is known that if $\lambda_k = k \in \mathbb{Z}$, then

$$\varphi(z) = \frac{1}{\pi} \sin \pi z.$$

The sinc function is defined as

$$\text{sinc} z = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}.$$

$L^p(\mathbb{R})$ is the space of functions for which the p-th power of the absolute value is Lebesgue integrable i.e.

$$\|f\|_p = \left(\int_{\mathbb{R}} |f|^p \, d\mu\right)^{1/p} < \infty,$$

and

$$\|f\|_\infty = \max_{x\in\mathbb{R}} |f(x)| < \infty.$$

The *Bernstein space*, denoted $\mathcal{B}_\sigma^p$, is the function space that consists of all functions in E_σ whose restrictions to the real line are in $L^p(\mathbb{R})$. The norm for $\mathcal{B}_\sigma^p$, $1 \leq p \leq \infty$ is given by $\|f\|_{\mathcal{B}_\sigma^p} = \|f\|_p$. The Bernstein spaces can be ordered as follows:

$$\mathcal{B}_\sigma^1 \subset \mathcal{B}_\sigma^2 \subset \ldots \subset \mathcal{B}_\sigma^\infty.$$

The general goal of this work is to estimate the peak value of a function f which has been reconstructed from its samples. In order to make the reconstruction of f from its samples possible, we make the common assumption that the functions are bandlimited, so that sampling theorems can be applied. We call a function f *bandlimited* if its Fourier transform $\hat{f}(\xi)$ vanishes for ξ outside a bounded interval $[-\sigma, \sigma]$. But such bandlimited functions are always also entire functions. According to the Payley-Wiener theorem, bandlimited functions are entire functions of exponential type $\leq \sigma$. The Bernstein spaces used in this paper consist of bandlimited functions. For example, the square integrable bandlimited functions are the elements of the Bernstein space $\mathcal{B}_\sigma^2$. The space $\mathcal{B}_\sigma^\infty$ is readily the wider class of functions that we can consider in this matter.

A set $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is called a *stable sampling* set for the space $\mathcal{B}_\sigma^\infty$ if there exists a constant K such that

$$\|f\|_\infty \leq K \sup_{k\in\mathbb{Z}} |f(\lambda_k)|$$

for all $f \in \mathcal{B}_\sigma^\infty$ [9, p. 1704]. After this definition, one can clearly see the importance of such a bound from the reconstruction standpoint regardless of the sharpness of the bound. However, having a smaller bound is more applicable to the peak value problem.

The below result is about the infinity norm estimate of a subclass of $\mathcal{B}_\pi^\infty$, that contains all trigonometric polynomials of degree $\leq n$ with real coefficients, that is

$$\mathcal{T}_n = \left\{f \mid f(t) = \tfrac{a_0}{2} + \textstyle\sum_{k=1}^n a_k \sin kt + b_k \cos kt, t, a_k, b_k \in \mathbb{R}\right\}$$

Theorem 1 (Ehlich and Zeller, [10])**.** *Let $m > n$, $m, n \in \mathbb{N}$ and $f \in \mathcal{T}_n$. Then,*

$$\|f\|_\infty \leq \frac{1}{\cos \frac{\pi n}{2m}} \max_{0\leq k<2m} \left|f\left(\frac{\pi}{m}k\right)\right|. \tag{3}$$

The estimate in (3) is sharp if and only if $n|m$.

Based on a previous result by Wunder and Boche [11], Jetter, Pfander and Zimmermann [12] deduced an estimate that is

$$\|f\|_\infty \leq \sqrt{\frac{m+n}{m-n}} \max_{0\leq k\leq N-1} \left|f\left(\frac{2\pi}{N}k\right)\right|$$

for $f \in \mathcal{T}_n$ where $m \geq n+1$ and $N \geq m+n$. In particular, for $N \geq 2n+1$, the choice $m = N-n$ gives

$$\sqrt{\frac{m+n}{m-n}} = \sqrt{\frac{N}{N-2n}}.$$

For a background of the problem in signal processing, see [5], [13] and [14].

3. The Solvability of Stable Sampling

In [15], Wunder and Boche obtained a result for the subclass of $\mathcal{B}^{\infty}_{\beta\pi,0}$, $0 < \beta < 1$ that is the restriction for all functions that are real when restricted to the real line. The result reads as follows:

Theorem 2 (Wunder and Boche, [15]). *If $0 < \beta < 1$ and $f \in \mathcal{B}^{\infty}_{\beta\pi,0}$, then*

$$\|f\|_{\infty} \leq \frac{1}{\cos(\beta\pi/2)} \sup_{k\in\mathbb{Z}} |f(\lambda_k)|. \tag{4}$$

In (4) the sampling set is $\mathbb{Z}$. If the sampling set $\mathbb{Z}$ in (4) is to be replaced by $a\mathbb{Z}$, then the factor $1/\cos(\beta\pi/2)$ has to be replaced by $1/\cos(\beta\pi a/2)$ provided that $0 < \beta < 1/a$. Here, $\mathcal{S}$ will denote the set of all sampling patterns $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ that are made of the zeros of a π-sine-type function. Accordingly, an interesting question is

Question 1. (Boche and Mönich, [16, p. 2218])
Let $0 < \beta < 1$ and $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \in \mathcal{S}$. Does there exist a constant $C = C(\beta)$ such that

$$\|f\|_{\infty} \leq C(\beta) \sup_{k\in\mathbb{Z}} |f(\lambda_k)|$$

for all $f \in \mathcal{B}^{\infty}_{\beta\pi}$?

In fact, a related question can be asked of whether the estimate in (4) holds true if the integers are replaced by a sampling set that satisfies the condition $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, $|\lambda_k - k| < 1/4$. This is a natural question since later on we will make a comparison with a new bound that implements the sampling set with the quarter condition.

There is a result by Beurling that shows the existence of such a constant, $C(\beta)$, under a sufficient and a necessary condition. Before we state the result, we define the following density condition:

$$D^{-}(\Lambda) = \lim_{A\to\infty} \lim_{r\to\infty} \frac{n_A^{-}(r)}{r} \tag{5}$$

where

$$n_A^{-}(r) = \min_{t\in\mathbb{R}} \#\left(\Lambda \cap \{x+iy | t < x < t+r, |y| < A\}\right).$$

For the zeros of a sine-type function the limit of A in (5) is insignificant since the zeros lie in a horizontal strip of a finite width. In that case we only require to have $A \geq \eta$.

Beurling's theorem reads as follows:

Theorem 3. (Beurling, [17, p. 346]) *Let $f \in \mathcal{B}^{\infty}_{\beta\pi}, 0 < \beta < 1$ and let $\{\lambda_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{R}$ be a uniformly discrete set. Then, we obtain*

$$\|f\|_{\infty} \leq K(\Lambda,\beta) \sup_{k\in\mathbb{Z}} |f(\lambda_k)|, \tag{6}$$

where $K(\Lambda,\beta) < \infty$ if and only if $D^{-}(\Lambda) > \beta$.

To answer Question 1, we use the following result by Horváth and Joó [18].

Theorem 4. (Horváth and Joó, [18, p. 268]) *Let F be a σ-sine-type function and let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ be its zeros. Then,*

$$\lim_{r\to\infty} \frac{n(t+r) - n(t)}{r} = \frac{\sigma}{\pi}$$

uniformly in $t \in \mathbb{R}$, where

$$n(t) := \sum_{n, Re\lambda_n \in [0,t]} 1.$$

The original statement in the proof of Theorem 4 is that for $\epsilon > 0$, there is $r > r_0(\epsilon)$ such that

$$\left| \frac{n(t+r) - n(t)}{r} - \frac{\sigma}{\pi} \right| < c\epsilon$$

for all $t \in \mathbb{R}$, where $c = c(F)$ is independent of ϵ, t and r. Thus,

$$\left| \frac{\min_{t\in\mathbb{R}} [n(t+r) - n(t)]}{r} - \frac{\sigma}{\pi} \right| < c\epsilon$$

still holds true for $r > r_0(\epsilon)$. It follows that

$$D^{-}(\Lambda) = \lim_{A\to\infty} \lim_{r\to\infty} \frac{n_A^{-}(r)}{r} = \frac{\sigma}{\pi} = \frac{\pi}{\pi} = 1 > \beta, \tag{7}$$

since $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is the set zeros of a π-sine-type function. Therefore, we conclude that

$$\|f\|_{\infty} \leq K(\Lambda,\beta) \sup_{k\in\mathbb{Z}} |f(\lambda_k)|$$

exists for all $f \in \mathcal{B}^{\infty}_{\beta\pi}, 0 < \beta < 1$, by virtue of Beurling's theorem, Theorem 3. This answers Question 1.

We end this section by extending the notion of stable sampling to the zeros of a sine-type function that are not necessarily real. We say that $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ is a set (a sequence) of stable sampling for $\mathcal{B}^{\infty}_{\sigma}$ if there exists K such that

$$\sup_{z\in\mathbb{C}} |f(z)| e^{-\sigma|\mathrm{Im} z|} \leq K \sup_{k\in\mathbb{Z}} |f(\lambda_k)| e^{-\sigma|\mathrm{Im}\lambda_k|}$$

for all $f \in \mathcal{B}^{\infty}_{\sigma}$ [19, p. 412]. The following result gives the necessary and sufficient condition for stable sampling. It reads as follows:

Theorem 5. (Ortega-Cerdá and Seip, [19, p. 413]) *A sequence Λ is a set of stable sampling for $\mathcal{B}^{\infty}_{\sigma}$ if and only if there exists a separated subsequence $\Lambda' \subset \Lambda$ such that*

$$D^{-}(\Lambda') > \frac{\sigma}{\pi}.$$

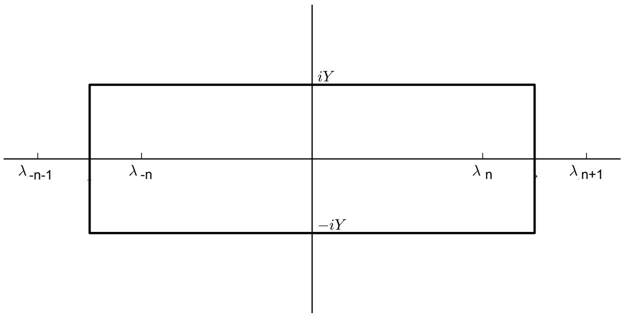

Figure 1: The contour for the contour integral method.

Now let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ be the zeros of a π-sine-type function. Then, Λ is a separated set being the set of zeros of a π-sine-type function. We choose Λ' to be Λ itself. Thus, from (7) we obtain

$$D^{-}(\Lambda') > \beta = \frac{\beta\pi}{\pi}.$$

Hence, the set of zeros of a π-sine-type function, $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$, forms a set of stable sampling for the space $\mathcal{B}^{\infty}_{\beta\pi}, 0<\beta<1$.

4. Generalized Valiron-Tschakaloff Sampling Series

In this section, we will generalize the Valiron-Tschakaloff sampling series. The generalization appears in Theorem 8 below. We will need to estimate a certain contour integral where the contour Γ_n is in Figure 1 and defined as $\Gamma_n = \eta_1 \cup \eta_2 \cup \gamma_1 \cup \gamma_2$, where $x_{-n} = \mathrm{Re}\,(\lambda_{-n}+\lambda_{-n-1})/2$, $x_n = \mathrm{Re}\,(\lambda_n+\lambda_{n+1})/2$

$$\eta_1 = \{x+iy_n \mid x_{-n} \le x \le x_n, y_n = x_n\},$$

$$\eta_2 = \{x-iy_n \mid x_{-n} \le x \le x_n, \, y_n = x_n\},$$

$$\gamma_1 = \{x_n + iy \mid \; |y| < x_n\},$$

and,

$$\gamma_2 = \{x_{-n} + iy \mid \; |y| < x_n\}.$$

The following two results are required for the proof of Theorem 8.

Theorem 6. (Levinson, [20, p. 56]) *Let φ be the canonical product in (2) with*

$$|\lambda_k - k| \le D < \frac{p-1}{2p}, \, 1<p\le 2.$$

Then,

$$A_p\,|\mathrm{Im}z|\,(|z|+1)^{-4D-1}e^{\pi|\mathrm{Im}z|} < |\varphi(z)| < B_p(|z|+1)^{4D}e^{\pi|\mathrm{Im}z|} \tag{8}$$

Theorem 7. (Phragmén-Lindelöf, [21, p. 39]) *If $f(z)$ is an entire function of exponential type π, and*

$$|f(x)| \le M, -\infty < x < \infty,$$

then

$$|f(x+iy)| \le Me^{\pi|y|},$$

in the whole complex plane $\mathbb{C}$.

The following is the generalization of Valiron-Tschakaloff sampling series. It will be used to prove the main result of this paper.

Theorem 8. *Let $f \in \mathcal{B}^{\infty}_{\pi}$ and let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ satisfy*

$$|\lambda_k - k| \le D < 1/4. \tag{9}$$

Then, for any $m \in \mathbb{Z}$ we obtain the following expansion

$$f(z) = L(z) + \sum_{k\in\mathbb{Z}\setminus\{m\}} f(\lambda_k)\frac{(z-\lambda_m)\,\varphi(z)}{(\lambda_k-\lambda_m)\,(z-\lambda_k)\,\varphi'(\lambda_k)}, \tag{10}$$

where φ is the canonical product (2) and $L(z)$ is

$$L(z) = \left\{\frac{\varphi(z)}{\varphi'(\lambda_m)}\right\} f'(\lambda_m) + \left\{\left[\frac{1}{(z-\lambda_m)\,\varphi'(\lambda_m)} - \frac{\frac{1}{2}\varphi''(\lambda_m)}{(\varphi'(\lambda_m))^2}\right]\varphi(z)\right\} f(\lambda_m).$$

The convergence is uniform over compact subsets $\mathcal{C}$ of $\mathbb{C}$.

Proof. Let $m \in \mathbb{Z}$ be arbitrary, $\mathcal{C} \subset \mathbb{C}$ compact, and $z \in \mathbb{C}$. Let n be sufficiently large such that both λ_m and $\mathcal{C}$ lie in the interior of the contour in Figure 1. Then, by the Cauchy integral formula we obtain

$$f(z) = \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{f(w)}{(w-z)}dw = \frac{1}{2\pi i}$$

$$\oint_{\Gamma_n}\frac{[(w-\lambda_m)\,\varphi(w) - ((z-\lambda_m)-(z-\lambda_m))\,\varphi(z)]f(w)}{(w-z)\,(w-\lambda_m)\,\varphi(w)}dw$$

$$= \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{[(w-\lambda_m)\,\varphi(w)-(z-\lambda_m)\,\varphi(z)]f(w)}{(w-z)\,(w-\lambda_m)\,\varphi(w)}dw + \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{[(z-\lambda_m)\,\varphi(z)]f(w)}{(w-z)\,(w-\lambda_m)\,\varphi(w)}dw. \tag{11}$$

For the first integral in (11) we obtain by the residue theorem for $k \ne m$, $|k| \le n$, that

$$\mathrm{Res}\left(\frac{[(w-\lambda_m)\,\varphi(w)-(z-\lambda_m)\,\varphi(z)]f(w)}{(w-\lambda_m)\,(w-z)\,\varphi(w)}, \lambda_k\right) = \frac{(z-\lambda_m)\,\varphi(z)f(\lambda_k)}{(\lambda_k-\lambda_m)\,(z-\lambda_k)\,\varphi'(\lambda_k)}.$$

In the next computation, the point λ_m is a singularity of order 2. After computing and simplifying, we obtain

$$\mathrm{Res}\left(\frac{[(w-\lambda_m)\,\varphi(w)-(z-\lambda_m)\,\varphi(z)]f(w)}{(w-\lambda_m)\,(w-z)\,\varphi(w)}, \lambda_m\right)$$

$$= \left\{ \frac{\varphi(z)}{\varphi'(\lambda_m)} \right\} f'(\lambda_m) + \left\{ \left[\frac{1}{(z-\lambda_m)\,\varphi'(\lambda_m)} - \frac{\frac{1}{2}\varphi''(\lambda_m)}{(\varphi'(\lambda_m))^2} \right] \varphi(z) \right\} f(\lambda_m).$$

Thus, we obtain

$$f(z) = \left\{ \frac{\varphi(z)}{\varphi'(\lambda_m)} \right\} f'(\lambda_m) + \left\{ \left[\frac{1}{(z-\lambda_m)\,\varphi'(\lambda_m)} - \frac{\frac{1}{2}\varphi''(\lambda_m)}{(\varphi'(\lambda_m))^2} \right] \varphi(z) \right\} f(\lambda_m) + \sum_{|k|\le n, k\neq m} f(\lambda_k) \frac{(z-\lambda_m)\,\varphi(z)}{(\lambda_k-\lambda_m)\,(z-\lambda_k)\,\varphi'(\lambda_k)} + E_n,$$

where

$$E_n = \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{[(z-\lambda_m)\,\varphi(z)]}{(w-z)} \frac{f(w)}{(w-\lambda_m)\,\varphi(w)} dw.$$

We now show that E_n goes to zero as n goes to infinity. For a sufficiently large n, we obtain $| y_n |> \eta$ (η as (ii) in the definition of σ-sine-type function) and $| y_n |>| \bar{y} |$ for all $z = \bar{x}+i\bar{y} \in \mathcal{C}$. Thus, we have $| f(x+iy) | \le \|f\|_\infty e^{\pi|y|}$ by Phragmén-Lindelöf, see Theorem 7, and also we have $| \varphi(x+iy) | \ge A\,|y|\,(|z|+1)^{-4D-1} e^{\pi|y|}$ as in (8). The error over the horizontal segment η_1 will be

$$\begin{aligned} &\left| \frac{1}{2\pi i} \oint_{\eta_1} \frac{[(z-\lambda_m)\,\varphi(z)]}{(w-z)} \frac{f(w)}{(w-\lambda_m)\,\varphi(w)} dw \right| \\ &\le \frac{M\,\|f\|_\infty}{2\pi A} \int_{x_{-n}}^{x_n} \frac{(|x+iy_n|+1)^{4D+1}}{|y_n|\,|\, x+iy_n-\bar{x}-i\bar{y} \,||\, x+iy_n-\lambda_m \,|} dx \\ &= O\left(\frac{1}{|y_n|^{1-4D}} \right), \end{aligned} \tag{12}$$

where $M = \sup_{z\in\mathcal{C}} |\,(z-\lambda_m)\,\varphi(z)\,|$. The quantity in (12) goes to zero as n goes to infinity. For the integral over the path γ_1, we only estimate the integral over the segment γ_1^+ in the first quadrant. The computation of the remaining segment γ_1^- in the fourth quadrant can be estimated in the same way using the symmetry relations (A.1) and (A.2). Also, the computation over the path γ_2 is similar to γ_1. For the canonical product φ we will use the estimate in [20, p. 57, eq. 16.08] that is

$$|\varphi\,(x+iy)| \ge \frac{(1+|\lambda_N-(x+iy)|)\,e^{\pi|y|}}{(1+|x+iy-N|)\,(1+|x+iy|)^{4D}},$$

where N is determined by

$$N - \frac{1}{2} \le |w| \sec\theta < N + \frac{1}{2},$$

$w = x+iy$ and $\theta = \mathrm{Arg}\,(w)$. Now,

$$\left| \frac{1}{2\pi i} \oint_{\gamma_1^+} \frac{[(z-\lambda_m)\,\varphi(z)]}{(w-z)} \frac{f(w)}{(w-\lambda_m)\,\varphi(w)} dw \right| \le \frac{M\,\|f\|_\infty}{2\pi} \int_0^{x_n} \frac{(1+|x_n+iy-N|)\,(|x_n+iy|+1)^{4D}}{|\, x_n+iy-\lambda_m \,||\, x_n+iy-\bar{x}-i\bar{y} \,|\,|\lambda_N-x_n-iy|} dy.$$

The quantity

$$\frac{1+|x_n+iy-N|}{|\lambda_N-x_n-iy|}$$

can be shown to be bounded for $y \in [0, x_n]$ and thus

$$\begin{aligned} &\left| \frac{1}{2\pi i} \oint_{\gamma_1} \frac{[(z-\lambda_m)\,\varphi\,(z)]}{(w-z)} \frac{f\,(w)}{(w-\lambda_m)\,\varphi\,(w)} dw \right| \\ &= O\left(\frac{1}{|x_n|^{1-4D}} \right). \end{aligned} \tag{13}$$

The right-hand side of (13) goes to zero as n goes to infinity. This completes the proof. ☐

The above result is obtained by adding one additional piece of information that can serve as oversampling. The required piece of information is the derivative value of the function f at λ_m.

We point out here that if $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is chosen to be the integers i.e. $\lambda_k = k \in \mathbb{Z}$ and $m = 0$, then we obtain the sampling series by Valiron-Tschakaloff [22, p. 91] and [23]. The Valiron-Tschakaloff sampling series reads as follows: For $f \in \mathcal{B}_\pi^\infty$, we have for all $z \in \mathbb{C}$

$$f\,(z) = f'\,(0)\,z \frac{\sin \pi z}{\pi z} + f\,(0) \frac{\sin \pi z}{\pi z} + \sum_{k\in\mathbb{Z}\setminus\{0\}} f\,(k) \frac{z}{k} \frac{\sin \pi\,(z-k)}{\pi\,(z-k)}.$$

5. Estimating the Infinity Norm

In this section, we will estimate the infinity norm of the class $\widetilde{\mathcal{B}}_{\beta\pi}^\infty, 0 < \beta < 1$, that is

$$\widetilde{\mathcal{B}}_{\beta\pi}^\infty = \left\{ f \in \mathcal{B}_{\beta\pi}^\infty \mid f\,(t) \in \mathbb{R} \text{ for } t \in \mathbb{R} \right\},$$

by knowing the information of the function over a set that requires a maximum gap condition from the integers. Accordingly, we state the following interesting question:

Question 2. *Let $0 < \beta < 1$ and $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ with $|\lambda_k - k| < \delta$ for all $k \in \mathbb{Z}$ for some $\delta < \frac{1}{4}$. Does there exist a constant $C = C(\Lambda, \beta)$ such that*

$$\|f\|_\infty \le C\,(\Lambda, \beta) \sup_{k\in\mathbb{Z}} |f\,(\lambda_k)| \tag{14}$$

for all $f \in \widetilde{\mathcal{B}}_{\beta\pi}^\infty$? If so, then can $C(\Lambda, \beta)$ be estimated?

For the first part of the question, we have D^- for $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ that satisfies $|\lambda_k - k| \le D < 1/4$ as

$$D^-\,(\Lambda) = 1 > \beta.$$

Thus, for all $f \in \widetilde{\mathcal{B}}_{\beta\pi}^\infty, 0 < \beta < 1$ with the condition, $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, $|\lambda_k - k| \le D < 1/4$ an estimate such as

(14) exists. It also means that the perturbed set from the integers above remains stable for the space $\widetilde{\mathcal{B}}_{\beta\pi}^{\infty}, 0<\beta<1$. In order to bound $C(\Lambda,\beta)$ in (14), we will use Theorem 8. It worths pointing out that the stability of the sampling set that follows from the Kadec 1/4-theorem is for the space $\mathcal{B}_{\pi}^{2}$.

Before stating the main result of this paper, we will need some definitions: We define $\|.\|_{\Lambda,\infty}$ as

$$\|f\|_{\Lambda,\infty}=\sup_{\lambda_k\in\Lambda}|f(\lambda_k)|,$$

and the quantity $C_\Lambda(\beta)$ as

$$C_\Lambda(\beta)=\sup_{f\in\widetilde{\mathcal{B}}_{\beta\pi}^{\infty},\|f\|_{\Lambda,\infty}\le 1}\|f\|_\infty .$$

Following the condition (9), we show that $C_\Lambda(\beta)>1$.

Example 1. *We consider the function $g(z)=c\,\mathrm{sinc}(\beta\pi z)$, $0<\beta<1$, where $c=\frac{1}{\mathrm{sinc}(\beta\pi/4)}>1$. Then, $|g(0)|>1$ and $\left|g\left(-\frac14\right)\right|=\left|g\left(\frac14\right)\right|=1$. Now, if $f(z)=g(z-t_{k_0})$, $t_{k_0}=(\lambda_{k_0}+\lambda_{k_0+1})/2$ for some fixed $k_0\in\mathbb{Z}$, then $\|f\|_{\Lambda,\infty}\le 1$ since $\lambda_{k+1}-\lambda_k>\frac12$ for all $k\in\mathbb{Z}$, while $\|f\|_\infty\ge|f(t_{k_0})|>1$. We conclude that $C_\Lambda(\beta)\ge\frac{1}{\mathrm{sinc}(\beta\pi/4)}>1$.*

For a more general set-up, we let $\Gamma\subset\mathbb{R}$, $\|f\|_\Gamma=\sup_{\gamma\in\Gamma}|f(\gamma)|$ and $d_\Gamma=\sup_{x\in\mathbb{R}}\mathrm{dist}(x,\Gamma)$, where $\mathrm{dist}(x,\Gamma)$ denotes the distance of the point x to the set Γ. This means the largest gap between two successive points in Γ has a width of $2d_\Gamma$. Then, we define $C_\Gamma(\beta)$ as

$$C_\Gamma(\beta)=\sup_{f\in\widetilde{\mathcal{B}}_{\beta\pi}^{\infty},\|f\|_{\Gamma}\le 1}\|f\|_\infty .$$

Now, we state the main result.

Theorem 9. *Let $f\in\widetilde{\mathcal{B}}_{\beta\pi}^{\infty}$, $0<\beta<1$, φ be as in (2), $\Lambda=\{\lambda_k\}_{k\in\mathbb{Z}}\subset\mathbb{R}$ be such that it satisfies $|\lambda_k-k|\le D<\frac14$, $\lambda_0=0$, $\varphi'(0)>0, |\varphi'(0)|=\inf_{\lambda_k\in\Lambda}|\varphi'(\lambda_k)|$. Let $E(t)=\varphi'(0)\varphi'(t)-\varphi''(0)\varphi(t)$ and $J=[\beta'a,\beta'b]$, $\lambda_{-1}\le a<0<b\le\lambda_1$, such that $d_0(\beta')=\min_{t\in J}E(t)>0$ for $0<\beta'<1$. If $\Gamma\subset\mathbb{R}$ with $d_\Gamma\le(b-a)/2$, then*

$$\|f\|_\infty\le\frac{(\varphi'(0))^2}{d_0(\beta)}\sup_{\gamma\in\Gamma}|f(\gamma)|. \tag{15}$$

Proof. Let $\epsilon>0$ be arbitrary. Then, there is $f_\epsilon\in\widetilde{\mathcal{B}}_{\beta\pi}^{\infty}$ with $\|f_\epsilon\|_\Gamma\le 1$ and $\|f_\epsilon\|_\infty>C_\Gamma(\beta)-\epsilon$. We now construct a related function $f_{\epsilon,\delta}$ such that $|f_{\epsilon,\delta}|$ assumes its maximum at a certain point. For this purpose let $h\in C_0^\infty(\mathbb{R})$ be an even non-negative function such that $h(t)=0$ for $|t|\ge1$ and $\int_{\mathbb{R}}h(t)\,dt=1$. For $\delta>0$ let $h_\delta(t)=h(t/\delta)/\delta$. Then, $\widehat{h_\delta}(t)\in\mathbb{R}$ for $t\in\mathbb{R}$, and $\left|\widehat{h_\delta}(t)\right|\le\int_{\mathbb{R}}|h_\delta(t)|\,dt=\int_{\mathbb{R}}|h(\tau)|\,d\tau=1$. Let $f_{\epsilon,\delta}(t)=f_\epsilon(t)\widehat{h_\delta}(t)$. Then, $|f_{\epsilon,\delta}(t)|\le|f_\epsilon(t)|$, $\lim_{|t|\to\infty}f_{\epsilon,\delta}(t)=0$, $f_{\epsilon,\delta}$ converges pointwise to f_ϵ as $\delta\to0$, $f_{\epsilon,\delta}\in\widetilde{\mathcal{B}}_{\beta'\pi}^{\infty}$ for $\beta'=\beta+\delta/\pi$, and $|f_{\epsilon,\delta}(t)|$ assumes its maximum at some point $t_{\epsilon,\delta}$. Now let δ be sufficiently small such that $\beta'=\beta+\delta/\pi<1$ as well as

$$|f_{\epsilon,\delta}(t_{\epsilon,\delta})|=\|f_{\epsilon,\delta}\|_\infty\ge\|f_\epsilon\|_\infty-\epsilon\ge C_\Gamma(\beta)-2\epsilon.$$

The inequality $\|f_{\epsilon,\delta}\|_\infty\ge\|f_\epsilon\|_\infty-\epsilon$ holds because there is some t_ϵ with $|f_\epsilon(t_\epsilon)|\ge\|f_\epsilon\|_\infty-\epsilon/2$ and since $f_{\epsilon,\delta}$ converges pointwise, one has for sufficiently small δ that $\|f_\epsilon\|_\infty\ge\|f_{\epsilon,\delta}\|_\infty\ge|f_{\epsilon,\delta}(t_\epsilon)|\ge|f_\epsilon(t_\epsilon)|-\epsilon/2\ge\|f_\epsilon\|_\infty-\epsilon$. Since $-f_{\epsilon,\delta}$ has the same desired properties as $f_{\epsilon,\delta}$, we may assume that $f_{\epsilon,\delta}(t_{\epsilon,\delta})>0$. Furthermore, since $f_{\epsilon,\delta}(t)$ is maximal at $t=t_{\epsilon,\delta}$, we obtain $f'_{\epsilon,\delta}(t_{\epsilon,\delta})=0$.

Let $\varphi(t)$ be the canonical product of the sampling set Λ and let $\varphi_{\beta'}(t)=\varphi(\beta't)$ and $\tilde g(\tau)=g(\tau/\beta')$ where

$$g(t)=f_{\epsilon,\delta}(t+t_{\epsilon,\delta})-\|f_{\epsilon,\delta}\|_\infty\frac{\varphi'_{\beta'}(t)}{\varphi'_{\beta'}(0)}.$$

Now we substitute the function $\tilde g$ in the sampling series (10) to obtain

$$\begin{aligned}\tilde g(\tau)&-\left\{\frac{\varphi(\tau)}{\varphi'(0)}\right\}\tilde g'(0)\\&=\tau\varphi(\tau)\left\{\sum_{k\in\mathbb{Z}\setminus\{0\}}\tilde g(\lambda_k)\frac{1}{\lambda_k(\tau-\lambda_k)\varphi'(\lambda_k)}\right\}.\end{aligned} \tag{16}$$

We have $\varphi'(\lambda_k)=(-1)^kc_k$, $c_k>0$ and $\lambda_k(\tau-\lambda_k)<0$ for $\tau\in(\lambda_{-1},\lambda_1)$. But, $\tilde g(\lambda_k)=d_k(-1)^{k+1}$, $d_k>0$ and if we substitute in (16) we obtain

$$\begin{aligned}\tilde g(\tau)&-\left\{\frac{\varphi(\tau)}{\varphi'(0)}\right\}\tilde g'(0)\\&=\tau\varphi(\tau)\left\{\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{(-1)^{k+1}d_k}{\lambda_k(\tau-\lambda_k)(-1)^kc_k}\right\}.\end{aligned}$$

It follows that

$$\tilde g(\tau)-\left\{\frac{\varphi(\tau)}{\varphi'(0)}\right\}\tilde g'(0)\ge0 \text{ for } \tau\in(\lambda_{-1},\lambda_1).$$

Now, let $\tau=\beta't$. Then, $\tilde g(\tau)=g(t)$ and thus

$$g(t)-\left\{\frac{\varphi(\beta't)}{\varphi'(0)}\right\}\frac{g'(0)}{\beta'}\ge0, \tag{17}$$

for $\beta't\in(\lambda_{-1},\lambda_1)$. We compute $g'(0)$ and it is

$$g'(0)=-\|f_{\epsilon,\delta}\|_\infty\frac{\beta'\varphi''(0)}{\varphi'(0)}.$$

By substituting the above in (17), we obtain

$$g(t)+\left\{\frac{\varphi(\beta't)}{\varphi'(0)}\right\}\|f_{\epsilon,\delta}\|_\infty\frac{\varphi''(0)}{\varphi'(0)}\ge0,$$

for $\beta't\in(\lambda_{-1},\lambda_1)$. Replacing g by its definition, we obtain

$$f_{\epsilon,\delta}(t+t_{\epsilon,\delta})-\|f_{\epsilon,\delta}\|_\infty\frac{\varphi'(\beta't)}{\varphi'(0)}+\|f_{\epsilon,\delta}\|_\infty\frac{\varphi''(0)\varphi(\beta't)}{(\varphi'(0))^2}\ge0,$$

for $\beta' t \in (\lambda_{-1}, \lambda_1)$. Then,

$$f_{\epsilon,\delta}(t + t_{\epsilon,\delta}) \geq \|f_{\epsilon,\delta}\|_\infty \frac{\varphi'(0)\varphi'(\beta' t) - \varphi''(0)\varphi(\beta' t)}{(\varphi'(0))^2},$$

for $\beta' t \in (\lambda_{-1}, \lambda_1)$. We have $E(\beta' t) = \varphi'(0)\varphi'(\beta' t) - \varphi''(0)\varphi(\beta' t) > d_0(\beta')$ for $t \in [a,b]$. Also, we have $2d_\Gamma \leq (b-a)$ and thus there is $t_\gamma \in [a,b]$ such that $t_\gamma + t_{\epsilon,\delta} = \gamma \in \Gamma$ and so $|f_{\epsilon,\delta}(t_\gamma + t_{\epsilon,\delta})| \leq 1$. By the intermediate value theorem (IVT), there exists t^* between 0 and t_γ such that $f_{\epsilon,\delta}(t^* + t_{\epsilon,\delta}) = 1$. Therefore,

$$\|f_{\epsilon,\delta}\|_\infty \leq \frac{f_{\epsilon,\delta}(t^* + t_{\epsilon,\delta})(\varphi'(0))^2}{E(\beta' t^*)} \leq \frac{(\varphi'(0))^2}{d_0(\beta')}.$$

This means that

$$C_\Gamma(\beta) \leq \|f_{\epsilon,\delta}\|_\infty + 2\epsilon \leq \frac{(\varphi'(0))^2}{d_0(\beta')} + 2\epsilon.$$

Letting both ϵ and δ go to zero completes the proof. □

A simple practically relevant subclass of sampling sequences for which the assumptions are fulfilled is the following:

Corollary 1. *Let $f \in \widetilde{\mathcal{B}}^\infty_{\beta\pi}$, $0 < \beta < 1$, and let $\Gamma \subset \mathbb{R}$ such that $2d_\Gamma \leq 1$, where $d_\Gamma = \sup_{x\in\mathbb{R}} \mathrm{dist}(x,\Gamma) = \sup_{x\in\mathbb{R}} \inf_{\gamma\in\Gamma} |x-\gamma|$. Then*

$$\| f \|_\infty \leq \frac{1}{\cos(\beta\pi d_\Gamma)} \sup_{\gamma\in\Gamma} |f(\gamma)|.$$

This result generalizes the estimate by Wunder and Boche (Theorem 2) to non-equidistant sampling sets Γ. Theorem 2 is recovered for $\Gamma = \mathbb{Z}$, in which case $d_\Gamma = 1/2$.

Proof. One applies Theorem 9 with $\Lambda = \mathbb{Z}$, $\lambda_k = k, k \in \mathbb{Z}$, $a = -d_\Gamma > \lambda_{-1}$, $b = d_\Gamma \leq \lambda_1$. The conditions $\lambda_0 = 0$, $|\lambda_k - k| \leq D < 1/4$, and $d_\Gamma \leq (b-a)/2$ are obviously satisfied. Using the representation

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right),$$

one has

$$\begin{aligned} \varphi(z) &= \frac{1}{\pi}\sin(\pi z) \\ \varphi'(z) &= \cos(\pi z), \quad \varphi'(0) = 1 > 0 \\ \varphi''(z) &= -\pi\sin(\pi z), \quad \varphi''(0) = 0. \end{aligned}$$

Since $\varphi'(\lambda_k) = \cos(\pi k) = (-1)^k$, one has $1 = |\varphi'(0)| = \inf_{k\in\mathbb{Z}} |\varphi'(k)| = \inf_{\lambda_k\in\Lambda} |\varphi'(\lambda_k)|$. Furthermore,

$$E(t) = \varphi'(0)\varphi'(t) - \varphi''(0)\varphi(t) = \cos(\pi t).$$

This gives

$$\begin{aligned} d_0(\beta') &= \min_{\beta' a \leq t \leq \beta' b} \cos(\pi t) \\ &= \min_{|t|\leq \beta' d_\Gamma} \cos(\pi t) \\ &= \cos(\pi\beta' d_\Gamma). \end{aligned}$$

Since $d_\Gamma \leq 1/2$, one has $d_0(\beta') > 0$ for $0 < \beta' < 1$. It now follows from Theorem 9 that

$$\begin{aligned} \| f \|_\infty &\leq \frac{(\varphi'(0))^2}{d_0(\beta)} \sup_{\gamma\in\Gamma} |f(\gamma)| \\ &= \frac{1}{\cos(\beta\pi d_\Gamma)} \sup_{\gamma\in\Gamma} |f(\gamma)| \end{aligned}$$

as claimed. □

In the next result, we will treat the critical case where $2d_\Gamma = (b-a)$. For that reason, we will restrict the bandwidth of the function f in $\widetilde{\mathcal{B}}^\infty_{\beta\pi}$.

Corollary 2. *Let φ be as in (2) and $\Lambda \subset \mathbb{R}$ be a perturbed set that satisfies the quarter condition such that $|\varphi'(0)| = \inf_{\lambda_k\in\Lambda} |\varphi'(\lambda_k)|$, $E(t) = \varphi'(0)\varphi'(t) - \varphi''(0)\varphi(t)$, $\beta < \min\{|t| \mid t \neq 0, E(t) = 0\}/d_\Lambda$ and $d_0 = E(\beta d_\Lambda)$. Then, for all $\widetilde{\mathcal{B}}^\infty_{\beta\pi}$ we have*

$$\|f\|_\infty \leq \frac{(\varphi'(0))^2}{d_0} \sup_{\gamma\in\Lambda} |f(\gamma)|.$$

Proof. Let $\beta < \min\{|t| \mid t \neq 0, E(t) = 0\}/d_\Lambda$ and $J = [\beta a, \beta b]$ with $a = -b = -d_\Lambda$. Then, for $t \in J$ we have

$$-|t_0| < -\beta d_\Lambda = \beta a \leq t \leq \beta b = \beta d_\Lambda < |t_0|,$$

where $|t_0| = \min\{|t| \mid t \neq 0, E(t) = 0\}$ and thus $\min_{t\in J} E(t) > 0$. Also, $b - a = 2d_\Lambda$. The hypothesis of Theorem 9 is satisfied and thus the conclusion follows. □

6. Numerical Example

In this section, we will give an example using a nonuniform sampling set. The example will use these sampling sets (i) $\Lambda_1 = \mathbb{Z} \setminus \{1\} \cup \{1-\delta\}$ and (ii) $\Lambda_2 = 2\mathbb{Z} \cup (2\mathbb{Z} + (1+\delta))$. In both cases, $2d_\Lambda = (1+\delta)$ is the maximum gap. It can be shown that for the sampling set (ii) we obtain

$$\varphi(t) = -\sin\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}(t-(1+\delta))\right),$$

$$\varphi'(0) = \frac{\pi}{2}\sin\left(\frac{\pi}{2}(1+\delta)\right),$$

and

$$\begin{aligned} E(t) = -\left(\frac{\pi}{2}\right)^2 \Big[&\sin\left(\frac{\pi}{2}(1+\delta)\right)\sin\left(\frac{\pi}{2}(2t-(1+\delta))\right) \\ &+ 2\cos\left(\frac{\pi}{2}(1+\delta)\right)\sin\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}(t-(1+\delta))\right)\Big]. \end{aligned}$$

In fact, one can show that when $\delta = 0$ (d_Λ in this case is 1/2), the factor $(\varphi'(0))^2/d_0$ simplifies to $1/\cos(\beta\pi/2)$ which is the factor for the bound estimate in Theorem 2.

Using these sampling sets, we will find an estimate of the infinity norm and demonstrate how the bound becomes sharper by considering denser sets. We need to point out that to use Wunder and Boche (W-B) estimate (4), we need a uniform sampling set. Hence, if we perturb the set Λ, then we need a subset $\Lambda' \subset \Lambda$ that is uniform and contains the origin. Obviously, the new sampling set Λ' has lower density than Λ. Consequently, we need to have a smaller bandwidth to be able to use the estimate (4). Specifically, if $\Lambda' = a\mathbb{Z}$, then the function has to have a bandwidth $\beta\pi$, $0 < \beta < 1/a$ which means the estimate (4) is not applicable otherwise. The following example demonstrates this delicate situation.

Example 2. *We consider Λ_1 and Λ_2 mentioned above and functions in the space $\widetilde{\mathcal{B}}^{\infty}_{\beta\pi}$, $0 < \beta < 1$. The functions are*

$$f(t) = \frac{3\pi}{5}\operatorname{sinc}\frac{6\pi}{10}(t-1) - \frac{\sqrt{5}\pi}{10}\operatorname{sinc}\frac{\sqrt{5}\pi}{4}(t-6), \quad (18)$$

$$g(t) = \frac{2\pi}{5}\operatorname{sinc}\frac{4\pi}{10}t - \frac{2\sqrt{3}\pi}{25}\operatorname{sinc}\frac{\sqrt{3}\pi}{5}(t-6). \quad (19)$$

We choose $a = -b = -d_\Lambda$ and choose $\delta = 0.23$ for Λ_1 and $\delta = 0.15$ for Λ_2. In the function f, we have $\beta = 0.6$ and it can be shown that $\beta d_\Lambda < |t_0|$ for both Λ_1 and Λ_2. It follows that $d_0 = E(\beta d_\Lambda) = E(\beta(1+\delta)/2)$ as suggested by Corollary 2. For the given sampling sets Λ_1 and Λ_2, it is clear that $\Lambda' = 2\mathbb{Z}$ and so the required bandwidth to apply (4) is $\beta\pi$, $0 < \beta < 1/2$. The bandwidth of function (18) is 0.6π and so the bound estimate (4) is not applicable. However, this will not be the case by using a denser set Λ such as ($\frac{1}{2}\Lambda$, and $\frac{1}{3}\Lambda$) or by considering a function with a smaller bandwidth such as (19). In the following table, we consider couple cases where Γ is equal to Λ, $\frac{1}{2}\Lambda$, and $\frac{1}{3}\Lambda$.

Γ	$\|f\|_\infty$	*W-B bound*	*Bound (15)*
$\Gamma = \Lambda_1$	1.8404	*N/A*	7.0876
$\Gamma = \frac{1}{2}\Lambda_1$	1.8404	3.1252	2.2992
$\Gamma = \frac{1}{3}\Lambda_1$	1.8404	2.1742	2.0196

Γ	$\|f\|_\infty$	*W-B bound*	*Bound (15)*
$\Gamma = \Lambda_2$	1.8404	*N/A*	4.1050
$\Gamma = \frac{1}{2}\Lambda_2$	1.8404	3.1252	2.1650
$\Gamma = \frac{1}{3}\Lambda_2$	1.8404	2.1742	1.9622

Γ	$\|g\|_\infty$	*W-B bound*	*Bound (15)*
$\Gamma = \Lambda_1$	1.2441	4.0139	1.9105
$\Gamma = \frac{1}{2}\Lambda_1$	1.2441	1.5332	1.3637
$\Gamma = \frac{1}{3}\Lambda_1$	1.2441	1.3578	1.2926

Γ	$\|g\|_\infty$	*W-B bound*	*Bound (15)*
$\Gamma = \Lambda_2$	1.2441	4.0139	1.6860
$\Gamma = \frac{1}{2}\Lambda_2$	1.2441	1.5332	1.3313
$\Gamma = \frac{1}{3}\Lambda_2$	1.2441	1.3578	1.2794

The tables show that the denser Γ is, the sharper bound we obtain. It is important to point out again that when we compute the W-B bound we consider Λ', $\frac{1}{2}\Lambda'$ and $\frac{1}{3}\Lambda'$ instead of Λ, $\frac{1}{2}\Lambda$ and $\frac{1}{3}\Lambda$. The function f in (18) and the suggested bounds by bound (15) are depicted in Figure 2.

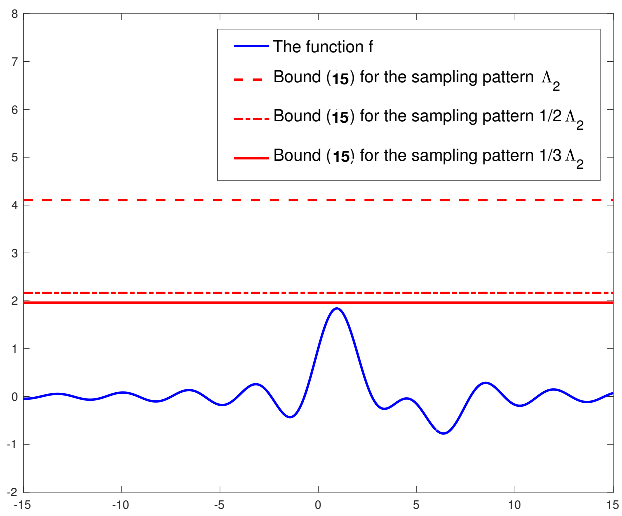

Figure 2: Bound (15) for the sampling patterns $\Gamma = \Lambda_2, \frac{1}{2}\Lambda_2$, and $\frac{1}{3}\Lambda_2$.

7. Conclusion

The problem of estimating the infinity norm of a function from its supremum sample values is important for the stable sampling of the space $\mathcal{B}^{\infty}_{\beta\pi}, 0 < \beta < 1$. Boche and Mönich asked the question of whether an estimate of the form

$$\|f\|_\infty \leq C(\beta) \sup_{\gamma\in\Gamma} |f(\gamma)|, \quad f \in \widetilde{\mathcal{B}}^{\infty}_{\beta\pi}, \quad 0 < \beta < 1, \quad (20)$$

exists if Γ is the set of zeros of a π-sine-type function. We provided an affirmative answer using the result of Beurling as well as some properties of the sine-type functions. Furthermore, we proved a generalization of Valiron-Tschakaloff sampling theorem for sampling sets that are perturbations of the integers by up to a quarter. This sampling theorem is subsequently used to derive an upper bound for $C(\beta)$ in (20) if Γ is a subset of $\mathbb{R}$ with a maximum gap of $2d_\Gamma$.

8. Appendix

Let $G(\{\lambda_k\}; z)$ denote the canonical product $\varphi(z)$ for the set of zeros $\{\lambda_k\}_{k\in\mathbb{Z}}$. If $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, one has the relations.

$$G(\{\lambda_k\}; \bar{z}) = \overline{G(\{\lambda_k\}; z)} \quad (A.1)$$

and

$$G(\{\lambda_k\}; -z) = s(\lambda_0) G(\{-\lambda_{-k}\}; z), \quad (A.2)$$

with

$$s(\lambda_0) = \begin{cases} 1 & \text{if} \quad \lambda_0 \neq 0 \\ -1 & \text{if} \quad \lambda_0 = 0 \end{cases}.$$

[1] H. Behmard, A. Faridani, Sampling of bandlimited functions on unions of shifted lattices, Journal of Fourier Analysis and Applications 8 (1) (2002) 43–58.

[2] G. Wunder, H. Boche, Peak value estimation of bandlimited signals from their samples, noise enhancement, and a local characterization in the neighborhood of an extremum, IEEE Transactions on Signal Processing 51 (3) (2003) 771–780.

[3] D. Wulich, Reduction of peak to mean ratio of multicarrier modulation using cyclic coding, Electronics Letters 32 (5) (1996) 432.

[4] Y. Wu, W. Y. Zou, Orthogonal frequency division multiplexing: A multi-carrier modulation scheme, IEEE Transactions on Consumer Electronics 41 (3) (1995) 392–399.

[5] K. G. Paterson, V. Tarokh, On the existence and construction of good codes with low peak-to-average power ratios, Information Theory, IEEE Transactions on 46 (6) (2000) 1974–1987.

[6] J. Tellado, L. Hoo, J. M. Cioffi, Maximum likelihood detection of nonlinearly distorted multicarrier symbols by iterative decoding, in: Global Telecommunications Conference, 1999. GLOBECOM'99, Vol. 5, IEEE, 1999, pp. 2493–2498.

[7] A. E. Jones, T. A. Wilkinson, S. Barton, Block coding scheme for reduction of peak to mean envelope power ratio of multicarrier transmission schemes, Electronics letters 30 (25) (1994) 2098–2099.

[8] Al-Hammali, Hussain, Nonuniform Sampling of Band-limited Functions, PhD dissertation, https://ir.library.oregonstate.edu /concern/graduate_thesis_or_dissertations/gb19f863p.

[9] H. Landau, Sampling, data transmission, and the nyquist rate, Proceedings of the IEEE 55 (10) (1967) 1701–1706.

[10] H. Ehlich, K. Zeller, Schwankung von polynomen zwischen gitterpunkten, Mathematische Zeitschrift 86 (1) (1964) 41–44.

[11] G. Wunder, H. Boche, Peak magnitude of oversampled trigonometrie polynomials, Frequenz 56 (5-6) (2002) 102–109.

[12] K. Jetter, G. Pfander, G. Zimmermann, The crest factor for trigonometric polynomials. part i: Approximation theoretical estimates, Revue d'Analyse Numerique et de Theorie de l'Approximation 30 (2) (2001) 179–196.

[13] V. Tarokh, H. Jafarkhani, On the computation and reduction of the peak-to-average power ratio in multicarrier communications, Communications, IEEE Transactions on 48 (1) (2000) 37–44.

[14] H. Minn, C. Tellambura, V. K. Bhargava, On the peak factors of sampled and continuous signals, IEEE Communications Letters 5 (4) (2001) 129–131.

[15] H. Boche, G. Wunder, Über eine verallgemeinerung eines resultats von riesz über trigonometrische polynome auf allgemeine bandbegrenzte funktionen, ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik 82 (5) (2002) 347–351.

[16] U. J. Mönich, H. Boche, Non-equidistant sampling for bounded bandlimited signals, Signal processing 90 (7) (2010) 2212–2218.

[17] A. Beurling, Balayage of fourier-stieltjes transforms, The Collected Works of Arne Beurling 2 (1989) 341–350.

[18] M. Horváth, I. Joó, On riesz bases ii, in: Annales Univ. Sci. Budapest, Vol. 33, 1990, pp. 267–271.

[19] J. Ortega-Cerdà, K. Seip, Multipliers for entire functions and an interpolation problem of beurling, Journal of functional analysis 162 (2) (1999) 400–415.

[20] N. Levinson, Gap and density theorems, Vol. 26, American Mathematical Soc., 1963.

[21] B. Y. Levin, Lectures on entire functions, translations of mathematical monographs, vol. 150, American Mathematical Society, Providence, RI 1 (1996) 996.

[22] F. Marvasti, Nonuniform Sampling: Theory and Practice, New York: Kluwer Academic/Plenum., 2001.

[23] P. Butzer, P. Ferreira, J. Higgins, G. Schmeisser, R. Stens, The sampling theorem, poisson's summation formula, general parseval formula, reproducing kernel formula and the paley–wiener theorem for bandlimited signals–their interconnections, Applicable Analysis 90 (3-4) (2011) 431–461.