SAMPLING IN PARALLEL-BEAM TOMOGRAPHY

Adel Faridani

Dept. of Mathematics
Oregon State University
Corvallis, OR 97331
faridani@math.orst.edu

September 30, 1997

Abstract

We pesent Shannon sampling theory for functions defined on $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ denotes the circle group, prove a new estimate for the aliasing error, and apply the result to parallel-beam diffraction tomography. The class of admissible sampling lattices is characterized and general sampling conditions are derived which lead to the identification of new efficient sampling schemes. Corresponding results for x-ray tomography are obtained in the high-frequency limit.

1. INTRODUCTION

Sampling theorems provide interpolation formulas for functions whose Fourier transform is compactly supported. If the Fourier transform does not have compact support, a so-called aliasing error occurs. In this paper we pesent a new estimate for the aliasing error for functions defined on $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ denotes the circle group, and work out its application to computed tomography.

In computed tomography (CT) an object is exposed to radiation which is measured after passing through the object. From these measurements a function f characterizing the interaction of the object with the radiation is reconstructed, thus providing an image of the interior of the object. In practice only finitely many measurements can be taken, and the question arises of how to sample the data in order to obtain a desired resolution in the reconstructed image. This question can be addressed using Shannon sampling theory [2, 29]. We will characterize the class of practically relevant sampling lattices, derive sampling conditions which ensure that the data function Pf is essentially determined by the sampled values, and identify new efficient sampling schemes.

In diagnostic radiology, the classical application of CT, x-rays are used as radiation. The measurements are then line integrals of the x-ray absorption coefficient f. This leads to the mathematical problem of reconstructing a function from its line integrals, i.e., the inversion of the x-ray transform. When radiation of lower frequencies such as ultrasound or microwaves is used the mathematical model has to be generalized in order to include diffraction effects. A suitable generalization is the inverse scattering problem for the Helmholtz equation. The model we will investigate here results from linearizing this problem via the Born- or Rytov-approximations. This is called diffraction tomography [5, 12, 13, 14, 16, 19]. This model encompasses the x-ray transform which is obtained in the high-frequency limit.

The paper is organized as follows: In the next section we develop the Shannon sampling theory suitable for tomography. We characterize a class of admissible sampling lattices and obtain a new estimate for the aliasing error. The mathematical model underlying diffraction tomography is described briefly in the following section and the data function Pf is introduced. In §4 the essential support of the Fourier transform of Pf is determined. This allows to derive the sampling conditions in §5. It turns out that the required amount of data varies considerably for different sampling lattices. We identify a number of lattices which are more efficient than the usually employed standard scheme. In §6 the corresponding results for the x-ray transform are derived by taking a high-frequency limit.

2. SAMPLING ON $\mathbb{T} \times \mathbb{R}$

In this section we present Shannon sampling theory for functions of two variables which are 2π periodic in the first argument. Such functions can be viewed as being defined on the group $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ denotes the circle group. We will use the interval $[0, 2\pi)$ with addition modulo 2π as a model for $\mathbb{T}$. Sampling theorems provide exact reconstruction formulas for bandlimited functions from function values on a discrete set. In the cases considered here the sampling sets are certain subgroups of $\mathbb{T} \times \mathbb{R}$. If the function to be reconstructed is not bandlimited, an aliasing error occurs. The classical estimate for this error is given in Theorem 2. However, this estimate is not always convenient to evaluate. It turns out that a slight restriction on the admissible sampling sets allows for a sharper and more convenient estimate; see Proposition 5 and Theorem 6.

We begin with standard definitions and notation: Let $\mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{C}$ denote the integers, and the natural, real and complex numbers, respectively. $\mathbb{R}^n$ consists of n-tuples of real numbers, usually designated by single letters, $x = (x_1, \ldots, x_n)^T$,$y = (y_1, \ldots, y_n)^T$, etc. The inner product and absolute value are defined by $x \cdot y = \sum_{i=1}^n x_i y_i$ and $|x| = \sqrt{x \cdot x}$. The unit sphere S^{n-1} consists of the points of absolute value 1. For $a \in \mathbb{R}$ let $\lfloor a \rfloor$ and $\lceil a \rceil$ denote the largest integer $\leq a$ and the smallest integer $\geq a$, respectively.

The Fourier transform on $\mathbb{R}^n$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx$$

for integrable functions f, and is extended to larger classes of functions or distributions by continuity or duality. The Fourier transform of a function $g \in L_1(\mathbb{T} \times \mathbb{R})$ is given by

$$\begin{aligned} \hat{g}(\zeta) &= (2\pi)^{-1} \int_{\mathbb{T}\times\mathbb{R}} g(z) e^{-i<z,\zeta>}\, dz \\ &= (2\pi)^{-1} \int_0^{2\pi} \int_{\mathbb{R}} g(\varphi, s) e^{-i(k\varphi+\sigma s)}\, ds\, d\varphi \end{aligned}$$

with $z = (\varphi, s) \in \mathbb{T} \times \mathbb{R}$, $\zeta = (k, \sigma) \in \mathbb{Z} \times \mathbb{R}$, and $< z, \zeta >= k\varphi + \sigma s$. The corresponding inverse Fourier transform of a function G defined on $\mathbb{Z} \times \mathbb{R}$ is given by

$$\tilde{G}(z) = (2\pi)^{-1} \int_{\mathbb{Z}\times\mathbb{R}} G(\zeta) e^{i<z,\zeta>}\, d\zeta$$

$$= (2\pi)^{-1} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} G(k, \sigma) e^{i(k\varphi + \sigma s)} d\sigma. \tag{1}$$

Definition 1 *Let W be a non-singular (2,2)-matrix such that $(2\pi, 0)^T \in W\mathbb{Z}^2$, where $W\mathbb{Z}^2 = \left\{ W \begin{pmatrix} k \\ l \end{pmatrix},\ k, l \in \mathbb{Z} \right\}$. The subgroup L of $T \times \mathbb{R}$ given by*

$$L := (W\mathbb{Z}^2) \cap ([0, 2\pi) \times \mathbb{R})$$

is called the sampling lattice generated by W. The lattice $L^\perp = 2\pi W^{-T} \mathbb{Z}^2$ is called the dual lattice with respect to L. The lattice L does not uniquely determine the generator matrix W, but it does determine the determinant of W as well as the dual lattice. The lattice constant C_L is defined by

$$C_L = (2\pi)^{-1} |\det W|.$$

A sampling lattice L containing an element $(0, s)^T \in \mathbb{T} \times \mathbb{R}$ for some $s \neq 0$ is called an admissible sampling lattice.

The following theorem is the analogue of the well-known Petersen-Middleton theorem [23] for sampling on $\mathbb{T} \times \mathbb{R}$.

Theorem 2 *Let $g \in C_0^\infty(\mathbb{T} \times \mathbb{R})$, L a sampling lattice and K be a compact subset of $\mathbb{Z} \times \mathbb{R}$ such that its translates $K + \eta, \eta \in L^\perp$ are disjoint. Let χ_K denote the characteristic function of K. For $z \in \mathbb{T} \times \mathbb{R}$ define*

$$Sg(z) = C_L \sum_{y \in L} \tilde{\chi}_K(z - y) g(y). \tag{2}$$

Then

$$|g(z) - Sg(z)| \leq \pi^{-1} \int_{(\mathbb{Z} \times \mathbb{R}) \setminus K} |\hat{g}(\zeta)| d\zeta. \tag{3}$$

Proof: See, e.g., [9] or [20,pp. 62-64].

The lattice $L^\perp$ has to be sufficiently sparse for the sets $K + \eta$, $\eta \in L^\perp$ to be disjoint. This translates into a condition for the lattice L to be sufficiently dense, since the densities of L and $L^\perp$ are inversely proportional to each other. The larger the set K, the more restrictive the requirement becomes. To allow for efficient sampling, K should be as small as possible, but on the other hand K must be sufficiently large for the right-hand side of the estimate (3) to be small. In this sense K can be viewed as essential support of $\hat{g}$.

Extensions of the theorem to sampling sets which are unions of cosets of a lattice have been proven in [8, 10]. In the remainder of this section we show that a sharper error estimate may be obtained for admissible sampling lattices. The next lemma gives a parameterization of the admissible lattices and provides two alternative normal forms for their generator matrices.

Lemma 3 *Let L be an admissible sampling lattice for $\mathbb{T} \times \mathbb{R}$. Then there exist $d > 0$, $P, M, N \in \mathbb{N} \cup \{0\}$ with M a divisor of P, $0 \leq N < M$, $gcd(M, N) = 1$, such that $L = L(d, P, M, N)$ is generated by the matrices*

$$W_1 = \begin{pmatrix} 2\pi/P & 0 \\ dN/M & d \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} 2\pi M/P & 2\pi N'/P \\ 0 & d/M \end{pmatrix} \tag{4}$$

where $0 \leq N' < M$, $NN' \equiv 1 mod\, M$. Hence the lattice constant equals $C_L = d/P$, and the dual lattice is generated by

$$2\pi W_1^{-T} = \begin{pmatrix} P & -PN/M \\ 0 & 2\pi/d \end{pmatrix} \text{ and by } 2\pi W_2^{-T} = \begin{pmatrix} P/M & 0 \\ -2\pi N'/d & 2\pi M/d \end{pmatrix} \tag{5}$$

Proof: It was shown in [9, Lemma 2.4] that there exist $d > 0$, $M, N', L' \in \mathbb{N}$, $0 \leq N' < M$, $\gcd(M, N') = 1$, such that L is generated by the matrix

$$W = \begin{pmatrix} 2\pi/L' & 2\pi N'/(ML') \\ 0 & d/M \end{pmatrix}.$$

With $P = L'M$ this matrix equals W_2. Now let N be given by $0 \leq N < M$, $NN' \equiv 1 \bmod M$. It remains to show that $W_1\mathbb{Z}^2 = W_2\mathbb{Z}^2$, i.e., that $W_1^{-1}W_2\mathbb{Z}^2 = \mathbb{Z}^2$. This is the case if and only if $W_1^{-1}W_2$ is an integer matrix and $|\det(W_1^{-1}W_2)| = 1$. These conditions are readily verified. □

It follows from Lemma 3 that the admissible lattices have the general form

$$\begin{aligned} L &= L(d, P, M, N) \\ &= \{(\varphi_j, s_{jl})^T \in T \times \mathbb{R} : \varphi_j = 2\pi j/P,\ j = 0, \dots P-1, \\ &\qquad s_{jl} = d(l + jN/M),\ l \in \mathbb{Z}\}, \end{aligned} \tag{6}$$

i.e., for each of the P evenly distributed angles φ_j there are equidistant values s_{jl} with spacing d, which are shifted by the amount jdN/M. If $N \neq 0$ this shift varies with j and cycles effectively through M different values, since shifts by integer multiples of d have no effect.

Definition 4 *For $g : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$ and $\varphi \in \mathbb{T}$ let $g_\varphi : \mathbb{R} \to \mathbb{C}$ be given by $g_\varphi(s) = g(\varphi, s)$. Define*

$$\epsilon^*(g,\tau) = (2\pi)^{-1/2} \sup_{\varphi\in\mathbb{T}} \int_{|\sigma|>\tau} |\widehat{g_\varphi}(\sigma)| d\sigma. \tag{7}$$

For $K \in \mathbb{Z} \times \mathbb{R}$ compact let $k_1 = \max\{|k| : (k,\sigma) \in K\}$, $\sigma_1 = \max\{|\sigma| : (k,\sigma) \in K\}$, and $M_1 = \{(k,\sigma) \in \mathbb{Z} \times \mathbb{R} : |\sigma| \leq \sigma_1\}\backslash K$; see Fig. 1.

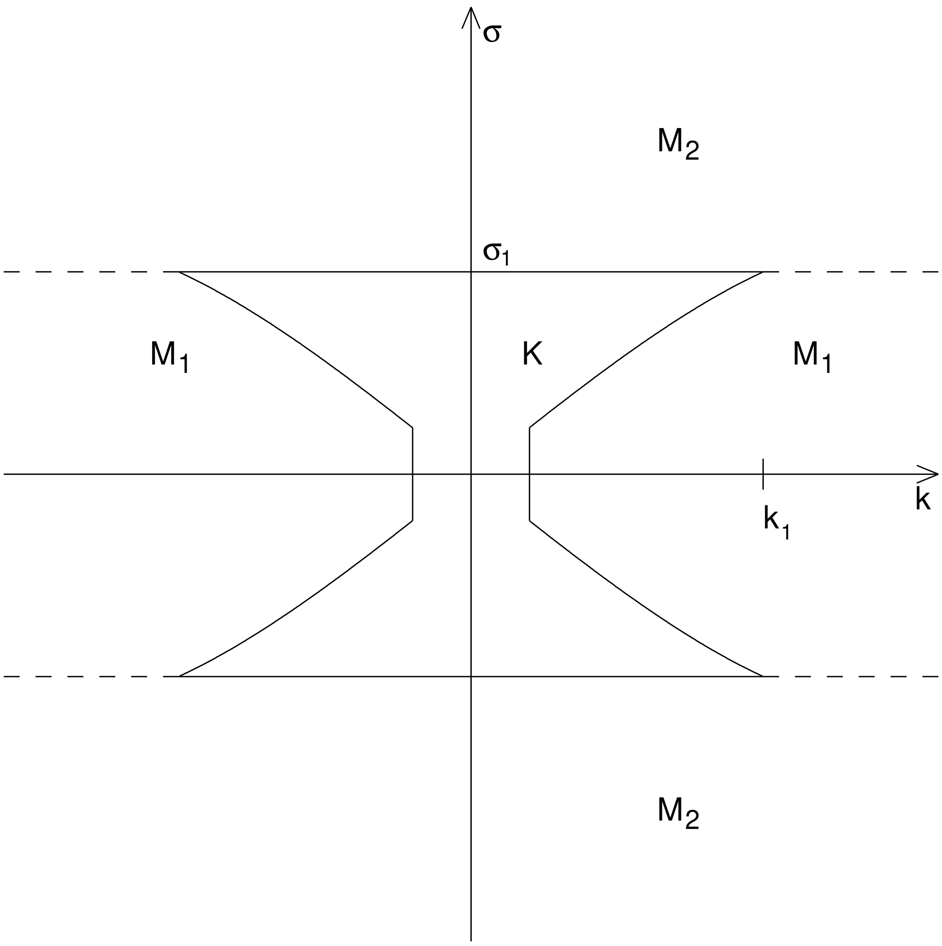

Figure 1. Example of a set K with the related quantities k_1, σ_1, M_1 as in Definition 4, and the set M_2 as used in the proof of Theorem 6.

We are now able to state the sampling theorem which will be used in later sections.

Proposition 5 *Let K be a compact subset of $\mathbb{Z} \times \mathbb{R}$ satisfying the condition that if $(k, \sigma) \in K$, then $(k', \sigma) \in K$ for all k' with $|k'| \leq |k|$. Let $L = L(d, P, M, N)$ be an admissible sampling lattice such that the translated sets $K + \eta$, $\eta \in L^{\perp}$ are disjoint, $g \in C_0^{\infty}(\mathbb{T} \times \mathbb{R})$, Sg as in (2), $z \in \mathbb{T} \times \mathbb{R}$, $k_1, \sigma_1, M_1, \epsilon^*$ as in Definition 4, and $\sigma^* = \max\{\sigma_1, \frac{2\pi}{d} - \sigma_1\}$. Then*

$$\begin{aligned} |g(z) - Sg(z)| \leq \quad & \pi^{-1} \int_{M_1} |\hat{g}(\zeta)| d\zeta \ + \ \epsilon^*(g, \sigma_1) \\ & + \ C(k_1, P) \left\lceil \frac{\sigma_1 d}{\pi} \right\rceil \epsilon^*(g, \sigma^*) \end{aligned}$$

with $C(k_1, P) = 1 + \frac{6k_1+3}{P} + \frac{2}{\pi} \ln(4k_1 + 2)$.

Proposition 5 is a simplified version of Theorem 6 below, but is all that will be used later on. Readers not interested in the more general result or the proofs may safely move on to the next section.

Theorem 6 *Let $g \in C_0^{\infty}(\mathbb{T} \times \mathbb{R})$, $K \subset \mathbb{Z} \times \mathbb{R}$ compact and $L = L(d, P, M, N)$ an admissible sampling lattice such that the translated sets $K + \eta$, $\eta \in L^{\perp}$ are disjoint. Let k_1, σ_1, M_1 as in Definition 4, $\sigma^* = \max\{\sigma_1, \frac{2\pi}{d} - \sigma_1\}$, and $\varphi_j = 2\pi j/P$, $j = 0, \ldots P-1$. Then*

$$\begin{aligned} & |g(\varphi, s) - Sg(\varphi, s)| \\ & \leq \pi^{-1} \int_{M_1} |\hat{g}(\zeta)| d\zeta \ + \ (2\pi)^{-1/2} \int_{\sigma > \sigma_1} |\widehat{g_\varphi}(\sigma)| \, d\sigma \\ & \quad + \ \frac{C(K, P, \varphi)}{\sqrt{2\pi}} \left\lceil \frac{\sigma_1 d}{\pi} \right\rceil \max_{j=0,\ldots,P-1} \int_{|\sigma| > \sigma^*} |\widehat{g_{\varphi_j}}(\sigma)| d\sigma \end{aligned} \tag{8}$$

with

$$C(K, P, \varphi) \ = \ \frac{1}{P} \sum_{j=0}^{P-1} \max_{|\sigma| \leq \sigma_1} \left| \sum_{k \in N(\sigma)} e^{ik(\varphi - \varphi_j)} \right|.$$

where $N(\sigma) = \{k : \ (k, \sigma) \in K\}$.

Proof: The Poisson summation formula for L reads

$$C_L \sum_{y \in L} g(y) e^{-i<y,\zeta>} \ = \ \sum_{\eta \in L^{\perp}} \hat{g}(\zeta - \eta).$$

It implies that

$$\widehat{Sg}(\zeta) \;=\; \chi_K(\zeta) \sum_{\eta\in L^\perp} \hat{g}(\zeta-\eta). \tag{9}$$

Note that $\widehat{Sg}$ vanishes outside K. The Fourier inversion formula now gives

$$\begin{aligned}
&|g(z) - Sg(z)| \\
&= \frac{1}{2\pi}\left|\int_{\mathbb{Z}\times\mathbb{R}} (\hat{g}(\zeta) - \widehat{Sg}(\zeta))\; e^{i<z,\zeta>}\; d\zeta\right| \\
&\le \frac{1}{2\pi}\left|\int_K (\hat{g}(\zeta) - \widehat{Sg}(\zeta))\; e^{i<z,\zeta>}\; d\zeta\right| \;+\; \frac{1}{2\pi}\left|\int_{(\mathbb{Z}\times\mathbb{R})\backslash K} \hat{g}(\zeta)e^{i<z,\zeta>}\; d\zeta\right|
\end{aligned} \tag{10}$$

Let $M_2 = \{(k,\sigma) \in \mathbb{Z}\times\mathbb{R} : |\sigma| > \sigma_1\}$. Then $M_1 \cap M_2 = \emptyset$, and $(\mathbb{Z}\times\mathbb{R})\backslash K = M_1 \cup M_2$; see Fig. 1. Splitting the integral over $(\mathbb{Z}\times\mathbb{R})\backslash K$ into contributions coming from M_1 and M_2 and using the relation

$$(2\pi)^{-1/2}\sum_{k\in\mathbb{Z}} \hat{g}(k,\sigma)e^{ik\varphi} \;=\; \widehat{g_\varphi}(\sigma)$$

yields

$$\begin{aligned}
&\frac{1}{2\pi}\left|\int_{(\mathbb{Z}\times\mathbb{R})\backslash K} \hat{g}(\zeta)e^{i<z,\zeta>}\; d\zeta\right| \\
&\le \frac{1}{2\pi}\int_{M_1} |\hat{g}(\zeta)|\; d\zeta \;+\; \frac{1}{2\pi}\left|\int_{|\sigma|>\sigma_1} \sum_{k\in\mathbb{Z}} \hat{g}(k,\sigma)e^{ik\varphi}e^{i\sigma s}\; d\sigma\right| \\
&\le \frac{1}{2\pi}\int_{M_1} |\hat{g}(\zeta)|\; d\zeta + \;(2\pi)^{-1/2}\int_{\sigma>\sigma_1} |\widehat{g_\varphi}(\sigma)|\; d\sigma
\end{aligned} \tag{11}$$

The theorem now follows from combining (10), (11), and the estimate

$$\begin{aligned}
&\frac{1}{2\pi}\left|\int_K (\hat{g}(\zeta) - \widehat{Sg}(\zeta))e^{i<z,\zeta>}\; d\zeta\right| \;=\; \frac{1}{2\pi}\left|\int_K \sum_{\eta\in L_0^\perp} \hat{g}(\zeta-\eta)e^{i<z,\zeta>}\; d\zeta\right| \\
&\le \frac{1}{2\pi}\int_{M_1} |\hat{g}(\zeta)|\; d\zeta \;+\; \frac{C(K,P,\varphi)}{\sqrt{2\pi}}\left\lceil\frac{\sigma_1 d}{\pi}\right\rceil \max_{j=0,\dots,P-1}\int_{|\sigma|>\sigma^*} |\widehat{g_{\varphi_j}}(\sigma)| d\sigma
\end{aligned} \tag{12}$$

where $L_0^\perp = L^\perp\backslash\{0\}$.

The first equation in (12) follows from (9). In order to prove the inequality note that $\zeta - \eta \in (\mathbb{Z}\times\mathbb{R})\backslash K = M_1 \cup M_2$ for $\zeta \in K$ and $\eta \in L_0^\perp$

since the sets $K+\eta$, $\eta \in L^\perp$ are disjoint. Hence

$$\frac{1}{2\pi}\left|\int_K \sum_{\eta \in L_0^\perp} \hat{g}(\zeta-\eta) e^{i<z,\zeta>}\, d\zeta\right| \le \frac{1}{2\pi}(I_1+I_2) \tag{13}$$

with

$$I_j = \left|\int_K \sum_{\eta \in L_0^\perp} \hat{g}(\zeta-\eta)\chi_{M_j}(\zeta-\eta) e^{i<z,\zeta>}\, d\zeta\right|, \quad j=1,2.$$

Disjointness of the sets $K+\eta$, $\eta \in L^\perp$ implies

$$I_1 \;\le\; \sum_{\eta \in L_0^\perp} \int_{K-\eta} |\hat{g}(\zeta)|\, \chi_{M_1}(\zeta) d\zeta \;\le\; \int_{M_1} |\hat{g}(\zeta)| d\zeta \tag{14}$$

In order to estimate I_2 note that $K \subseteq [-k_1,k_1]\times[-\sigma_1,\sigma_1]$, and that by (5) any $\eta \in L^\perp$ has the form $\eta = (Pj - PNl/M,\ 2\pi l/d) = \eta_{jl}$ for some $j,l \in \mathbb{Z}$. Therefore, if $\zeta = (k,\sigma) \in K$ and $\eta_{jl} \in L_0^\perp$, then $\zeta - \eta_{jl} \in M_2$ if and only if $|\sigma - 2\pi l/d| > \sigma_1$. This is possible only if $l \neq 0$. It follows that

$$\begin{aligned} I_2 = \Bigg|\int_{-\sigma_1}^{\sigma_1} \sum_{k\in N(\sigma)} \sum_{l\neq 0} \sum_{j \in \mathbb{Z}} \hat{g}\left(k - \frac{PNl}{M} - Pj,\ \sigma - \frac{2\pi l}{d}\right) e^{ik\varphi + i\sigma s} \\ \chi_{\mathbb{R}\setminus[-\sigma_1,\sigma_1]}\left(\sigma - \frac{2\pi l}{d}\right) d\sigma\Bigg| \end{aligned} \tag{15}$$

The next step consists in transforming the sum over j using the Poisson summation formula for functions defined on $\mathbb{T}$, i.e.,

$$\frac{2\pi}{P}\sum_{j=0}^{P-1} G\left(\frac{2\pi j}{P}\right) e^{-2\pi i jk/P} \;=\; \sqrt{2\pi}\sum_{j\in\mathbb{Z}} \hat{G}(k+Pj).$$

Let

$$G_\tau(\varphi) = (2\pi)^{-1/2}\int_{\mathbb{R}} g(\varphi,s) e^{-is\tau} ds \;=\; \widehat{g_\varphi}(\tau).$$

With $\tau = \sigma - \frac{2\pi l}{d}$ and $\varphi_j = 2\pi j/P$ we have

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} \hat{g}\left(k-\frac{PNl}{M}-Pj,\ \sigma-\frac{2\pi l}{d}\right) &= \sum_{j\in\mathbb{Z}} \widehat{G_\tau}\left(k-\frac{PNl}{M}-Pj\right)\\
&= \frac{\sqrt{2\pi}}{P}\sum_{j=0}^{P-1} G_\tau\left(\frac{2\pi j}{P}\right) e^{-2\pi i j(k-PNl/M)/P}\\
&= \frac{\sqrt{2\pi}}{P}\sum_{j=0}^{P-1} \widehat{g_{\varphi_j}}\left(\sigma-\frac{2\pi l}{d}\right) e^{i(PNl/M-k)\varphi_j}
\end{aligned}
\tag{16}
$$

Combining (15) and (16) yields

$$
\begin{aligned}
I_2 &\leq \frac{\sqrt{2\pi}}{P}\sum_{j=0}^{P-1}\left(\max_{|\sigma|\leq\sigma_1}\left|\sum_{k\in N(\sigma)} e^{ik(\varphi-\varphi_j)}\right|\right)\\
&\qquad \sum_{l\neq 0}\int_{-\sigma_1}^{\sigma_1}\left|\widehat{g_{\varphi_j}}\left(\sigma-\frac{2\pi l}{d}\right)\right| \chi_{\mathbb{R}\setminus[-\sigma_1,\sigma_1]}\left(\sigma-\frac{2\pi l}{d}\right) d\sigma\\
&\leq \sqrt{2\pi}C(K,P,\varphi)\left\lceil\frac{\sigma_1 d}{\pi}\right\rceil \max_{j=0,\ldots,P-1}\int_{|\sigma|>\sigma^*}|\widehat{g_{\varphi_j}}(\sigma)|\, d\sigma
\end{aligned}
\tag{17}
$$

The last inequality is obtained by observing that for almost all σ, σ' such that $|\sigma|\leq\sigma_1$, $|\sigma'|>\sigma^*$, there are at most $\lceil\frac{\sigma_1 d}{\pi}\rceil$ different values of l such that $(\sigma-2\pi l/d)=\sigma'$. The estimate (12) now follows from (13), (14) and (17). $\square$

An obvious but coarse estimate for the quantity $C(K,P,\varphi)$ is $C(K,P,\varphi) \leq 2k_1+1$. The next lemma provides an estimate for $C(K,P,\varphi)$ in case the set K satisfies the condition of Proposition 5, i.e., if $(k,\sigma)\in K$, then $(k',\sigma)\in K$ for all k' with $|k'|\leq|k|$. In this case $N(\sigma)=\{-n,\ldots,n\}$ for some $0\leq n\leq k_1$, hence

$$
\begin{aligned}
C(K,P,\varphi) &\leq \frac{1}{P}\sum_{j=0}^{P-1}\max_{0\leq n\leq k_1}\left|\sum_{k=-n}^{n} e^{ik(\varphi-\varphi_j)}\right|\\
&= \frac{1}{P}\sum_{j=0}^{P-1}\max_{0\leq n\leq k_1}\left|\frac{\sin((2n+1)(\varphi-\varphi_j)/2)}{\sin((\varphi-\varphi_j)/2)}\right|.
\end{aligned}
\tag{18}
$$

Lemma 7 *Let*

$$g(\varphi) = \max_{0\le n\le k_1} \left| \frac{\sin((2n+1)\varphi/2)}{\sin(\varphi/2)} \right| .$$

Then

$$\frac{1}{P}\sum_{j=0}^{P-1} g(\varphi - 2\pi j/P) \;\le\; \min\left(2k_1+1,\; 1+\frac{6k_1+3}{P} \;+\; \frac{2}{\pi}\,\ln(4k_1+2)\right). \tag{19}$$

Proof: Since $0\le g(\varphi)\le 2k_1+1$ it is clear that $P^{-1}\sum_{j=0}^{P-1} g(\varphi-2\pi j/P)\le 2k_1+1$. Let $\Delta\varphi = 2\pi/P$, $\varphi^* = \min\left(\pi,\; 2\arcsin\left(\frac{1}{2k_1+1}\right)+\Delta\varphi\right)$, and $\varphi_j = j\Delta\varphi$. Consider first the case $\varphi^*=\pi$. Then

$$\pi = \varphi^* \le 2\arcsin\left(\frac{1}{2k_1+1}\right) + \frac{2\pi}{P}.$$

Multiplying by $(2k_1+1)/\pi$ and observing that $x\arcsin(1/x)\le \pi/2$ for $x\ge 1$ gives $2k_1+1\le 1+(4k_1+2)/P$. Hence for $\varphi^*=\pi$ the right-hand side of (19) equals $2k_1+1$ and the assertion holds.

Now assume $\varphi^*<\pi$. Since g is 2π-periodic we may consider it a function on the interval $[-\pi,\pi)$. For $\varphi\in\mathbb{R}$ let $\varphi'\in[-\pi,\pi)$ such that $\varphi'\equiv\varphi \bmod 2\pi$. Clearly,

$$g(\varphi) \;=\; g(\varphi') \;\le \begin{cases} 2k_1+1 & \text{for} \quad |\varphi'|<\varphi^* \\ \frac{1}{\sin(\varphi'/2)} & \text{for} \quad \varphi^*\le|\varphi'|\le\pi \end{cases} .$$

Since $|(\varphi-\varphi_j)'|<\varphi^*$ for at most $\left\lceil\frac{2\varphi^*}{\Delta\varphi}\right\rceil$ terms,

$$\begin{aligned}
&\frac{1}{P}\sum_{j=0}^{P-1} g(\varphi-\varphi_j)\\
&\le \frac{2k_1+1}{P}\left\lceil\frac{2\varphi^*}{\Delta\varphi}\right\rceil \;+\; \frac{1}{P}\sum_{\varphi^*\le|(\varphi-\varphi_j)'|\le\pi} \frac{1}{\sin((\varphi-\varphi_j)'/2)}\\
&\le \frac{2k_1+1}{P}\left(\frac{2P\arcsin\left(\frac{1}{2k_1+1}\right)}{\pi}+3\right) \;+\; \frac{2}{P}\sum_{l=0}^{\lfloor\frac{\pi-\varphi^*}{\Delta\varphi}\rfloor} \frac{1}{\sin((\varphi^*+l\Delta\varphi)/2)}\\
&\le 1 \;+\; \frac{6k_1+3}{P} \;+\; \frac{1}{\pi}\int_{\varphi^*-\Delta\varphi}^{\pi}\frac{1}{\sin(\varphi/2)}\,d\varphi.
\end{aligned}$$

Evaluating the integral gives

$$\begin{aligned}\int_{\varphi^*-\Delta\varphi}^{\pi} \frac{1}{\sin(\varphi/2)}\, d\varphi &= -2\ln\left(\tan\left((\varphi^*-\Delta\varphi)/4\right)\right) \\ &= 2\ln\left(2k_1+1+\sqrt{(2k_1+1)^2-1}\right) \\ &\leq 2\ln(4k_1+2). \quad \Box\end{aligned}$$

It is now clear that Proposition 5 follows from Theorem 6, Lemma 7 and (18).

3. BRIEF INTRODUCTION TO DIFFRACTION TOMOGRAPHY

In order to obtain a mathematical model accounting for diffraction effects in a tomographic experiment, we assume that the Fourier amplitude $u(x,\omega)$ of the wavefield $U(x,t)$ satisfies the reduced wave equation

$$\Delta u + k_0^2(1+f(x))u = 0 \tag{20}$$

together with the Sommerfeld radiation condition. Here Δ denotes the Laplacian, and k_0 is the wavenumber of the incoming radiation. The compactly supported complex-valued function f describes the influence of the scatterer and is related to the index of refraction $n(x)$ by $f(x) = n^2(x) - 1$. Reconstructing f from measured values of u leads to the nonlinear inverse scattering problem for (20). In the framework of diffraction tomography the inverse scattering problem is usually linearized by means of the Born- or Rytov-approximations. Following [14, 18] these approximations can be motivated as follows:

Let w be a C^2-function to be specified later, u_0 an incoming plane wave,

$$u_0(x) = e^{ik_0\theta\cdot x},\ \theta \in S^{n-1}$$

and

$$v := u_0\, w(u/u_0)$$

with u a solution of (20).

With $\alpha := \frac{u}{u_0}$ we have with (20):

$$(\Delta + k_0^2)v = -k_0^2 u_0 \left[f + (\frac{dw}{d\alpha}\alpha - 1)f - k_0^{-2}\frac{d^2w}{d\alpha^2}(\nabla\alpha)^2 \right] \tag{21}$$

If $w(\alpha) = \alpha - 1$, the third term in the bracket vanishes. Assuming $|u - u_0| \ll |u_0|$ and therefore neglecting the second term leads to the Born approximation with $v = u - u_0$ and

$$(\Delta + k_0^2)v = -k_0^2 u_0 f. \tag{22}$$

Setting $w(\alpha) = \ln \alpha$ leads to the Rytov approximation. Here the second term on the right-hand side of (21) vanishes. The third term is neglected, assuming that $|\nabla \ln \frac{u}{u_0}|^2 \ll k_0^2 |f|$. We again get equation (22) with $v = u_0 \ln \frac{u}{u_0}$. The relative merits of these two approximations have been discussed extensively in the literature. Since both lead to the same equation (22) which is the starting point for our further considerations, we do not need to consider this point here.

In the following we confine ourselves to two spatial dimensions and the so called 'parallel-beam' measurement geometry sketched in Fig. 2. The cross-section to be reconstructed is represented by the function f which is supported in the disk D_r of radius r with center in the origin. The object is irradiated with plane waves $u_0(x) = e^{ik_0\theta\cdot x}$ with wavenumber k_0 propagating in direction $\theta = (\cos\varphi, \sin\varphi)^T$. The scattered radiation is measured behind the object on a line orthogonal to θ with distance $R > r$ from the origin. So the detectors are located at points $R\theta + s\theta^\perp$ with $\theta^\perp = (-\sin\varphi, \cos\varphi)^T$ and $|s| \leq A$. Since for each direction θ one wishes to take more than one measurement, the admissible lattices defined in the previous section are precisely the lattices suitable for this case. We assume the aperture A to be sufficiently large, so that the effects of A being finite may be neglected. The data are assumed to consist of measured values of

$$Pf(\varphi, s) := k_0^{-1} \frac{v(R\theta + s\theta^\perp)}{u_0(R\theta + s\theta^\perp)} = k_0^{-1} e^{-ik_0 R}\, v(R\theta + s\theta^\perp)$$

with v being the solution of (22) corresponding to the given boundary conditions. The appropriate Green's function is given by $-\frac{i}{4}H_0(k_0|x - x'|)$ where H_0 denotes the Hankel function of the first kind of order zero. This leads to the following definition of the 'propagation operator' P [6], which maps f into the data function Pf:

Definition 8 *Throughout this paper, the letters θ and φ are related by θ being the unit vector in $\mathbb{R}^2$ with polar angle φ, i.e., $\theta = (\cos\varphi, \sin\varphi)^T$, and $\theta^\perp = (-\sin\varphi, \cos\varphi)^T$. The propagation operator $P : L_1(D_r) \to C^\infty(\mathbb{R}^2)$ is given by*

$$Pf(\varphi, s) = \frac{i}{4} k_0 e^{-ik_0 R} \int_{D_r} H_0(k_0|R\theta + s\theta^\perp - y|) f(y) e^{ik_0\theta\cdot y} dy \tag{23}$$

Sometimes φ is kept fixed and Pf is considered a function of s alone. In this case we write $P_\varphi f(s)$ for $Pf(\varphi, s)$. Particularly useful is the Fourier transform

$$\widehat{P_\varphi f}(\sigma) = (2\pi)^{-1/2} \int_{\mathbb{R}} Pf(\varphi, s) e^{-i\sigma s} ds.$$

In diffraction tomography one tries to reconstruct an approximation to f from measured values of Pf. The following 'generalized projection theorem' [28] gives a relationship between the Fourier transforms of f and $P_\varphi f$:

Theorem 9 *For $\sigma \in \mathbb{R}$ let $a(\sigma) := \sqrt{k_0^2 - \sigma^2}$ so that $Im\, a(\sigma) \geq 0$. If $f \in L_1(D_r)$,*

$$\widehat{P_\varphi f}(\sigma) = i\sqrt{\frac{\pi}{2}} k_0 \frac{e^{iR(a(\sigma)-k_0)}}{a(\sigma)} \hat{f}((a(\sigma) - k_0)\theta + \sigma\theta^\perp). \tag{24}$$

For a proof see e. g. [20, Chap. I].

For $-k_0 \leq \sigma \leq k_0$ the arguments $\xi = (a(\sigma) - k_0)\theta + \sigma\theta^\perp$ of $\hat{f}$ run over a semicircle with center at $-k_0\theta$, radius k_0, and top at the origin; see Fig. 3. By varying the direction θ the values of $\hat{f}(\xi)$ for $|\xi| \leq \sqrt{2}k_0$ can thus be obtained from the data. By filtering and performing an inverse Fourier transform we can compute low-pass filtered versions of f with bandwidth $b \leq \sqrt{2}k_0$. This is the basic idea underlying the standard reconstruction algorithms [5, 14, 22]. The achievement of resolution corresponding to a bandwidth greater than $\sqrt{2}k_0$ has been investigated in [26].

4. THE ESSENTIAL SUPPORT OF $\widehat{Pf}$

We wish to apply the results of §2 to the data function Pf. This requires determination of a suitable set K and formulation of conditions on the lattice parameters d, P, M, N so that the translated sets $K + \eta$, $\eta \in L^\perp$ are disjoint.

Definition 10 *For $k_0 > 0$ and $a(\sigma)$ as in Theorem 9 we define the functions $\rho : [-k_0, k_0] \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}$ as follows:*

$$\rho(\sigma) := ((a(\sigma) - k_0)^2 + \sigma^2)^{\frac{1}{2}}, \quad |\sigma| \leq k_0$$

$$\gamma(\sigma) := \begin{cases} \sigma\sqrt{1 - (\frac{\sigma}{2k_0})^2} & |\sigma| \leq \sqrt{2}k_0 \\ k_0 \text{sign}(\sigma) & |\sigma| > \sqrt{2}k_0 \end{cases}$$

Note that $\rho(\gamma(\sigma)) = |\sigma|$ *for* $|\sigma| \leq \sqrt{2}k_0$ *and* $\gamma(\rho(\sigma)) = |\sigma|$ *for* $|\sigma| \leq k_0$.

For $\vartheta, r, b, k_0 > 0$, $\vartheta < 1 \leq rb < \sqrt{2}rk_0$ *define the set*

$$K(\vartheta, b) := \left\{(k, \sigma) \in \mathbb{Z} \times \mathbb{R} : \ |\sigma| \leq \gamma(b), \quad |k| < \frac{r}{\vartheta} \max(\rho(\sigma), (1-\vartheta)b)\right\} \tag{25}$$

(see Fig. 4).

The next theorem will show that this is the set we are looking for.

Theorem 11 *For* $0 < b < \sqrt{2}k_0$, $0 < \vartheta < 1$ *let* $K = K(\vartheta, b)$ *as in (25). Let* $L = L(d, P, M, N)$ *be an admissible sampling lattice such that the sets* $K + \eta$, $\eta \in L^\perp$ *are disjoint. If* $f \in L_1(D_r)$, $g = Pf$, Sg *as in (2), and* ϵ^* *as in (7), then*

$$\begin{aligned} |g(\varphi, s) - Sg(\varphi, s)| \leq{} & 2^{-1/2} \parallel f \parallel_{L_1} \frac{be^{-\beta(\vartheta)rb}}{1 - e^{-\alpha(\vartheta)}} \\ & + \ \epsilon^*(g, \gamma(b)) + c(\vartheta, b, P) \left\lceil \frac{\gamma(b)d}{\pi} \right\rceil \epsilon^*(g, \gamma^*). \end{aligned} \tag{26}$$

where $\alpha(\vartheta) = (1 - \vartheta^2)^{3/2}/3$, $\beta(\vartheta) = \alpha(\vartheta)(1-\vartheta)/\vartheta$, $c(\vartheta, b, P) = 1 + (3 + 6rb/\vartheta)/P + (2/\pi)\ln(2 + 4rb/\vartheta)$, *and* $\gamma^* = \max\{\gamma(b), \frac{2\pi}{d} - \gamma(b)\}$.

As discussed in conjunction with Theorem 2, a good choice for K should be small enough for the sampling to be efficient and large enough for the aliasing error to be small. The first term on the right-hand side of (26) decreases exponentially with b. It can be shown using (24) that $|\widehat{P_\varphi f}(\sigma)|$ decreases exponentially for $|\sigma| > k_0$ (evanescent waves), because f is supported in D_r and $R > r$. Hence $\epsilon^*(Pf, \gamma(b))$ will be small if $|\widehat{P_\varphi f}(\sigma)|$ is also small for $\gamma(b) \leq |\sigma| \leq k_0$. From (24) it follows that this will be the case if $|\hat{f}(\xi)|$ is small for $b \leq |\xi| \leq \sqrt{2}k_0$. This provides the criterion for chosing the parameter b, which may be interpreted as essential bandwidth of the function f.

It should be noted that the set K given here is minimal under the assumption that the only a priori information available on f is its essential bandwidth b. If more is known about f, smaller sets K may be possible. For example, in the extreme case that f is a radial function, $\widehat{Pf}(k, \sigma) = 0$ for $k \neq 0$ and the set K could be chosen as $\{(0, \sigma) : \ |\sigma| \leq \gamma(b)\}$.

The proof of Theorem 11 will occupy the rest of this section.

Lemma 12 *Under the hypotheses of Theorem 11 and with* $M_1 = \{(k,\sigma) \in \mathbb{Z} \times \mathbb{R} : |\sigma| \le \gamma(b)\} \backslash K$,

$$\int_{M_1} |\hat{g}(\zeta)|\, d\zeta \;\le\; \frac{\pi}{\sqrt{2}} \parallel f \parallel_{L_1} \frac{be^{-\beta(\vartheta)rb}}{1-e^{-\alpha(\vartheta)}}.$$

Proof: Let J_k denote the Bessel function of the first kind of integer order k. An integral representation is furnished by

$$J_k(w) = \frac{i^{-k}}{2\pi} \int_0^{2\pi} e^{iw\cos\varphi - ik\varphi}\, d\varphi, \quad w \in \mathbb{C}, \tag{27}$$

see [20, Formula VII.3.16] or [1, Formula 9.1.21]. Combining the upper bound [1, Formula 9.1.63] and a derivation given in [20, p. 198] yields the estimate

$$|J_k(tk)| \;\le\; e^{-|k|(1-t^2)^{3/2}/3} \;=\; e^{-\alpha(t)|k|}, \quad 0 \le t \le 1. \tag{28}$$

For $|k| \ge b' = (1-\vartheta)rb/\vartheta$ let $I_k = \{\sigma \in \mathbb{R} : |\sigma| \le \gamma\left(\min(\vartheta|k|/r, b)\right)\}$. Then

$$\int_{M_1} |\hat{g}(\zeta)|\, d\zeta \;=\; \sum_{|k|\ge b'} \int_{I_k} |\hat{g}(k,\sigma)|\, d\sigma. \tag{29}$$

Using Theorem 9 and equation (27) one obtains for $|\sigma| \le k_0$

$$\begin{aligned} |\hat{g}(k,\sigma)| &= \frac{k_0}{2a(\sigma)} \left| \int_0^{2\pi} \hat{f}((a(\sigma)-k_0)\theta + \sigma\theta^\perp) e^{-ik\varphi} d\varphi \right| \\ &= \frac{k_0}{2a(\sigma)} \left| \int_0^{2\pi} \hat{f}(\rho(\sigma)\theta) e^{-ik\varphi} d\varphi \right| \qquad (30) \\ &\le \frac{k_0}{2\sqrt{{k_0}^2 - \sigma^2}} \int_{D_r} |f(y) J_k\left(|y|\rho(\sigma)\right)|\, dy. \qquad (31) \end{aligned}$$

Using the substitution $u = r\rho(\sigma)$ and (28) we get

$$\begin{aligned} &\int_{I_k} |\hat{g}(k,\sigma)|\, d\sigma \\ \le\; &\frac{k_0}{2} \|f\|_{L_1} \sup_{0\le t\le 1} \int_{I_k} |J_k(tr\rho(\sigma))| \frac{d\sigma}{\sqrt{{k_0}^2 - \sigma^2}} \\ =\; &\frac{k_0}{2} \|f\|_{L_1} \sup_{0\le t\le 1} 2 \int_0^{\min(\vartheta|k|, rb)} (r^2{k_0}^2 - u^2/4)^{-\frac{1}{2}} |J_k(tu)|\, du \end{aligned}$$

$$\leq k_0\|f\|_{L_1}\left(\sup_{0\leq t\leq 1}|J_k(t\vartheta k)|\right)\int_0^{\min(\vartheta|k|,rb)}(r^2{k_0}^2-u^2/4)^{-\frac{1}{2}}\,du$$
$$\leq 2k_0\arcsin\left(\frac{b}{2k_0}\right)\|f\|_{L_1}e^{-\alpha(\vartheta)|k|}$$
$$\leq \frac{\pi b}{2\sqrt{2}}\|f\|_{L_1}e^{-\alpha(\vartheta)|k|}. \qquad (32)$$

The result now follows by inserting (32) into (29) and evaluating the sum over k. □

Lemma 13 $\widehat{Pf}\in L_1(\mathbb{Z}\times\mathbb{R})$.

Proof: Let $g=Pf$. We show that $\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|\hat{g}(k,\sigma)|d\sigma<\infty$ by establishing that $\int_{\mathbb{R}}|\hat{g}(k,\sigma)|d\sigma=O(|k|^{-n})$ for all $n\in\mathbb{N}$. Let

$$q(\varphi,\sigma,x):=-ix\cdot\left((a(\sigma)-k_0)\theta+\sigma\theta^{\perp}\right).$$

Then

$$\begin{aligned}|k^n\hat{g}(k,\sigma)| &= \left|(2\pi)^{-1/2}\int_0^{2\pi}\left(\frac{d^n}{d\varphi^n}\widehat{g_\varphi}(\sigma)\right)e^{-ik\varphi}d\varphi\right|\\ &= \frac{k_0}{2}|a(\sigma)|^{-1}\left|e^{iRa(\sigma)}\right|\\ &\qquad\times\left|\int_0^{2\pi}\left(\frac{d^n}{d\varphi^n}\hat{f}\left((a(\sigma)-k_0)\theta+\sigma\theta^{\perp}\right)\right)e^{-ik\varphi}d\varphi\right|\\ &\leq \frac{k_0}{2}\parallel f\parallel_{L_1}|a(\sigma)|^{-1}\left|e^{iRa(\sigma)}\right|\sup_{x\in D_r,\varphi\in T}\left|\frac{d^n}{d\varphi^n}e^q\right|.\end{aligned}$$

We have $\sup_{x,\varphi}\left|\frac{d^l}{d\varphi^l}q\right|\leq\sqrt{2}r|\sigma|$. The real part of q satisfies

$$|\mathrm{Re}\,q|\begin{cases} = & 0 & |\sigma|\leq k_0\\ \leq & r\sqrt{\sigma^2-k_0^2} & |\sigma|>k_0\end{cases}.$$

Therefore

$$\sup_{x\in D_r,\varphi\in\mathbb{T}}\left|\frac{d^n}{d\varphi^n}e^q\right|\leq\begin{cases} P_n(\sqrt{2}r|\sigma|) & \sigma\leq k_0\\ P_n(\sqrt{2}r|\sigma|)e^{r\sqrt{\sigma^2-k_0^2}} & \sigma>k_0\end{cases}$$

with P_n a polynomial of degree at most n. Now let $h(\sigma) = 1$ for $\sigma \le k_0$ and $h(\sigma) = e^{-(R-r)\sqrt{\sigma^2 - k_0^2}}$ for $\sigma > k_0$. Then

$$\begin{aligned}\int_{\mathbb{R}} |\hat{g}(k,\sigma)d\sigma &\le |k|^{-n}\frac{k_0}{2} \parallel f \parallel_{L_1} \int_{\mathbb{R}} |a(\sigma)|^{-1} P_n(\sqrt{2}r|\sigma|)h(\sigma)d\sigma \\ &= C(n)|k|^{-n}. \quad \square\end{aligned}$$

Proof of Theorem 11: For $\epsilon > 0$ define $g^\epsilon(\varphi, s) = g(\varphi, s)\psi(\epsilon s)$ with $\psi \in C_0^\infty(\mathbb{R})$, $0 \le \psi(s) \le 1$, $\psi(s) = 1$ for $|s| \le 1$, and $\psi(s) = 0$ for $|s| > 2$. The theorem follows by applying Proposition 5 to g^ϵ with $\sigma_1 = \gamma(b)$, $k_1 = rb/\vartheta$, letting $\epsilon \to 0$, and using Lemma 12. Note that in the limit $\epsilon \to 0$ the estimates for g^ϵ turn into their counterparts for g because of Lemma 13 and the fact that $\widehat{g_\varphi^\epsilon}$ and $\widehat{g^\epsilon}(k, \cdot)$ converge in the L_1 -sense to $\widehat{g_\varphi}$ and $\hat{g}(k, \cdot)$, respectively. $\square$

An earlier version of Theorem 11 was given in [7] with a proof based on Theorem 2. Proposition 5 allows both the sharper estimate and the considerably less complicated proof presented here.

5. SAMPLING CONDITIONS

In this section we translate the condition that the sets $K + \eta$, $\eta \in L^\perp$ be disjoint into conditions on the lattice parameters d, P, M, N. (Recall the general form (6) of an admissible lattice $L(d, P, M, N)$.) In order to avoid technicalities coming from working with subsets of $\mathbb{Z} \times \mathbb{R}$ we introduce the following subset of $\mathbb{R}^2$:

$$K'(\vartheta, b) := \left\{(k,\sigma) \in \mathbb{R}^2 : \ |\sigma| \le \gamma(b), \quad |k| \le \frac{r}{\vartheta}\max(\rho(\sigma), (1-\vartheta)b)\right\}, \tag{33}$$

with $0 < \vartheta < 1 \le rb < \sqrt{2}rk_0$. Clearly, $K' \cap \mathbb{Z} \times \mathbb{R}$ contains the set K of (25), and disjointness of $K' + \eta$, $\eta \in L^\perp$ implies disjointness of $K + \eta$, $\eta \in L^\perp$. We obtain the following sampling conditions.

Proposition 14 *Let $L = L(d, P, M, N)$ be an admissible sampling lattice and $K' = K'(\vartheta, b)$ as in (33). The sets $K' + \eta$, $\eta \in L^\perp$ are disjoint if and only if*

$$d < \frac{\pi M}{\gamma(b)} \quad \textit{and} \tag{34}$$

$$P > \max_{0 \le l \le l_0} c_l^{-1}\left(F(\gamma(b)) + F\left(\frac{2\pi l}{d} - \gamma(b)\right)\right), \tag{35}$$

where

$$l_0 = \lfloor d\gamma(b)/\pi \rfloor,$$
$$F(\sigma) = \frac{r}{\vartheta}\max\{\rho(\sigma),\ (1-\vartheta)b\}, \quad |\sigma| \le \gamma(b),$$

and

$$c_l \ = \ \begin{cases} 1, & l = 0 \\ \min_{k\in\mathbb{Z}} \left|k - \frac{Nl}{M}\right|, & l \neq 0 \end{cases}.$$

Before proving the Proposition we discuss some of its consequences. First it is worthwhile to note that F is an even function and

$$F(\gamma(b)) = \frac{rb}{\vartheta}.$$

Since always $l_0 \ge 0$, a necessary condition on the number of views is $P > c_0^{-1}(F(\gamma(b)) + F(-\gamma(b))$, i.e.,

$$P \ > \ 2rb/\vartheta. \tag{36}$$

On the other hand, sufficient conditions are

$$d \ < \ \frac{\pi}{\gamma(b)}, \quad P \ > \ \frac{2rb}{\vartheta}, \tag{37}$$

independently of M and N. This follows from (34) and (35) since $d < \pi/\gamma(b)$ implies $l_0 = 0$. We may use the lattice constant

$$C_L = (2\pi)^{-1}|\det W| = d/P$$

as a measure of the overall sampling efficiency, since it is inversely proportional to the density of sampling points. The larger C_L, the fewer points are needed, and hence the more efficient the lattice.

The most often used lattice which we call the standard lattice is obtained for $M = 1$, $N = 0$. The condition (34) for d reads $d < \pi/\gamma(b)$. As shown above this means that the condition for P simplifies to $P > 2rb/\vartheta$. Hence the sampling conditions for the standard lattice are exactly the conditions (37) which are always sufficient. Hence the standard lattice, although it needs a minimal number of views, is not the most efficient lattice. The sampling conditions imply that for the standard lattice

$$C_L \ < \ \frac{\pi\vartheta}{2rb\gamma(b)}. \tag{38}$$

Choosing $M = 2$ and $N = 1$ gives a lattice which we call the efficient lattice. The condition (34) becomes $d < 2\pi/\gamma(b)$, hence d may be chosen up to twice as large as in case of the standard lattice. If $\pi/\gamma(b) \le d < 2\pi/\gamma(b)$ then $l_0 = 1$, $c_0 = 1$, $c_1 = 1/2$ and (35) reads

$$P \; > \; \frac{2rb}{\vartheta} \; + \; 2F\left(\frac{2\pi}{d} - \gamma(b)\right). \tag{39}$$

If d is sufficiently close to its maximum value of $2\pi/\gamma(b)$, then $F\left(\frac{2\pi}{d} - \gamma(b)\right) = (1-\vartheta)rb/\vartheta$, and the condition on P reads $P > (2-\vartheta)2rb/\vartheta$. For the efficiency it follows that for these values of d and P

$$C_L \; < \; \frac{2}{2-\vartheta}\,\frac{\pi\vartheta}{2rb\gamma(b)}. \tag{40}$$

Hence, for ϑ close to 1 this lattice is nearly twice as efficient as the standard lattice.

The truly two-dimensional nature of the sampling conditions is manifested by the dependence of the condition (35) for P on the parameter d. As a consequence, increasing the amount of sampled data may make things worse instead of better in some cases. E.g., consider the efficient lattice with d close to $2\pi/\gamma(b)$, and P close to $(2-\vartheta)2rb/\vartheta$. If P is kept fixed and d is sufficiently decreased but still larger than $\pi/\gamma(b)$, the condition (39) will be violated. This effect is caused by the non-rectangular shape of the set K'.

In order to find new lattices of high efficiency, let $M > 1$ and observe that if d satisfies (34) then $l_0 \le M - 1$, and $c_l \in \{j/M,\ j = 1, \dots, M\}$ for $0 \le l \le l_0$. Hence the smallest possible value for c_l is $1/M$. Furthermore,

$$c_l^{-1}\left(F(\gamma(b)) + F\left(\frac{2\pi l}{d} - \gamma(b)\right)\right) \left\{ \begin{array}{ll} > MF(\gamma(b)) & \text{if } c_l = 1/M \\ \le MF(\gamma(b)) & \text{if } c_l \ge 2/M \end{array} \right. .$$

Assume that d is chosen close to its maximum value of $M\pi/\gamma(b)$. Then $l_0 = M-1$ and the maximum in (35) is assumed for $l \in \{l_1, l_2\}$ where l_1, l_2 are the two values of l for which $c_l = 1/M$. Since $F(\sigma)$ increases with $|\sigma|$, it follows that the parameter N should be chosen such that $\max_{j=1,2} |2\pi l_j/d - \gamma(b)|$ is small. This means that l_1, l_2 should both be close to $M/2$. A class of lattices which satisfies this condition is obtained by chosing

$$M > 1, \quad M \text{ odd}, \quad N = 2. \tag{41}$$

In this case, if d is close $M\pi/\gamma(b)$, $c_l = 1/M$ for $l \in \{(M-1)/2,\ (M+1)/2\}$ and (35) reads

$$P > M\left(F(\gamma(b)) \; + \; F\left(\pi(M+1)/d - \gamma(b)\right)\right)$$

In the limiting case $d = M\pi/\gamma(b)$ this becomes

$$P > M\left(F(\gamma(b)) \; + \; F\left(\gamma(b)/M\right)\right)$$

If M is large enough such that $\rho(\gamma(b)/M) < (1-\vartheta)b$, then the lattice is as efficient as the efficient lattice.

Sampling conditions for the standard lattice have also been derived in [22, 27]. The qualitative argument used in [22] for the condition on P has been made precise in Lemma 12. The approach taken in [27] assumes that $\hat{f}$ has already been computed via the generalized projection theorem. Hence only the condition for the number of views can be compared to the results derived here. Translated into our terminology, the condition is $P \geq 2r\gamma(b)$ for the minimum number of views. For b close to $\sqrt{2}k_0$ this is significantly lower than our condition $p > 2rb$, while the two conditions differ not very much for $b < k_0$. From the theory developed above it follows that there could be aliasing in the data when the sampling is such that the sets $K+\eta,\ \eta \in L^\perp$ overlap and if $\widehat{Pf}(\zeta)$ is not negligibly small in the part of K being overlapped. In such a case, our theoretical as well as numerical results indicate that the condition $p > 2rb$ cannot be weakened. We attribute the discrepancy to the result of [27] to the fact that the approach taken there is based on the approximative Carson's rule rather than on an exact theorem. However, for many objects the essential support of $\widehat{Pf}$ does not occupy all of K, so that some overlap of the sets $K+\eta,\ \eta \in L^\perp$ can be accepted without consequences. Then the required number of views for the standard lattice may be much lower than $2rb$.

The remainder of this section will be devoted to the proof of Proposition 14.

Lemma 15 *Let K' and F be as in Proposition 14 and $\eta = (\eta_1, \eta_2) \in \mathbb{Z} \times \mathbb{R}$. Then $K' \cap (K'+\eta) = \emptyset$ if and only if one of the following two conditions is satisfied.*

$$i) \quad |\eta_2| \; > \; 2\gamma(b) \tag{42}$$

$$ii) \quad |\eta_2| \; \leq \; 2\gamma(b), \quad |\eta_1| \; > \; F(\gamma(b)) + F(|\eta_2| - \gamma(b)) \tag{43}$$

Proof: Since $(k,\sigma)\in K'$ implies $|\sigma|\le\gamma(b)$ it is clear that condition $i)$ is sufficient. Consider the case that $i)$ does not hold, i.e., that $0\le|\eta_2|\le 2\gamma(b)$. Assume first that $\eta_1\ge 0$ and $0<\eta_2\le 2\gamma(b)$. Let S be the strip

$$S=\mathbb{R}\times[\eta_2-\gamma(b),\gamma(b)].$$

Clearly, $K'\cap(K'+\eta)\subset S$. The left and right boundaries of the set K' are partly flat and partly curved towards the interior of K', cf. Fig. 4. Therefore the part of K' intersecting S lies to the left of the line segment joining the two boundary points $(F(\eta_2-\gamma(b)),\ \eta_2-\gamma(b))$ and $(F(\gamma(b)),\ \gamma(b))$ of K'. Similarly, the part of $K'+\eta$ intersecting S lies to the right of the line segment joining the two boundary points $(\eta_1-F(\gamma(b)),\ \eta_2-\gamma(b))$, $(\eta_1-F(\eta_2-\gamma(b)),\ \gamma(b))$ of $K'+\eta$. Since these two line segments are parallel, a sufficient condition for disjointness of K' and $K'+\eta$ is $\eta_1-F(\gamma(b))>F(\eta_2-\gamma(b))$, i.e., $\eta_1>F(\gamma(b))+F(\eta_2-\gamma(b))$. If this condition is violated, then the line segment joining $(-F(\gamma(b)),\ \gamma(b))$ and $(F(\gamma(b)),\ \gamma(b))$ which is contained in K' intersects the segment joining $(\eta_1-F(\eta_2-\gamma(b)),\ \gamma(b))$ and $(\eta_1+F(\eta_2-\gamma(b)),\ \gamma(b))$ which is contained in $K'+\eta$, hence $K'\cap(K'+\eta)\ne\emptyset$.

Because of the symmetry of the set K', analogous arguments can be used to prove the assertion for the cases $\eta_1\ge 0$, $-2\gamma(b)\le\eta_2\le 0$, and $\eta_1<0$, $|\eta_2|\le 2\gamma(b)$. $\square$

Proof of Proposition 14: Since $(K'+\eta)\cap(K'+\eta')\ne\emptyset$ implies $K'\cap(K'+(\eta'-\eta))\ne\emptyset$, it suffices to find conditions such that $K'\cap(K'+\eta)=\emptyset$ for $0\ne\eta\in L^\perp$. By Lemma 15 $K'\cap(K'+(\eta_1,\eta_2))=\emptyset$ if and only if (42) or (43) are satisfied.

Note that $L^\perp=2\pi W_1^{-T}\mathbb{Z}^2=2\pi W_2^{-T}\mathbb{Z}^2$ with W_1,W_2 as in Lemma 3. Using W_2, any $\eta\in L^\perp$ can be written as $\eta=(Pk/M,\ 2\pi(lM-kN')/d)$ for some $k,l\in\mathbb{Z}$. For $k=0$, $l=1$ this gives $\eta=(\eta_1,\eta_2)=(0,2\pi M/d)$. Hence (43) is not satisfied and therefore (42) yields the necessary condition $d<\pi M/\gamma(b)$. Now assume that this condition is satisfied and represent $\eta\in L^\perp$ using W_1, i.e., $\eta=(P(k-Nl/M),\ 2\pi l/d)$ for some $k,l\in\mathbb{Z}$. We have $|\eta_2|=|2\pi l/d|>2\gamma(b)$ if and only if $|l|>\lfloor d\gamma(b)/\pi\rfloor=l_0$. For $|l|\le l_0$ condition (43) has to be satisfied. Since $d<\pi M/\gamma(b)$, $l_0\le M-1$. For $|l|\le l_0$ and $0\ne\eta\in L^\perp$ we have

$$\min_{\substack{k\in\mathbb{Z}\\(k,l)\ne(0,0)}}|\eta_1| \;=\; P\min_{\substack{k\in\mathbb{Z}\\(k,l)\ne(0,0)}}\left|k-\frac{Nl}{M}\right| \;=\; Pc_l.$$

Hence the required condition is

$$P \; > \; c_l^{-1}\left(F(\gamma(b)) + F\left(\frac{2\pi|l|}{d} - \gamma(b)\right)\right), \quad |l| \leq l_0.$$

Observing that $c_{-l} = c_l$ gives the condition (35). □

6. SAMPLING CONDITIONS FOR THE X-RAY TRANSFORM

In the high-frequency limit $k_0 \to \infty$ diffraction effects are negligible and the radiation can be assumed to propagate along lines. Accordingly, the propagation operator turns into a line integral of f. This is the model underlying x-ray tomography. For an overview over the mathematical aspects of x-ray tomography see, e.g., [11, 20, 24] and the references given there. Because of its practical importance it is worthwhile to state our results for this case explicitly. For simplicity assume that $f \in C_0^\infty$.

Definition 16 *Let $f \in C_0^\infty(\mathbb{R}^2)$. Then*

$$\tilde{P}f(\varphi, s) \; = \; \int_{\mathbb{R}} f(s\theta^\perp + t\theta)dt \tag{44}$$

is called the x-ray transform of f.

Clearly $\tilde{P}f$ is the integral of f over the line with direction θ and signed distance s from the origin. Note that

$$\lim_{k_0\to\infty}(-2i)Pf(\varphi, s) = \tilde{P}f(\varphi, s), \tag{45}$$

$$\lim_{k_0\to\infty}\rho(\sigma) = |\sigma|, \tag{46}$$

$$\lim_{k_0\to\infty}\gamma(\sigma) = \sigma. \tag{47}$$

A proof of (45) can be found in [20, p. 7]. Equations (46) and (47) imply that the set $K'(\vartheta, b)$ given in (33) turns into the set

$$\tilde{K}(\vartheta, b) = \left\{(k, \sigma) \in \mathbb{R}^2 : \; |\sigma| \leq b, \quad |k| \leq \frac{r}{\vartheta}\max\left(|\sigma|, \; (1 - \vartheta)b\right)\right\}, \tag{48}$$

which corresponds to the set defined in [20, p. 71, Theorem III.3.1]. Taking the high-frequency limit in Theorem 11 yields

Corollary 17 *For* $0 < \vartheta < 1 \leq rb$ *let* $\tilde{K}(\vartheta, b)$ *as in (48). Let* $L = L(d, P, M, N)$ *be an admissible sampling lattice such that the sets* $\tilde{K} + \eta$, $\eta \in L^\perp$ *are disjoint. If* $f \in C_0^\infty(D_r)$, $g = \tilde{P}f$, *and* Sg *as in (2), then*

$$|g(\varphi, s) - Sg(\varphi, s)| \leq 2^{1/2} \parallel f \parallel_{L_1} \frac{be^{-\beta(\vartheta)rb}}{1 - e^{-\alpha(\vartheta)}} + \epsilon^*(g, b) + c(\vartheta, b, P) \left\lceil \frac{bd}{\pi} \right\rceil \epsilon^*(g, b^*), \qquad (49)$$

where $\alpha(\vartheta) = (1 - \vartheta^2)^{3/2}/3$, $\beta(\vartheta) = \alpha(\vartheta)(1 - \vartheta)/\vartheta$, $c(\vartheta, b, P) = 1 + (3 + 6rb/\vartheta)/P + (2/\pi)\ln(2 + 4rb/\vartheta)$, *and* $b^* = \max\{b, \frac{2\pi}{d} - b\}$.

The estimate (49), which is based on Proposition 5 sharpens the corresponding estimate in [20, Theorem III.3.1] which was derived using Theorem 2. This can most easily be seen by observing that the quantity $\epsilon_0(f, b)$ in [20, Theorem III.3.1] becomes infinite if f has discontinuities, while the right-hand side of (49) remains finite for some discontinuous functions, e.g. for f being the characteristic function of a disk. The general sampling conditions for the x-ray transform read as follows:

Corollary 18 *Let* $L = L(d, P, N, M)$ *be an admissible sampling lattice and* $\tilde{K}(\vartheta, b)$ *as in (48). The sets* $\tilde{K} + \eta$, $\eta \in L^\perp$ *are disjoint if and only if*

$$d < \frac{\pi M}{b} \quad \textit{and} \qquad (50)$$

$$P > \max_{0 \leq l \leq l_0} c_l^{-1} \left(F(b) + F\left(\frac{2\pi l}{d} - b\right)\right), \qquad (51)$$

where

$$l_0 = \lfloor db/\pi \rfloor,$$
$$F(\sigma) = \frac{r}{\vartheta} \max\{|\sigma|,\ (1 - \vartheta)b\}, \quad |\sigma| \leq b,$$

and c_l *as in Proposition 14.*

The discussion of the various lattices and their relative efficiency given in the previous section applies here as well. In particular, the choice of parameters (41) again leads to new efficient lattices.

While sampling the x-ray transform has received considerable attention, investigations in the literature concentrated on the standard and the efficient lattice. The general sampling conditions and the discussion of other lattices given here seem to be new. The efficient lattice was first suggested in [3] based on a geometrical argument. It was rediscovered in [17, 25] by using Shannon sampling theory, and a rigorous mathematical treatment was given in [20]. Error estimates for reconstructions from x-ray data sampled on the efficient lattice have been derived in [15, 9] and additional numerical experiments can be found in [11]. Use of the efficient lattice for diffraction tomography was suggested and supported by numerical experiments in [7]. Parallel-beam sampling schemes were the sampling set is a union of cosets of a lattice where investigated in [8] and applied in [4]. Sampling conditions for the so-called fan-beam geometry were derived in [21].

Acknowledgment

This research has been supported by NSF grant DMS-9404436.

REFERENCES

1. M. Abramowitz and I. A. Stegun. "Handbook of Mathematical Functions," U.S. Dept. of Commerce (1972).
2. P. L. Butzer, W. Splettstößer and R. L. Stens, The sampling theorem and linear prediction in signal analysis, Jahresber. Deutsch. Math.-Verein. 90:1 (1988).
3. A. M. Cormack, Sampling the Radon transform with beams of finite width, *Phys. Med. Biol.* 23:1141 (1978).
4. L. Desbat, Efficient sampling on coarse grids in tomography, *Inverse Problems* 9:251 (1993).
5. A. J. Devaney, A filtered backpropagation algorithm for diffraction tomography, *Ultrasonic Imaging* 4:336 (1982).
6. A. J. Devaney, Reconstructive tomography with diffracting wavefields, *Inverse Problems* 2:161 (1986)
7. A. Faridani. "Abtastbedingungen und Auflösung in der Beugungstomographie." Ph.D. Dissertation, University of Münster, Germany (1988).
8. A. Faridani, An application of a multidimensional sampling theorem to computed tomography, *in*: "Integral Geometry and Tomography," E. Grinberg and E.T Quinto, eds., Contemp. Math., vol. 113, Amer. Math. Soc., Providence, R.I., (1990), pp.65-80.
9. A. Faridani, Reconstructing from efficiently sampled data in parallel-beam tomography, *in*: "Inverse Problems and Imaging," G.F. Roach, ed., Pitmans Research Notes in Mathematics Series, vol. 245, (1991), pp. 68-102.

10. A. Faridani, A generalized sampling theorem for locally compact abelian groups, *Math. Comp.* 63:307 (1994).
11. A. Faridani, Results, old and new, in computed tomography, *in*: "Inverse Problems in Wave Propagation," G. Chavent et al., ed., Springer, New York (1997), pp. 167-193.
12. J. F. Greenleaf, Computerized tomography with ultrasound, *Proc. IEEE* 71:330 (1983).
13. A. C. Kak and M. Slaney. " Principles of Computerized Tomographic Imaging," IEEE Press, New York (1988).
14. M. Kaveh , M. Soumekh and J. F. Greenleaf, Signal processing for diffraction tomography, *IEEE Trans. Sonics and Ultrasonics* 31:230 (1984).
15. H. Kruse, Resolution of reconstruction methods in computerized tomography, *SIAM J. Sci. Stat. Comput.* 10:447 (1989).
16. K. J. Langenberg, Applied inverse problems for acoustic, electromagnetic and elastic wave scattering, *in*: "Basic Methods of Tomography and Inverse Problems," P. C. Sabatier, ed., Adam Hilger, Bristol (1987), pp. 125-467.
17. A. G. Lindgren and P. A. Rattey, The inverse discrete Radon transform with applications to tomographic imaging using projection data, *Advances in Electronics and Electron Physics* 56:359 (1981).
18. Z. Q. Lu, Multidimensional structure diffraction tomography for varying object orientation through generalized scattered waves, *Inverse Problems* 1:339 (1985).
19. R. K. Mueller, M. Kaveh and G. Wade, Reconstructive tomography and applications to ultrasonics, *Proc. IEEE* 67:567 (1979).
20. F. Natterer. "The Mathematics of Computerized Tomography," Wiley, New York (1986).
21. F. Natterer, Sampling in fan-beam tomography, *SIAM J. Appl. Math.* 53:358 (1993).
22. S. X. Pan and A. C. Kak, A computational study of reconstruction algorithms for diffraction tomography: interpolation versus filtered backpropagation, *IEEE Trans. Acoust. Speech Signal Processing* 31:1262 (1983).
23. D. P. Petersen and D. Middleton, Sampling and reconstruction of wave-number-limited functions in N-dimensional euclidean space, *Inf. Control* 5:279 (1962).
24. A. G. Ramm and A. I. Katsevich. "The Radon Transform and Local Tomography," CRC Press, Boca Raton (1996).
25. P. A. Rattey and A. G. Lindgren, Sampling the 2-D Radon transform, *IEEE Trans. Acoust. Speech Signal Processing* 29:994 (1981).
26. A. Schatzberg and A. J. Devaney, Super-resolution in diffraction tomography, *Inverse Problems* 8:149 (1992).
27. M. Soumekh and J.-H. Choi, Reconstruction in diffraction imaging, *IEEE Trans. Ultrason. Ferroelec. Freq. Contr.* 36:93 (1989).
28. E. Wolf, Three-dimensional structure determination of semi-transparent objects from holographic data, *Opt. Commun.* 1:153 (1969).
29. A. I. Zayed. "Advances in Shannon's Sampling Theory," CRC Press, Boca Raton (1993).

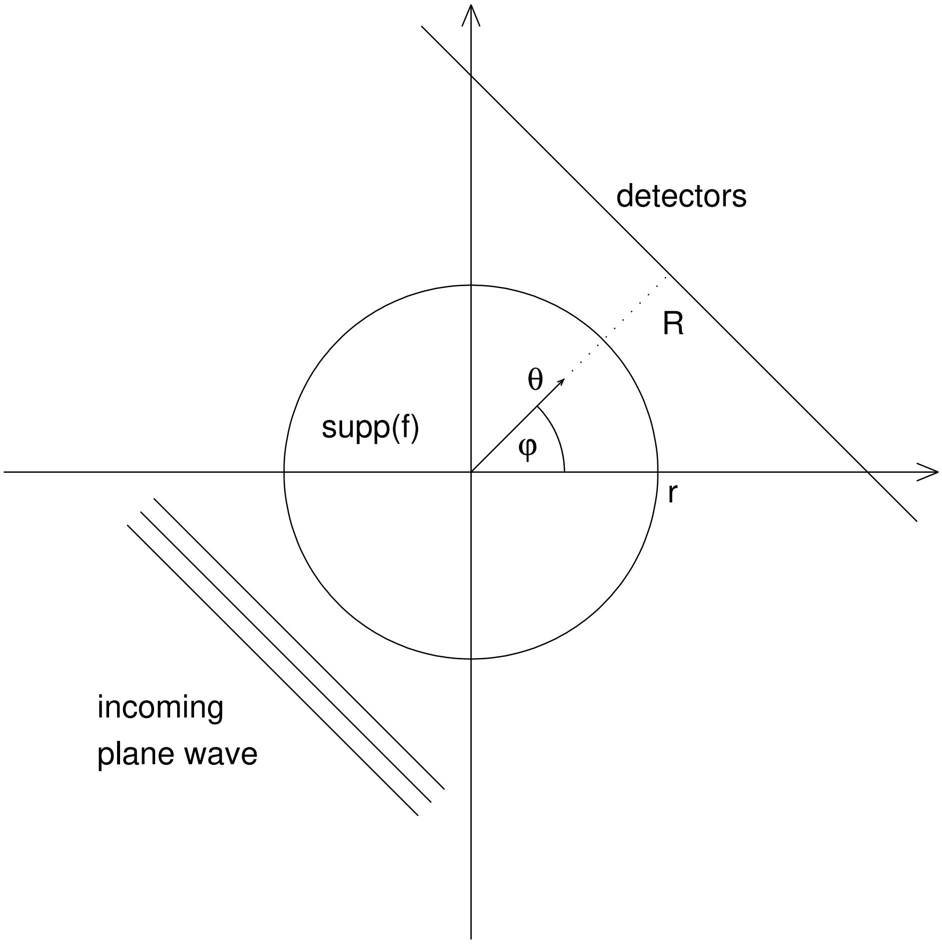

Figure 2. Parallel measurement geometry.

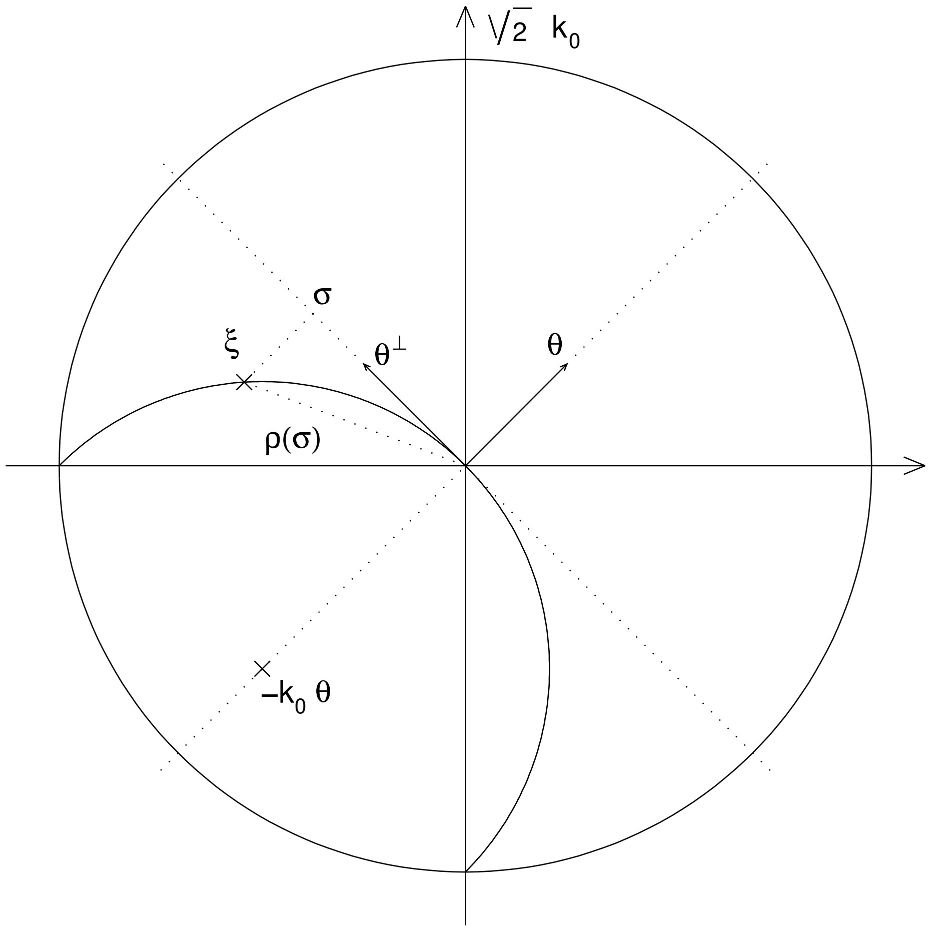

Figure 3. The Fourier transform $\widehat{P_\varphi f}(\sigma)$ is related to the Fourier transform of f at the point $\xi = (a(\sigma) - k_0)\theta + \sigma\theta^\perp$. For $-k_0 \leq \sigma \leq k_0$ the point ξ traverses a semicircle. As θ runs through S^1, this semicircle sweeps out a disk of radius $\sqrt{2}k_0$. The function ρ given in Definition 10 satisfies $\rho(\sigma) = |\xi|$.

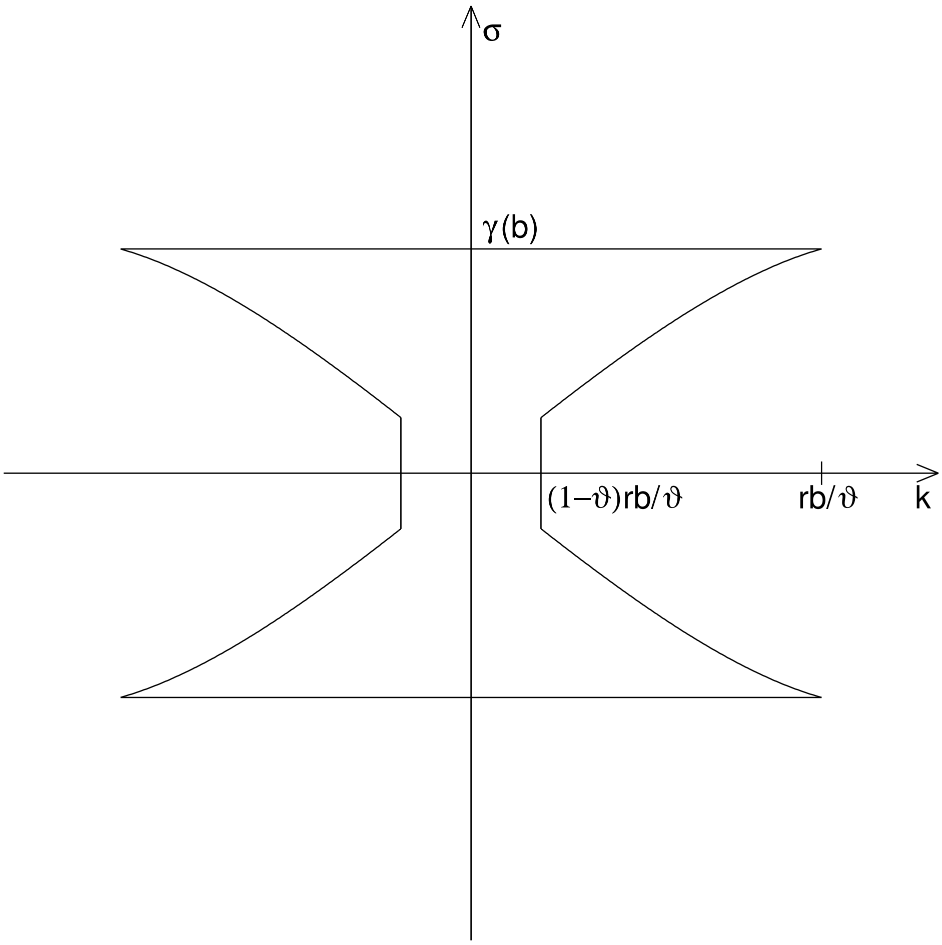

Figure 4. The set $K(\vartheta, b)$ of Definition 10 for $\vartheta = 0.8$ and $b = 1.2\,k_0$.