Exam 1: MAP 4403*

November 5, 2004

Name:
Student ID:

Note: Calculators are allowed but not needed. This is a **closed book** exam: Formula sheets are **not** allowed.

1. Find
$$\int_C \mathrm{Re}(z^2)dz,$$
where C is the horizontal line segment from the point i to the point $1+i$.

   Put $z(t) = t + i$ with $t \in [0,1]$. Then $\mathrm{Re}(z^2) = t^2 - 1$ and $dz = dt$, so
$$\int_C \mathrm{Re}(z^2)dz = \int_0^1 t^2 - 1dt = t^3/3 - t|_0^1 = -2/3.$$

2. Find and draw all cubic roots of $8\,\mathrm{e}^{i\frac{\pi}{2}}$.

   $2\,\mathrm{e}^{i(\frac{\pi}{6}+k\frac{2\pi}{3})}$, $k = 0, 1, 2$. These are on the circle of radius 2 with center 0. The root corresponding to $k = 0$ has argument equal to $\pi/6$ and the second and third are found by rotating the first over an angle of $+2\pi/3$, respectively $+4\pi/3$.

3. Find the principal value of
$$\int_{-\infty}^{+\infty} \frac{dx}{(x-1)(x+2)(x^2-2x+2)}.$$
(Hint: Recall the formula
$$\text{pr. v. } \int_{-\infty}^{+\infty} f(x)dx = 2\pi i \sum \mathrm{Res} f(z) + \pi i \sum \mathrm{Res} f(z)$$
where the first sum of residues is taken over singularities of f in the upper half plane and the second over simple poles of f on the real axis; this holds if f is rational with a denominator of degree at least two higher than the numerator.)

   The integrand has 4 poles, two real ones (in 1 and -2) and a pair of complex conjugate poles, given by the roots of z^2-2z+2. These are $1+i$ and $1-i$. Clearly the former is in the upper half plane while the latter is in the lower half plane. Finally notice that the integrand is rational and that the degree of its denominator is 4 higher than the degree of its numerator.

   We will need to compute three residues. Since all poles are simple, we can get them from:
$$\mathrm{Res}_{z=1+i} f(z) = \lim_{z \to 1+i} \frac{1}{(z-1)(z+2)(z-(1-i))} = \frac{-3+i}{20},$$
$$\mathrm{Res}_{z=1} f(z) = \lim_{z \to 1} \frac{1}{(z+2)(z^2-2z+2)} = \frac{1}{3},$$

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and

$$\text{Res}_{z=-2} f(z) = \lim_{z \to -2} \frac{1}{(z-1)(z^2-2z+2)} = -\frac{1}{30}.$$

Plugging this into the given formula, yields

$$\int_{-\infty}^{+\infty} \frac{dx}{(x-1)(x+2)(x^2-2x+2)} = \pi i (2\frac{-3+i}{20} + \frac{1}{3} - \frac{1}{30}) = -\frac{\pi}{10}.$$

4. Verify that the function

$$u(x,y) = \cosh(x)\sin(y)$$

is harmonic and find a conjugate harmonic function. (Hint: Recall that $\cosh(x) = (\mathrm{e}^x + \mathrm{e}^{-x})/2$ for a real variable x.)

Clearly u has continuous partial derivatives of of order 2 and

$$u_x = \sinh(x)\sin(y), u_{xx} = \cosh(x)\sin(y) \text{ and } u_y = \cosh(x)\cos(y), u_{yy} = -\cosh(x)\sin(y),$$

which implies that $u_{xx} + u_{yy} = 0$ and thus that u is harmonic. To find a conjugate harmonic function $v(x,y)$, we solve the Cauchy-Riemann equations for v:

$$(u_x =) \sinh(x)\sin(y) = v_y \text{ and } (u_y =) \cosh(x)\cos(y) = -v_x.$$

Integrating the first equation with respect to y gives:

$$v = -\sinh(x)\cos(y) + c(x),$$

for some function $c(x)$ having second order continuous partial derivatives. Plugging this into the second equation gives:

$$\cosh(x)\cos(y) = +\cosh(x)\cos(y) + dc/dx,$$

and thus $dc/dx = 0$ or equivalently $c(x) \equiv C$ for some constant C and thus a class of conjugate harmonic functions v is given by

$$v(x,y) = -\sinh(x)\cos(y) + C.$$

5. Find the Laurent series at $z_0 = 0$ of the function (and specify the region of convergence!):

$$f(z) = \frac{1}{z^4} Ln(1+z)$$

Then determine the residue of f at $z_0 = 0$.

First determine the Taylor series of $f(z) = Ln(1+z)$ at $z_0 = 0$:

$$f(0) = 0, f'(0) = +1, f''(0) = -1, f'''(0) = +2, f^{(iv)}(0) = -3.2, \dots,$$

or

$$f^{(n)}(0) = (-1)^{n+1}(n-1)! \text{ when } n \geq 1$$

and thus

$$Ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

The radius of convergence can be computed from Cauchy-Hadamard's formula:

$$R = \lim_{n \to \infty} \frac{n+1}{n} = 1,$$

and thus the region of convergence is given by

$$|z| < 1.$$

(Notice that for $z = -1$, the function $Ln(1+z)$ is not defined). All this implies

$$f(z) = \frac{1}{z^4} Ln(1+z) = \frac{1}{z^3} - \frac{1}{2z^2} + \frac{1}{3z} - \frac{1}{4} + \dots, \quad |z| < 1$$

and thus the residue of f at $z = 0$, the coefficient of the power $1/z$ in the above Laurent series is:

$$Res_{z_0=0} f(z) = \frac{1}{3}.$$