

# VANISHING VISCOSITY IN THE PLANE FOR NONDECAYING VELOCITY AND VORTICITY

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ABSTRACT. Assuming that initial velocity and initial vorticity are bounded in the plane, we show that on a sufficiently short time interval the unique solutions of the Navier-Stokes equations converge uniformly to the unique solution of the Euler equations as viscosity approaches zero. We also establish a rate of convergence.

## 1. INTRODUCTION

We consider the Navier-Stokes equations modeling incompressible viscous fluid flow, given by

$$(NS) \quad \begin{cases} \partial_t v_\nu + v_\nu \cdot \nabla v_\nu - \nu \Delta v_\nu = -\nabla p_\nu \\ \operatorname{div} v_\nu = 0 \\ v_\nu|_{t=0} = v_\nu^0, \end{cases}$$

and the Euler equations modeling incompressible non-viscous fluid flow, given by

$$(E) \quad \begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p \\ \operatorname{div} v = 0 \\ v|_{t=0} = v^0. \end{cases}$$

In this paper, we study the vanishing viscosity limit. The question of vanishing viscosity assumes  $v_\nu^0$  equals  $v^0$  and addresses whether or not the velocity  $v_\nu$  solving  $(NS)$  converges in some norm to the velocity  $v$  solving  $(E)$  as viscosity tends to 0. This area of research is active both for solutions in a bounded domain and for weak solutions in the plane. We focus our attention on the latter case.

The vanishing viscosity problem is closely tied to uniqueness of solutions to the Euler equations, because the methods used to prove uniqueness can often be applied to show vanishing viscosity. One of the most important uniqueness results in the plane is due to Yudovich. He establishes in [19] the uniqueness of a solution  $(v, p)$  to  $(E)$  in the space  $C(\mathbb{R}; L^2(\mathbb{R}^2)) \times L_{loc}^\infty(\mathbb{R}; L^2(\mathbb{R}^2))$  when  $v^0$  belongs to  $L^2(\mathbb{R}^2)$  and

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1991 *Mathematics Subject Classification.* Primary 76D05, 76C99.

*Key words and phrases.* Fluid mechanics, Inviscid limit.

the vorticity  $\omega(v^0) = \partial_1 v_2^0 - \partial_2 v_1^0$  belongs to  $L^p(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$  for some  $p < \infty$ . For this uniqueness class, Chemin proves in [2] that the vanishing viscosity limit holds in the  $L^p$ -norm, and he establishes a rate of convergence. (In fact, the author only considers the case  $p = 2$ ; however, the proof of the result can easily be generalized to any  $p < \infty$ .)

In this paper, we consider the case where initial velocity and initial vorticity are bounded and do not necessarily belong to  $L^p(\mathbb{R}^2)$  for any  $p < \infty$ . The existence and uniqueness of solutions to  $(NS)$  without any decay assumptions on the initial velocity is considered by Giga, Inui, and Matsui in [8]. The authors establish the short-time existence and uniqueness of mild solutions  $v_\nu$  to  $(NS)$  in the space  $C([0, T_0]; BUC(\mathbb{R}^n))$  when initial velocity is in  $BUC(\mathbb{R}^n)$ ,  $n \geq 2$ . Here  $BUC(\mathbb{R}^n)$  denotes the space of bounded uniformly continuous functions on  $\mathbb{R}^n$  (see Theorem 1 for details). In [9], Giga, Matsui, and Sawada prove that when  $n = 2$ , the unique solution can be extended globally in time.

Under the assumption that both initial velocity and initial vorticity belong to  $L^\infty(\mathbb{R}^2)$ , and with additional assumptions on the pressure, Serfati shows in [14] that a unique weak solution  $(v, p)$  to  $(E)$  exists in  $L^\infty([0, T]; L^\infty(\mathbb{R}^2)) \times L^\infty([0, T]; C(\mathbb{R}^2))$  (see Theorem 2).

We prove that the vanishing viscosity limit holds for short time in the  $L^\infty$ -norm when initial velocity and initial vorticity belong to  $L^\infty(\mathbb{R}^2)$  (see Theorem 3). To establish the result, we consider low, middle, and high frequencies of the difference between the solutions to the Navier-Stokes and Euler equations separately. Specifically, we define  $S_n v = \chi(2^{-n}D)v$ , where  $\chi$  is identically 1 on the unit ball and 0 outside of the ball of radius two, and we write

$$(1.1) \quad \begin{aligned} \|v_\nu - v\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^2))} &\leq \|S_{-n}(v_\nu - v)\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^2))} \\ &+ \|(S_n - S_{-n})(v_\nu - v)\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^2))} \\ &+ \|(Id - S_n)(v_\nu - v)\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^2))}. \end{aligned}$$

Letting  $n$  be a function of  $\nu$  so that as  $n$  approaches infinity,  $\nu$  tends to 0, we show that each of the three terms in (1.1) converges to 0 as  $n$  approaches infinity. The estimate of the term  $\|S_{-n}(v_\nu - v)\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^2))}$  relies on the structure of the integral equation satisfied by the solution  $v_\nu$  to  $(NS)$  (see (3.1) in Section 3). To bound the high frequency term, given by  $\|(Id - S_n)(v_\nu - v)\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^2))}$ , we apply Bernstein's Lemma and Calderon-Zygmund theory.

To estimate the  $L^\infty$ -norm of  $(S_n - S_{-n})(v_\nu - v)$ , we observe that this term has Fourier support in an annulus with inner and outer radius of order  $2^{-n}$  and  $2^n$ , respectively. Breaking this support into  $2n$  smaller

annuli, we use the definition of the Besov space  $\dot{B}_{\infty,\infty}^0$  (see Section 2) to conclude that

$$(1.2) \quad \|(S_n - S_{-n})(v_\nu - v)(t)\|_{L^\infty} \leq Cn\|(v_\nu - v)(t)\|_{\dot{B}_{\infty,\infty}^0}$$

for each  $t > 0$ . We proceed to show that  $v_\nu$  converges to  $v$  in the  $\dot{B}_{\infty,\infty}^0$ -norm as viscosity approaches 0 (see Lemma 8). One expects this problem to be easier than convergence in  $L^\infty(\mathbb{R}^2)$  since  $L^\infty(\mathbb{R}^2)$  embeds into  $\dot{B}_{\infty,\infty}^0(\mathbb{R}^2)$ ; however, the convergence rate in the  $\dot{B}_{\infty,\infty}^0$ -norm must be sufficiently fast to ensure that  $\|(S_n - S_{-n})(v_\nu - v)(t)\|_{L^\infty(\mathbb{R}^2)}$  converges to 0 as well, despite the extra factor of  $n$  on the right hand side of (1.2).

By reducing the problem of vanishing viscosity in the  $L^\infty$ -norm to that in the  $\dot{B}_{\infty,\infty}^0$ -norm, we eliminate some of the general difficulties one encounters when proving estimates for the fluid equations in  $L^\infty(\mathbb{R}^n)$ . For example, Calderon-Zygmund operators are not bounded on  $L^\infty(\mathbb{R}^n)$  but are bounded on  $\dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$  for all  $s \in \mathbb{R}$  (see [17]). We can then use the relation  $\nabla v = \nabla \nabla^\perp \Delta^{-1} \omega(v)$ , the boundedness of Calderon-Zygmund operators on  $\dot{B}_{\infty,\infty}^s$ , and Lemma 2 below to see that the  $\dot{B}_{\infty,\infty}^0$ -norm of  $v$  and the  $\dot{B}_{\infty,\infty}^{-1}$ -norm of  $\omega(v)$  are equivalent. This equivalence allows us to prove the main result by using the vorticity formulations of  $(NS)$  and  $(E)$  to show that the vorticities  $\omega(v_\nu)$  converge to  $\omega(v)$  in the  $\dot{B}_{\infty,\infty}^{-1}$ -norm as viscosity tends to 0.

The paper is organized as follows. In Section 2, we introduce the Littlewood-Paley theory, Bony's paraproduct decomposition, Besov spaces, and some useful technical lemmas. In Section 3, we discuss known properties of nondecaying solutions to  $(NS)$  and  $(E)$ . In Section 4 and Section 5, we state and prove the main result; we devote Section 5 entirely to showing that the vanishing viscosity limit holds in the  $\dot{B}_{\infty,\infty}^0$ -norm.

## 2. A FEW DEFINITIONS AND TECHNICAL LEMMAS

We first define the Littlewood-Paley operators (see [7] for further discussion of these operators). We let  $\varphi \in S(\mathbb{R}^n)$  satisfy  $\text{supp } \varphi \subset \{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ , and for every  $j \in \mathbb{Z}$  we let  $\varphi_j(\xi) = \varphi(2^{-j}\xi)$  (so  $\varphi_j(x) = 2^{jn}\varphi(2^jx)$ ). Observe that, if  $|j - j'| \geq 2$ , then  $\text{supp } \varphi_j \cap \text{supp } \varphi_{j'} = \emptyset$ . We define  $\psi_n \in S(\mathbb{R}^n)$  by the equality

$$\psi_n(\xi) = 1 - \sum_{j \geq n} \varphi_j(\xi)$$

for all  $\xi \in \mathbb{R}^n$ , and for  $f \in S'(\mathbb{R}^n)$  we define the operator  $S_n f$  by

$$S_n f = \check{\psi}_n * f.$$

In the following sections we will make frequent use of both the homogeneous and the inhomogeneous Littlewood-Paley operators. For  $f \in S'(\mathbb{R}^n)$  and  $j \in \mathbb{Z}$ , we define the homogeneous Littlewood-Paley operators  $\dot{\Delta}_j$  by

$$\dot{\Delta}_j f = \check{\varphi}_j * f,$$

and we define the inhomogeneous Littlewood-Paley operators by

$$\Delta_j f = \begin{cases} 0, & j < -1, \\ \check{\psi}_0 * f, & j = -1, \\ \check{\varphi}_j * f, & j > -1. \end{cases}$$

We remark that the operators  $\Delta_j$  and  $\dot{\Delta}_j$  coincide when  $j \geq 0$ , but differ when  $j \leq -1$ .

In the proof of the main theorem we use the paraproduct decomposition introduced by J.-M. Bony in [1]. We recall the definition of the paraproduct and remainder used in this decomposition.

**Definition.** Define the paraproduct of two functions  $f$  and  $g$  by

$$T_f g = \sum_{\substack{i,j \\ i \leq j-2}} \Delta_i f \Delta_j g = \sum_{j=1}^{\infty} S_{j-1} f \Delta_j g.$$

We use  $R(f, g)$  to denote the remainder.  $R(f, g)$  is given by the following bilinear operator:

$$R(f, g) = \sum_{\substack{i,j \\ |i-j| \leq 1}} \Delta_i f \Delta_j g.$$

Bony's decomposition then gives

$$fg = T_f g + T_g f + R(f, g).$$

We now define the homogeneous Besov spaces.

**Definition.** Let  $s \in \mathbb{R}$ ,  $(p, q) \in [1, \infty] \times [1, \infty)$ . We define the homogeneous Besov space  $\dot{B}_{p,q}^s(\mathbb{R}^n)$  to be the space of tempered distributions  $f$  on  $\mathbb{R}^n$  such that

$$\|f\|_{\dot{B}_{p,q}^s} := \left( \sum_{j \in \mathbb{Z}} 2^{jq_s} \|\dot{\Delta}_j f\|_{L^p}^q \right)^{\frac{1}{q}} < \infty.$$

When  $q = \infty$ , write

$$\|f\|_{\dot{B}_{p,\infty}^s} := \sup_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j f\|_{L^p}.$$

We also define the inhomogeneous Zygmund spaces.

**Definition.** The inhomogeneous Zygmund space  $C_*^s(\mathbb{R}^n)$  is the set of all tempered distributions  $f$  on  $\mathbb{R}^n$  such that

$$\|f\|_{C_*^s} := \sup_{j \geq -1} 2^{js} \|\Delta_j f\|_{L^\infty} < \infty.$$

It is well-known that  $C_*^s(\mathbb{R}^n)$  coincides with the classical Holder space  $C^s(\mathbb{R}^n)$  when  $s$  is not an integer and  $s > 0$ .

We will make frequent use of Bernstein's Lemma. We refer the reader to [3], chapter 2, for a proof of the lemma.

**Lemma 1.** (*Bernstein's Lemma*) *Let  $r_1$  and  $r_2$  satisfy  $0 < r_1 < r_2 < \infty$ , and let  $p$  and  $q$  satisfy  $1 \leq p \leq q \leq \infty$ . There exists a positive constant  $C$  such that for every integer  $k$ , if  $u$  belongs to  $L^p(\mathbb{R}^n)$ , and  $\text{supp } \hat{u} \subset B(0, r_1\lambda)$ , then*

$$(2.1) \quad \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^k \lambda^{k+n(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}.$$

Furthermore, if  $\text{supp } \hat{u} \subset C(0, r_1\lambda, r_2\lambda)$ , then

$$(2.2) \quad C^{-k} \lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^k \lambda^k \|u\|_{L^p}.$$

As a result of Bernstein's Lemma, we have the following lemma regarding the homogeneous Besov spaces.

**Lemma 2.** *Let  $u \in S'$ ,  $s \in \mathbb{R}$ , and  $p, q \in [1, \infty]$ . For any  $k \in \mathbb{Z}$  there exists a constant  $C_k$  such that whenever  $|\alpha| = k$ ,*

$$C^{-k} \|\partial^\alpha u\|_{\dot{B}_{p,q}^s} \leq \|u\|_{\dot{B}_{p,q}^{s+k}} \leq C^k \|\partial^\alpha u\|_{\dot{B}_{p,q}^s}.$$

We also make use of the following technical lemma. We refer the reader to [6] for a detailed proof.

**Lemma 3.** *Let  $v$  be a divergence-free vector field with vorticity  $\omega$ , and let  $v$  and  $\omega$  satisfy the relation  $\nabla v = \nabla \nabla^\perp \Delta^{-1} \omega$ . Then there exists an absolute constant  $C$  such that for all  $j \in \mathbb{Z}$ ,*

$$\|\dot{\Delta}_j \nabla v\|_{L^\infty} \leq C \|\dot{\Delta}_j \omega\|_{L^\infty}.$$

We will need a uniform bound in time on the  $L^\infty$ -norms of the vorticities corresponding to the solutions of (NS) and (E). For fixed  $\nu \geq 0$ , we have that

$$(2.3) \quad \|\omega_\nu(t)\|_{L^\infty} \leq \|\omega_\nu^0\|_{L^\infty}$$

for all  $t \geq 0$ . One can prove this bound by applying the maximum principle to the vorticity formulations of  $(NS)$  and  $(E)$ . We refer the reader to Lemma 3.1 of [11] for a detailed proof.

### 3. PROPERTIES OF NONDECAYING SOLUTIONS TO THE FLUID EQUATIONS

In this section, we summarize what is known about nondecaying solutions to  $(NS)$  and  $(E)$ . We begin with the mild solutions to  $(NS)$  established in [8]. By a mild solution to  $(NS)$ , we mean a solution  $v_\nu$  of the integral equation

$$(3.1) \quad v_\nu(t, x) = e^{t\nu\Delta}v_\nu^0 - \int_0^t e^{(t-s)\nu\Delta}\mathbf{P}(v_\nu \cdot \nabla v_\nu)(s)ds.$$

In (3.1),  $e^{\tau\nu\Delta}$  denotes convolution with the Gauss kernel; that is, for  $f \in S'$ ,  $e^{\tau\nu\Delta}f = G_{\tau\nu} * f$ , where  $G_{\tau\nu}(x) = \frac{1}{4\pi\tau\nu} \exp\{\frac{-|x|^2}{4\tau\nu}\}$ . Also,  $\mathbf{P}$  denotes the Helmholtz projection operator with  $ij$ -component given by  $\delta_{ij} + R_i R_j$ , where  $R_l = (-\Delta)^{-\frac{1}{2}}\partial_l$  is the Riesz operator. In [8], Giga, Inui, and Matsui prove the following result regarding mild solutions in  $\mathbb{R}^n$ ,  $n \geq 2$ .

**Theorem 1.** *Let  $BUC$  denote the space of bounded, uniformly continuous functions, and assume  $v_\nu^0$  belongs to  $BUC(\mathbb{R}^n)$  for fixed  $n \geq 2$ . There exists a  $T_0 > 0$  and a unique solution to (3.1) in the space  $C([0, T_0]; BUC(\mathbb{R}^n))$  with initial data  $v_\nu^0$ . Moreover, if we assume  $\operatorname{div} v_\nu^0 = 0$ , and if we define  $p_\nu(t) = \sum_{i,j=1}^2 R_i R_j v_{\nu i} v_{\nu j}(t)$  for each  $t \in [0, T_0]$ , then  $v_\nu$  and  $p_\nu$  belong to  $C^\infty([0, T_0] \times \mathbb{R}^n)$ , and  $(v_\nu, p_\nu)$  is the unique solution to  $(NS)$  with initial data  $v_\nu^0$ .*

*Remark 3.2.* In the above theorem, one can assume  $v_\nu^0 \in L^\infty(\mathbb{R}^n)$  and draw similar conclusions. Indeed, with this weaker assumption the theorem is still true as long as one replaces  $C([0, T_0]; BUC(\mathbb{R}^n))$  with  $C_w([0, T_0]; BUC(\mathbb{R}^n))$ , where  $C_w$  denotes the space of weakly continuous functions. For the main theorem of this paper, we assume that  $\omega^0$  is bounded on  $\mathbb{R}^2$ , which, by Lemma 4, implies that  $v_\nu^0$  belongs to  $C^\alpha(\mathbb{R}^2)$  for every  $\alpha < 1$ . Therefore, the statement of the theorem with  $v_\nu^0 \in BUC(\mathbb{R}^2)$  applies in our case.

*Remark 3.3.* Note that when the velocity  $v_\nu$  belongs to  $L^r(\mathbb{R}^n)$  for  $r < \infty$ , the pressure can be determined from  $v_\nu$  up to a constant, giving uniqueness of  $(v_\nu, \nabla p_\nu)$ . However, without any decay assumption on the velocity, uniqueness of  $\nabla p_\nu$  does not necessarily follow. As a result, in [8] Giga, Inui, and Matsui include the relation  $p_\nu(t) =$

$\sum_{i,j=1}^2 R_i R_j v_{\nu i} v_{\nu j}(t)$  in the statement of Theorem 1 to ensure uniqueness of  $(v_\nu, \nabla p_\nu)$  without decay of the velocity.

In [9], Giga, Matsui, and Sawada show that when  $n = 2$ , the solution to  $(NS)$  established in Theorem 1 can be extended to a global-in-time smooth solution. Moreover, in [11], Sawada and Taniuchi show that if  $v_\nu^0$  and  $\omega_\nu^0$  belong to  $L^\infty(\mathbb{R}^2)$ , then the following exponential estimate holds:

$$(3.4) \quad \|v_\nu(t)\|_{L^\infty} \leq C \|v_\nu^0\|_{L^\infty} e^{Ct \|\omega_\nu^0\|_{L^\infty}}.$$

For ideal incompressible fluids, Serfati proves the following existence and uniqueness result in [14].

**Theorem 2.** *Let  $v^0$  and  $\omega^0$  belong to  $L^\infty(\mathbb{R}^2)$ , and let  $c \in \mathbb{R}$ . For every  $T > 0$  there exists a unique solution  $(v, p)$  to  $(E)$  in the space  $L^\infty([0, T]; L^\infty(\mathbb{R}^2)) \times L^\infty([0, T]; C(\mathbb{R}^2))$  with  $\omega \in L^\infty([0, T]; L^\infty(\mathbb{R}^2))$ ,  $p(0) = c$ , and with  $\frac{p(t, x)}{|x|} \rightarrow 0$  as  $|x| \rightarrow \infty$ .*

The unique solution to  $(E)$  given in Theorem 2 satisfies an integral representation analogous to that for mild solutions to  $(NS)$ . Specifically, the Serfati solution satisfies the equation

$$(3.5) \quad v(t, x) = v(0, x) - \int_0^t \mathbf{P}(v \cdot \nabla v)(s) ds.$$

Serfati also establishes an estimate analogous to (3.4) for the Euler equations. He proves the bound

$$(3.6) \quad \|v(t)\|_{L^\infty} \leq C \|v^0\|_{L^\infty} e^{Ct \|\omega^0\|_{L^\infty}}.$$

Before we state the main theorem of the paper, we prove a result giving Holder regularity of the velocity vector fields  $v_\nu$  and  $v$  which solve  $(NS)$  and  $(E)$ , respectively, when initial velocity and initial vorticity belong to  $L^\infty(\mathbb{R}^2)$ . Specifically, we prove that under these assumptions on the initial data,  $v_\nu$  and  $v$  belong to the Zygmund space  $C_*^1$ . We prove the lemma only for  $v$ . The proof for  $v_\nu$  is identical.

**Lemma 4.** *Let  $(v, p)$  be the unique solution to  $(E)$  given by Theorem 2. Then the following estimate holds:*

$$\|v(t)\|_{C_*^1} \leq C \|v^0\|_{L^\infty} e^{Ct \|\omega^0\|_{L^\infty}} + C \|\omega^0\|_{L^\infty}.$$

*Proof.* Write

$$\|v(t)\|_{C_*^1} \leq C \|S_0 v(t)\|_{L^\infty} + \sup_{j \geq 0} 2^j \|\dot{\Delta}_j v(t)\|_{L^\infty}.$$

We first use Young's inequality to bound the low frequency term by  $C\|v(t)\|_{L^\infty}$ . We then apply the bound given in (3.6). For the high frequency terms, we apply Bernstein's Lemma and the uniform estimate on the  $L^\infty$ -norm of the vorticity given in (2.3) to bound the supremum by  $C\|\omega^0\|_{L^\infty}$ . This completes the proof.  $\square$

#### 4. STATEMENT AND PROOF OF THE MAIN RESULT

We are now prepared to state the main theorem.

**Theorem 3.** *Let  $(v_\nu, p_\nu)$  and  $(v, p)$  be the unique solutions to  $(NS)$  and  $(E)$ , respectively, given by Theorems 1 and 2, with the same bounded initial velocity  $v^0$  and bounded initial vorticity  $\omega^0$ . There exist constants  $C$  and  $C_1$ , depending only on  $\|v^0\|_{L^\infty}$  and  $\|\omega^0\|_{L^\infty}$ , such that the following estimate holds for any fixed  $\alpha \in (0, 1)$  and for  $\nu$  sufficiently small:*

$$(4.1) \quad \|v_\nu - v\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^2))} \leq C(T+1)e^{C_1 T}(\nu)^{\frac{\alpha}{2}} \{e^{C(e^{C_1 T} - 1)}\}^{-\frac{1}{2} \log_2 \nu}.$$

*Remark 4.2.* The constant  $C_1$  in (4.1) is equal to  $A\|\omega^0\|_{L^\infty}$ , where  $A$  is an absolute constant. Therefore, one can conclude from Theorem 3 that the vanishing viscosity limit holds on a time interval with length inversely proportional to the size of  $\|\omega^0\|_{L^\infty}$ . Moreover, the smallness of  $\nu$  required to conclude (4.1) depends on  $T$  (and therefore on  $\|\omega^0\|_{L^\infty}$ ). Specifically, larger  $T$  requires smaller  $\nu$  for (4.1) to hold (see Remark 4.10).

*Remark 4.3.* In what follows, to simplify notation we refer to the solution of  $(NS)$  by  $v_\nu$  rather than  $(v_\nu, p_\nu)$ ; similarly, for the solution of  $(E)$  we write  $v$  in place of  $(v, p)$ . It is important to keep in mind, however, that the solution to  $(NS)$  is actually the pair  $(v_\nu, p_\nu)$  given to us in Theorem 1 and that the solution to  $(E)$  is the pair  $(v, p)$  given in Theorem 2.

*Proof.* Let  $v$  be the unique solution to  $(E)$  with bounded initial velocity and vorticity. In what follows, we let  $v_n = S_n v$ , and  $\omega_n = S_n \omega(v)$ . We have the following inequality:

$$(4.4) \quad \begin{aligned} \|v_\nu - v\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^2))} &\leq \|S_{-n}(v_\nu - v)\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^2))} \\ &+ \|(Id - S_{-n})(v_\nu - v_n)\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^2))} \\ &+ \|(Id - S_{-n})(v_n - v)\|_{L^\infty([0, T]; L^\infty(\mathbb{R}^2))}. \end{aligned}$$

We will estimate each of the three terms on the right hand side of the inequality in (4.4). We begin with the third term, since it is the easiest



to handle. We use the definition of  $v_n$ , Bernstein's Lemma, Lemma 3, and (2.3) to obtain the inequality

$$(4.5) \quad \|(Id - S_{-n})(v_n - v)\|_{L^\infty([0,T];L^\infty(\mathbb{R}^2))} \leq C2^{-n}\|\omega^0\|_{L^\infty}.$$

To bound the first term on the right hand side of (4.4), we will use the integral representation of the  $L^\infty(\mathbb{R}^2)$  solution of (NS) given in (3.1), as well as the integral representation of the Serfati solution to (E) given in (3.5). We prove the following proposition.

**Proposition 5.** *Let  $v_\nu$  and  $v$  be solutions to (NS) and (E), respectively, satisfying the assumptions of Theorem 3. Then for any  $\alpha \in (0, 1)$  and for any  $\delta > 0$ ,*

$$\|S_{-n}(v_\nu - v)(t)\|_{L^\infty} \leq Ce^{C_1 t} \left\{ 2^{-n} + \delta^\alpha + \exp\left(-\frac{\delta^2}{4\nu t}\right) \right\},$$

where  $C$  and  $C_1$  depend only on  $\|v^0\|_{L^\infty}$  and  $\|\omega^0\|_{L^\infty}$ .

*Proof.* To prove Proposition 5, we apply the operator  $S_{-n}$  to (3.1) and (3.5), we subtract the modification of (3.5) from that of (3.1), and we take the  $L^\infty$ -norm to get

$$\begin{aligned} \|S_{-n}(v_\nu - v)(t)\|_{L^\infty} &\leq \|S_{-n}(e^{t\nu\Delta}v^0 - v^0)\|_{L^\infty} \\ &+ \int_0^t \|S_{-n}e^{(t-s)\nu\Delta}\mathbf{P}(v_\nu \cdot \nabla v_\nu)(s)\|_{L^\infty} ds + \int_0^t \|S_{-n}\mathbf{P}(v \cdot \nabla v)(s)\|_{L^\infty} ds. \end{aligned}$$

We first estimate  $\|S_{-n}e^{(t-s)\nu\Delta}\mathbf{P}(v_\nu \cdot \nabla v_\nu)(s)\|_{L^\infty}$ . We follow an argument of Taniuchi in [16] which uses the boundedness of the Helmholtz projection operator on the Hardy space  $\mathcal{H}^1(\mathbb{R}^2)$ . We have

$$(4.6) \quad \begin{aligned} &\|S_{-n}e^{(t-s)\nu\Delta}\mathbf{P}(v_\nu \cdot \nabla v_\nu)(s)\|_{L^\infty} \\ &\leq C2^{-n}\|v_\nu(s)\|_{L^\infty}^2, \end{aligned}$$

after applying Holder's inequality and the series of inequalities given by  $\|\nabla\mathbf{P}\psi_{-n}\|_{L^1(\mathbb{R}^2)} \leq \|\nabla\mathbf{P}\psi_{-n}\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq \|\nabla\psi_{-n}\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C2^{-n}$ . Proofs of (4.6) can also be found in [11] and [12].

Similarly, we have

$$(4.7) \quad \|S_{-n}\mathbf{P}(v \cdot \nabla v)(s)\|_{L^\infty} \leq C2^{-n}\|v(s)\|_{L^\infty}^2.$$

Using (3.4) and (3.6), we can bound  $\|v_\nu(s)\|_{L^\infty}$  and  $\|v(s)\|_{L^\infty}$  with a constant depending on  $t$ ,  $\|v^0\|_{L^\infty}$ , and  $\|\omega^0\|_{L^\infty}$ .

It remains to estimate  $\|S_{-n}(e^{t\nu\Delta}v^0 - v^0)\|_{L^\infty}$ . To bound this difference, we use the following lemma.

**Lemma 6.** *Let  $u$  belong to the Holder space  $C^\alpha(\mathbb{R}^2)$ . Then for any fixed  $\delta > 0$ , the following estimate holds:*

$$\|e^{t\nu\Delta}u - u\|_{L^\infty} \leq C\|u\|_{L^\infty} \frac{1}{\exp(\frac{\delta^2}{4\nu t})} + \delta^\alpha\|u\|_{C^\alpha}.$$

*Proof.* We fix  $\delta > 0$  and write

$$(4.8) \quad \begin{aligned} |e^{t\nu\Delta}u(x) - u(x)| &\leq \left| \int_{|y|\leq\delta} G_{t\nu}(y)u(x-y)dy - u(x) \right| \\ &\quad + \left| \int_{|y|>\delta} G_{t\nu}(y)u(x-y)dy \right|. \end{aligned}$$

After integrating, we can bound the second term on the right hand side of (4.8) by

$$(4.9) \quad C\|u\|_{L^\infty} \exp\left(-\frac{\delta^2}{4\nu t}\right).$$

To estimate the first term on the right hand side of (4.8), we use the property that the integral over  $\mathbb{R}^2$  of the Gauss kernel is equal to one to write

$$\begin{aligned} &\left| \int_{|y|\leq\delta} G_{t\nu}(y)u(x-y)dy - u(x) \right| \\ &\leq \left| \int_{|y|\leq\delta} G_{t\nu}(y)u(x-y)dy - \int_{\mathbb{R}^2} G_{t\nu}(y)u(x)dy \right| \\ &\leq \int_{|y|\leq\delta} G_{t\nu}(y)|u(x-y) - u(x)|dy + \int_{|y|>\delta} G_{t\nu}(y)|u(x)|dy \\ &\leq \sup_{|y|\leq\delta} |u(x-y) - u(x)| + C\|u\|_{L^\infty} \exp\left(-\frac{\delta^2}{4\nu t}\right), \end{aligned}$$

where we utilized the bound given in (4.9) on the second term. We now use the membership of  $u$  to  $C^\alpha(\mathbb{R}^2)$  for every  $\alpha < 1$  to bound the first term by  $\delta^\alpha\|u\|_{C^\alpha}$ . This completes the proof.  $\square$

We apply Lemma 6 with  $u = v^0$ , and we combine the resulting estimate with (4.6), (4.7), and Lemma 4 to complete the proof of Proposition 5.  $\square$

*Remark 4.10.* If we let  $\nu = 2^{-2n}$  and  $\delta = 2^{-n\alpha}$ , then, since  $\alpha \in (0, 1)$  is arbitrary, the estimate in Proposition 5 reduces to

$$(4.11) \quad \|S_{-n}(v_\nu - v)(t)\|_{L^\infty} \leq Ce^{C_1 t} 2^{-n\alpha}$$

for  $n \geq N$ , with  $N$  sufficiently large. We remark here that, because of the appearance of  $t$  on the right hand side of the inequality in Lemma 6,

the size of  $N$  necessary to make (4.11) hold depends on  $t$ . In particular, larger  $t$  requires larger  $N$ . In Theorem 3, we work on a finite time interval  $[0, T]$ . Therefore, we can choose  $N$  large enough so that (4.11) holds for all  $t \in [0, T]$ .

It remains to bound the second term on the right hand side of (4.4), given by  $\|(Id - S_{-n})(v_\nu - v_n)(t)\|_{L^\infty}$ . We prove the following estimate.

**Proposition 7.** *Let  $v_\nu$  and  $v$  be solutions to (NS) and (E), respectively, satisfying the properties of Theorem 3. Then there exist constants  $C$  and  $C_1$ , depending only on the initial data, such that the following estimate holds for any fixed  $\alpha \in (0, 1)$  and for sufficiently large  $n$ :*

$$(4.12) \quad \|(Id - S_{-n})(v_\nu - v_n)(t)\|_{L^\infty} \leq C(t+1)e^{C_1 t} 2^{-n\alpha} e^{Cn(e^{C_1 t} - 1)}.$$

*Remark 4.13.* In the proof of Proposition 7, we let  $\nu = 2^{-2n}$  as in Remark 4.10. Therefore, the dependence of the right hand side of (4.12) on  $\nu$  is hidden in its dependence on  $n$ .

*Proof.* We begin with the series of inequalities

$$(4.14) \quad \begin{aligned} \|(Id - S_{-n})(v_\nu - v_n)(t)\|_{L^\infty} &\leq \|(Id - S_n)(v_\nu - v_n)(t)\|_{L^\infty} \\ &+ \|(S_n - S_{-n})(v_\nu - v_n)(t)\|_{L^\infty} \\ &\leq C2^{-n}\|\omega^0\|_{L^\infty} + Cn\|(v_\nu - v_n)(t)\|_{\dot{B}_{\infty,\infty}^0}. \end{aligned}$$

The second inequality follows from an application of Bernstein's Lemma, Lemma 3, and the uniform bound on the vorticity given in (2.3).

To complete the proof of Theorem 3, we must show that the vanishing viscosity limit holds in the  $\dot{B}_{\infty,\infty}^0$ -norm, and that the convergence rate is sufficiently fast to counter the growth of the factor of  $n$  in the last line of (4.14). We need the following lemma, whose proof we postpone until the next section.

**Lemma 8.** *Let  $v_\nu$  and  $v$  be solutions to (NS) and (E), respectively, satisfying the properties of Theorem 3. Then there exist constants  $C$  and  $C_1$ , depending only on the initial data, such that the following estimate holds for any fixed  $\alpha \in (0, 1)$ :*

$$\begin{aligned} \|(v_\nu - v_n)(t)\|_{\dot{B}_{\infty,\infty}^0} &\leq C(t+1)e^{C_1 t} 2^{-n\alpha} \\ &+ \int_0^t Cne^{C_1 s} \|(v_\nu - v_n)(s)\|_{\dot{B}_{\infty,\infty}^0} ds. \end{aligned}$$

Assuming this lemma holds, we can apply Gronwall's Lemma and integrate in time to conclude that

$$\|(v_\nu - v_n)(t)\|_{\dot{B}_{\infty,\infty}^0} \leq C(t+1)e^{C_1 t} 2^{-n\alpha} e^{Cn(e^{C_1 t} - 1)}.$$

Since  $\alpha \in (0, 1)$  is arbitrary, we can write

$$n\|(v_\nu - v_n)(t)\|_{\dot{B}_{\infty,\infty}^0} \leq C(t+1)e^{C_1 t} 2^{-n\alpha} e^{Cn(e^{C_1 t}-1)}$$

for sufficiently large  $n$ . Combining this estimate with the estimate given in (4.14) yields Proposition 7.  $\square$

To complete the proof of Theorem 3, we combine (4.4), Remark 4.10, Proposition 7, and (4.5) to get the following estimate for large  $n$ :

$$\|(v_\nu - v)(t)\|_{L^\infty} \leq C(t+1)e^{C_1 t} 2^{-n\alpha} e^{Cn(e^{C_1 t}-1)}.$$

Using the equality  $n = -\frac{1}{2} \log_2 \nu$ , we obtain (4.1). This completes the proof of Theorem 3. We devote the next section to the proof of Lemma 8.  $\square$

## 5. PROOF OF LEMMA 8

We begin with some notation. For the proof of Lemma 8, we let  $\bar{\omega}_n = \omega_\nu - \omega_n$  and  $\bar{v}_n = v_\nu - v_n$ .

To prove the lemma, we localize the frequencies of the vorticity formulations of  $(E)$  and  $(NS)$ , and we consider the difference of the two resulting equations. After localizing the frequency of the vorticity formulation of  $(E)$ , we see that  $\dot{\Delta}_j \omega_n$  satisfies the following equation:

$$(5.1) \quad \partial_t \dot{\Delta}_j \omega_n + v_n \cdot \nabla \dot{\Delta}_j \omega_n + [\dot{\Delta}_j, v_n \cdot \nabla] \omega_n = \nabla \cdot \dot{\Delta}_j \tau_n(v, \omega),$$

where

$$\tau_n(v, \omega) = r_n(v, \omega) - (v - v_n)(\omega - \omega_n)$$

and

$$r_n(v, \omega) = \int \check{\psi}(y)(v(x - 2^{-n}y) - v(x))(\omega(x - 2^{-n}y) - \omega(x))dy.$$

This equation is utilized by Constantin and Wu in [5] and by Constantin, E, and Titi in a proof of Onsager's conjecture in [4].

If  $v_\nu$  is a solution to  $(NS)$ , we can localize the frequency of the vorticity formulation of  $(NS)$  to see that  $\dot{\Delta}_j \omega_\nu$  satisfies

$$(5.2) \quad \partial_t \dot{\Delta}_j \omega_\nu + v_\nu \cdot \nabla \dot{\Delta}_j \omega_\nu + [\dot{\Delta}_j, v_\nu \cdot \nabla] \omega_\nu = \nu \Delta \dot{\Delta}_j \omega_\nu.$$

We subtract (5.1) from (5.2). This yields

$$(5.3) \quad \begin{aligned} \partial_t \dot{\Delta}_j \bar{\omega}_n + v_n \cdot \nabla \dot{\Delta}_j \bar{\omega}_n - \nu \Delta \dot{\Delta}_j \bar{\omega}_n &= -\dot{\Delta}_j (\bar{v}_n \cdot \nabla \omega_\nu) \\ &+ \nu \Delta \dot{\Delta}_j \omega_n - \nabla \cdot \dot{\Delta}_j \tau_n(v, \omega) - [\dot{\Delta}_j, v_n \cdot \nabla] \bar{\omega}_n. \end{aligned}$$

We observe that  $v_n$  is a divergence-free Lipschitz vector field and apply the following lemma, which is proved in [10].

**Lemma 9.** *Let  $p \in [1, \infty]$ , and let  $u$  be a divergence-free vector field belonging to  $L^1_{loc}(\mathbb{R}^+; Lip(\mathbb{R}^d))$ . Moreover, assume the function  $f$  belongs to  $L^1_{loc}(\mathbb{R}^+; L^p(\mathbb{R}^d))$  and the function  $a^0$  belongs to  $L^p(\mathbb{R}^d)$ . Then any solution  $a$  to the problem*

$$\begin{cases} \partial_t a + u \cdot \nabla a - \nu \Delta a = f, \\ a|_{t=0} = a^0 \end{cases}$$

satisfies the following estimate:

$$\|a(t)\|_{L^p} \leq \|a^0\|_{L^p} + \int_0^t \|f(s)\|_{L^p} ds.$$

An application of Lemma 9 to (5.3) yields

$$\begin{aligned} \|\dot{\Delta}_j \bar{\omega}_n(t)\|_{L^\infty} &\leq \|\dot{\Delta}_j \bar{\omega}_n^0\|_{L^\infty} + C \int_0^t \|(-\dot{\Delta}_j(\bar{v}_n \cdot \nabla \omega_\nu) + \nu \Delta \dot{\Delta}_j \omega_n \\ &\quad - \nabla \cdot \dot{\Delta}_j \tau_n(v, \omega) - [\dot{\Delta}_j, v_n \cdot \nabla] \bar{\omega}_n)(s)\|_{L^\infty} ds. \end{aligned}$$

Multiplying through by  $2^{-j}$  and taking the supremum over  $j \in \mathbb{Z}$ , we obtain the inequality

(5.4)

$$\begin{aligned} \|\bar{\omega}_n(t)\|_{\dot{B}^{-1}_{\infty, \infty}} &\leq \|\bar{\omega}_n^0\|_{\dot{B}^{-1}_{\infty, \infty}} + C \int_0^t \{ \|\bar{v}_n \omega_\nu(s)\|_{\dot{B}^0_{\infty, \infty}} + \|\nu \nabla \omega_n(s)\|_{\dot{B}^0_{\infty, \infty}} \\ &\quad + \|\tau_n(v, \omega)(s)\|_{\dot{B}^0_{\infty, \infty}} + \sup_{j \in \mathbb{Z}} 2^{-j} \|[\dot{\Delta}_j, v_n \cdot \nabla] \bar{\omega}_n(s)\|_{L^\infty} \} ds, \end{aligned}$$

where we repeatedly used the divergence-free condition on  $v$  and Lemma 2.

To complete the proof of Lemma 8, we will estimate each term on the right hand side of (5.4), apply Gronwall's inequality, and utilize the equivalence of  $\|\bar{v}_n(t)\|_{\dot{B}^0_{\infty, \infty}}$  and  $\|\bar{\omega}_n(t)\|_{\dot{B}^{-1}_{\infty, \infty}}$ . The main difficulty in the remainder of the proof will be the estimate of the commutator term, which is given in Lemma 10 below. The estimates for the other terms on the right hand side of (5.4) are relatively straightforward and require nothing more than basic properties of Littlewood-Paley operators and the space  $\dot{B}^0_{\infty, \infty}$ . We therefore estimate these terms first, followed by the commutator term.

We begin by estimating  $\|\bar{\omega}_n^0\|_{\dot{B}^{-1}_{\infty, \infty}}$ . Using the definition of  $\bar{\omega}_n$  as well as the definition of the Besov space  $\dot{B}^{-1}_{\infty, \infty}$ , we see that

$$\|\bar{\omega}_n^0\|_{\dot{B}^{-1}_{\infty, \infty}} \leq \sup_{j \geq n} 2^{-j} \|\dot{\Delta}_j(\omega^0 - S_n \omega^0)\|_{L^\infty} \leq C 2^{-n} \|\omega^0\|_{L^\infty}.$$

To bound  $\|\bar{v}_n \omega_\nu(s)\|_{\dot{B}^0_{\infty, \infty}}$ , we observe that  $L^\infty$  is continuously embedded in  $\dot{B}^0_{\infty, \infty}$  and that the  $L^\infty$ -norm of vorticity is uniformly bounded

in time by (2.3) to write

$$(5.5) \quad \|\bar{v}_n \omega_\nu(s)\|_{\dot{B}_{\infty,\infty}^0} \leq C \|\bar{v}_n(s)\|_{L^\infty} \|\omega^0\|_{L^\infty}.$$

To estimate the  $L^\infty$ -norm of  $\bar{v}_n(s)$ , we use Remark 4.10, Bernstein's Lemma, (2.3), and the definition of the  $\dot{B}_{\infty,\infty}^0$ -norm to write

$$(5.6) \quad \begin{aligned} \|\bar{v}_n(s)\|_{L^\infty} &\leq \|S_{-n}(v_\nu - v)(s)\|_{L^\infty} + \|S_{-n}(v - v_n)(s)\|_{L^\infty} \\ &\quad + \|(S_n - S_{-n})(v_\nu - v_n)(s)\|_{L^\infty} + \|(Id - S_n)(v_\nu - v_n)(s)\|_{L^\infty} \\ &\leq C e^{C_1 s} 2^{-n\alpha} + C n \|(v_\nu - v_n)(s)\|_{\dot{B}_{\infty,\infty}^0} \end{aligned}$$

for fixed  $\alpha \in (0, 1)$ . Combining (5.5) and (5.6) gives

$$(5.7) \quad \|\bar{v}_n \omega_\nu(s)\|_{\dot{B}_{\infty,\infty}^0} \leq C e^{C_1 s} 2^{-n\alpha} + C n \|(v_\nu - v_n)(s)\|_{\dot{B}_{\infty,\infty}^0}.$$

To bound  $\nu \|\nabla \omega_n(s)\|_{\dot{B}_{\infty,\infty}^0}$ , we again use Bernstein's Lemma, the definition of  $\omega_n$ , and (2.3) to conclude that

$$\nu \|\nabla \omega_n(s)\|_{\dot{B}_{\infty,\infty}^0} \leq C \nu 2^n \|\omega^0\|_{L^\infty}.$$

If we let  $\nu = 2^{-2n}$  as in Remark 4.10, we obtain the inequality

$$\nu \|\nabla \omega_n(s)\|_{\dot{B}_{\infty,\infty}^0} \leq C 2^{-n} \|\omega^0\|_{L^\infty}.$$

Finally, we estimate  $\|\tau_n(v, \omega)(s)\|_{\dot{B}_{\infty,\infty}^0}$ . We begin by estimating  $\|(v - v_n)(\omega - \omega_n)(s)\|_{\dot{B}_{\infty,\infty}^0}$ . We again use the embedding  $L^\infty \hookrightarrow \dot{B}_{\infty,\infty}^0$ , Bernstein's Lemma, and (2.3) to write

$$(5.8) \quad \begin{aligned} \|(v - v_n)(\omega - \omega_n)(s)\|_{\dot{B}_{\infty,\infty}^0} &\leq \|(v - v_n)(\omega - \omega_n)(s)\|_{L^\infty} \\ &\leq C \|\omega^0\|_{L^\infty} 2^{-n} \|\omega^0\|_{L^\infty}. \end{aligned}$$

In order to bound  $\|r_n(v, \omega)(s)\|_{\dot{B}_{\infty,\infty}^0}$ , we use the membership of  $v$  to  $C^\alpha(\mathbb{R}^2)$  for any  $\alpha \in (0, 1)$  to write

$$(5.9) \quad |v(s, x - 2^{-n}y) - v(s, x)| \leq C 2^{-n\alpha} |y|^\alpha \|v(s)\|_{C^\alpha}.$$

Since  $|y|^\alpha |\tilde{\psi}(y)|$  is integrable, we can apply (5.9) and Holder's inequality to conclude that

$$(5.10) \quad \|r_n(v, \omega)(s)\|_{\dot{B}_{\infty,\infty}^0} \leq C 2^{-n\alpha} \|\omega^0\|_{L^\infty} \|v(s)\|_{C^\alpha} \leq C e^{C_1 s} 2^{-n\alpha},$$

where we used Lemma 4 to get the last inequality. Here the constants  $C$  and  $C_1$  depend only on  $\|v^0\|_{L^\infty}$  and  $\|\omega^0\|_{L^\infty}$ . Combining (5.8) and (5.10) yields

$$\|\tau_n(v, \omega)(s)\|_{\dot{B}_{\infty,\infty}^0} \leq C e^{C_1 s} 2^{-n\alpha}.$$

It remains to bound the commutator term on the right hand side of (5.4). We prove the following lemma.

**Lemma 10.** *Let  $v$  be a solution to (E) with vorticity  $\omega = \omega(v)$ , and assume  $v$  and  $\omega$  satisfy the conditions of Theorem 3. Then the following commutator estimate holds:*

$$(5.11) \quad \sup_{j \in \mathbb{Z}} 2^{-j} \|[\dot{\Delta}_j, v_n \cdot \nabla] \bar{\omega}_n(s)\|_{L^\infty} \leq C e^{C_1 s} 2^{-n\alpha} + C e^{C_1 s} n \|\bar{\omega}_n(s)\|_{\dot{B}_{\infty, \infty}^{-1}},$$

where  $C$  and  $C_1$  depend only on  $\|v^0\|_{L^\infty}$  and  $\|\omega^0\|_{L^\infty}$ .

*Proof.* We first use Bony's paraproduct decomposition to write

$$(5.12) \quad \begin{aligned} [\dot{\Delta}_j, v_n \cdot \nabla] \bar{\omega}_n &= \sum_{m=1}^2 [\dot{\Delta}_j, T_{v_n^m} \partial_m] \bar{\omega}_n \\ &\quad + [\dot{\Delta}_j, T_{\partial_m \cdot v_n^m}] \bar{\omega}_n + [\dot{\Delta}_j, \partial_m R(v_n^m, \cdot)] \bar{\omega}_n. \end{aligned}$$

To bound the  $L^\infty$ -norm of the first term on the right hand side of (5.12), we consider the cases  $j < 0$  and  $j \geq 0$  separately. For  $j \geq 0$ , we use the definition of the paraproduct and properties of the partition of unity to establish the following equality:

$$(5.13) \quad [\dot{\Delta}_j, T_{v_n} \partial_m] = \sum_{j'=\max\{1, j-4\}}^{j+4} [\dot{\Delta}_j, S_{j'-1}(v_n)] \Delta_{j'} \partial_m.$$

We then express the operator  $\dot{\Delta}_j$  as a convolution with  $\check{\varphi}_j$ , write out the commutator on the right hand side of (5.13), and change variables. This yields

$$(5.14) \quad \begin{aligned} \|[\dot{\Delta}_j, T_{v_n} \partial_m] \bar{\omega}_n\|_{L^\infty} &\leq \sum_{j': |j-j'| \leq 4} \left\| \int \check{\varphi}(y) (S_{j'-1} v_n(x - 2^{-j} y) \right. \\ &\quad \left. - S_{j'-1} v_n(x)) \Delta_{j'} \partial_m \bar{\omega}_n(x - 2^{-j} y) dy \right\|_{L^\infty} \\ &\leq \sum_{j': |j-j'| \leq 4} 2^{j'-j} \|S_{j'-1} \nabla v_n\|_{L^\infty} \|\Delta_{j'} \bar{\omega}_n\|_{L^\infty} \int |\check{\varphi}(y)| |y| dy, \end{aligned}$$

where we used Bernstein's Lemma to get the last inequality. To complete the argument for this term, we must estimate the growth of  $\|\nabla v_n\|_{L^\infty}$  with  $n$ . We break  $\nabla v_n$  into low and high frequencies and recall the definition of  $v_n$  to write

$$(5.15) \quad \|\nabla v_n\|_{L^\infty} \leq \|\Delta_{-1} \nabla v_n\|_{L^\infty} + \sum_{k=0}^n \|\Delta_k \nabla v_n\|_{L^\infty}.$$

For the low frequency term, we bound  $\|\Delta_{-1}\nabla v_n\|_{L^\infty}$  with  $\|v\|_{L^\infty}$  and apply (3.6). For the high frequencies, we apply Lemma 3 and (2.3). This yields

$$(5.16) \quad \|\nabla v_n(s)\|_{L^\infty} \leq Cne^{C_1s},$$

where  $C$  and  $C_1$  depend only on  $\|v^0\|_{L^\infty}$  and  $\|\omega^0\|_{L^\infty}$ . Plugging this bound into (5.14), multiplying (5.14) by  $2^{-j}$ , and taking the supremum over  $j \geq 0$  gives

$$\sup_{j \geq 0} 2^{-j} \|[\dot{\Delta}_j, T_{v_n} \partial_m] \bar{\omega}_n(s)\|_{L^\infty} \leq Cne^{C_1s} \|\bar{\omega}_n(s)\|_{\dot{B}_{\infty,\infty}^{-1}}.$$

For the case  $j < 0$ , we apply a different strategy. We first reintroduce the sum over  $m$  and expand the commutator. We then use the properties of our partition of unity, the assumption that  $j < 0$ , and the divergence-free assumption on  $v$  to establish the following series of inequalities:

$$(5.17) \quad \begin{aligned} \sum_{m=1}^2 \|[\dot{\Delta}_j, T_{v_n} \partial_m] \bar{\omega}_n\|_{L^\infty} &\leq \sum_{m=1}^2 \sum_{j'=1}^2 \|\partial_m \dot{\Delta}_j (S_{j'-1} v_n \Delta_{j'} \bar{\omega}_n)\|_{L^\infty} \\ &+ \sum_{m=1}^2 \sum_{j'=1}^2 \|S_{j'-1} v_n \Delta_{j'} \dot{\Delta}_j \partial_m \bar{\omega}_n\|_{L^\infty} \\ &\leq C \sum_{j'=1}^2 2^j \|v\|_{L^\infty} \|\Delta_{j'} \bar{\omega}_n\|_{L^\infty}, \end{aligned}$$

where we applied Bernstein's Lemma to get the factor of  $2^j$  in the last inequality. We bound  $\|\Delta_{j'} \bar{\omega}_n\|_{L^\infty}$  with  $2^{j'} \|\Delta_{j'} \bar{v}_n\|_{L^\infty}$ , again by Bernstein's Lemma, we multiply (5.17) by  $2^{-j}$ , and we take the supremum over  $j < 0$ . This gives

$$\sup_{j < 0} 2^{-j} \sum_{m=1}^2 \|[\dot{\Delta}_j, T_{v_n} \partial_m] \bar{\omega}_n\|_{L^\infty} \leq C \|v\|_{L^\infty} \|\bar{v}_n\|_{L^\infty}.$$

We now bound  $\|v\|_{L^\infty}$  using (3.6), and we bound  $\|\bar{v}_n\|_{L^\infty}$  as in (5.6). We obtain the desired estimate:

$$\sup_{j < 0} \sum_{m=1}^2 \|[\dot{\Delta}_j, T_{v_n} \partial_m] \bar{\omega}_n(s)\|_{L^\infty} \leq Ce^{C_1s} 2^{-n\alpha} + Cne^{C_1s} \|\bar{\omega}_n(s)\|_{\dot{B}_{\infty,\infty}^{-1}}.$$

We now estimate the  $L^\infty$ -norm of  $[\dot{\Delta}_j, T_{\partial_m \cdot v_n}] \bar{\omega}_n$ . We write out the commutator and estimate the  $L^\infty$ -norm of  $\dot{\Delta}_j (T_{\partial_m \bar{\omega}_n} v_n)$  and  $T_{\partial_m \dot{\Delta}_j \bar{\omega}_n} v_n$



separately. By the definition of the paraproduct and by properties of our partition of unity, we have

$$(5.18) \quad \begin{aligned} \|T_{\partial_m \dot{\Delta}_j \bar{\omega}_n} v_n\|_{L^\infty} &= \left\| \sum_{l \geq 1} S_{l-1} \partial_m \dot{\Delta}_j \bar{\omega}_n \Delta_l v_n \right\|_{L^\infty} \\ &\leq \sum_{l=\max\{1, j\}}^{\infty} \|S_{l-1} \partial_m \dot{\Delta}_j \bar{\omega}_n \Delta_l v_n\|_{L^\infty} \leq C \|\dot{\Delta}_j \bar{\omega}_n\|_{L^\infty} \|\nabla v_n\|_{L^\infty}, \end{aligned}$$

where we applied Bernstein's Lemma and took the sum to get the second inequality. We bound  $\|\nabla v_n\|_{L^\infty}$  as in (5.16), we multiply (5.18) by  $2^{-j}$ , and we take the supremum over  $j \in \mathbb{Z}$ . This yields

$$\sup_{j \in \mathbb{Z}} 2^{-j} \|T_{\partial_m \dot{\Delta}_j \bar{\omega}_n} v_n(s)\|_{L^\infty} \leq C n e^{C_1 s} \|\bar{\omega}_n(s)\|_{\dot{B}_{\infty, \infty}^{-1}}.$$

Moreover, since the Fourier transform of  $S_{l-1} \partial_m \bar{\omega}_n \Delta_l v_n$  has support in an annulus with inner and outer radius of order  $2^l$ , we have for  $j \geq 0$

$$(5.19) \quad \begin{aligned} \|\dot{\Delta}_j (T_{\partial_m \bar{\omega}_n} v_n)\|_{L^\infty} &= \|\dot{\Delta}_j \left( \sum_{l \geq 1} S_{l-1} \partial_m \bar{\omega}_n \Delta_l v_n \right)\|_{L^\infty} \\ &\leq \sum_{l=\max\{1, j-4\}}^{j+4} \sum_{k \leq l} 2^{2k} 2^{-l} \|\Delta_k \bar{v}_n\|_{L^\infty} \|\Delta_l \nabla v_n\|_{L^\infty} \leq C 2^j \|\bar{v}_n\|_{L^\infty} \|\omega^0\|_{L^\infty}, \end{aligned}$$

where we used Bernstein's Lemma to get the first inequality, and we used Lemma 3 and (2.3) to get the second inequality. For the case  $j < 0$ ,  $\|\dot{\Delta}_j (T_{\partial_m \bar{\omega}_n} v_n)\|_{L^\infty}$  is identically 0. Therefore (5.19) still holds. We bound  $\|\bar{v}_n\|_{L^\infty}$  as in (5.6), we multiply (5.19) by  $2^{-j}$ , and we take the supremum over  $j \in \mathbb{Z}$ , which yields

$$\sup_{j \in \mathbb{Z}} 2^{-j} \|\dot{\Delta}_j (T_{\partial_m \bar{\omega}_n} v_n)(s)\|_{L^\infty} \leq C e^{C_1 s} 2^{-n\alpha} + C n \|\bar{\omega}_n(s)\|_{\dot{B}_{\infty, \infty}^{-1}}$$

for any fixed  $\alpha \in (0, 1)$ .

To estimate the remainder, we again expand the commutator and consider each piece separately. We break  $v_n$  into a low-frequency term and high-frequency term, and we consider  $\|\dot{\Delta}_j (\partial_m R((Id - S_0)v_n, \bar{\omega}_n))\|_{L^\infty}$ .

We have

$$\begin{aligned}
& \sup_{j \in \mathbb{Z}} 2^{-j} \|\dot{\Delta}_j(\partial_m R((Id - S_0)v_n, \bar{\omega}_n))\|_{L^\infty} \\
& \leq C \sum_l \sum_{i=-1}^1 \|\Delta_{l-i}(Id - S_0)v_n\|_{L^\infty} \|\Delta_l \bar{\omega}_n\|_{L^\infty} \\
& \leq C \|\bar{\omega}_n\|_{\dot{B}_{\infty, \infty}^{-1}} \sum_l \sum_{i=-1}^1 \|\Delta_{l-i} \nabla v_n\|_{L^\infty},
\end{aligned}$$

where we used Bernstein's Lemma to get the first inequality and the second inequality. We now apply the arguments in (5.15) and (5.16) to conclude that

$$\sup_{j \in \mathbb{Z}} 2^{-j} \|\dot{\Delta}_j(\partial_m R((Id - S_0)v_n, \bar{\omega}_n))(s)\|_{L^\infty} \leq C e^{C_1 s n} \|\bar{\omega}_n(s)\|_{\dot{B}_{\infty, \infty}^{-1}}.$$

To bound the low frequencies, we again apply Bernstein's Lemma and the definition of the remainder term. We write

$$\begin{aligned}
\sup_{j \in \mathbb{Z}} 2^{-j} \|\dot{\Delta}_j(\partial_m R(S_0 v_n, \bar{\omega}_n))\|_{L^\infty} & \leq C \sum_{l \leq 1} \sum_{i=-1}^1 \|\Delta_{l-i} S_0 v_n\|_{L^\infty} \|\Delta_l \bar{\omega}_n\|_{L^\infty} \\
& \leq C \|v\|_{L^\infty} \|\bar{v}_n\|_{L^\infty}.
\end{aligned}$$

The last inequality follows from the bound  $\|\Delta_l \bar{\omega}_n\|_{L^\infty} \leq \|\Delta_l \nabla \bar{v}\|_{L^\infty}$ , Bernstein's Lemma, and the observation that  $l \leq 1$ . We now bound  $\|v\|_{L^\infty}$  using (3.6) and we bound  $\|\bar{v}_n\|_{L^\infty}$  as in (5.6). This yields

$$\begin{aligned}
& \sup_{j \in \mathbb{Z}} 2^{-j} \|\dot{\Delta}_j(\partial_m R(S_0 v_n, \bar{\omega}_n))(s)\|_{L^\infty} \\
& \leq C e^{C_1 s} 2^{-n\alpha} + C e^{C_1 s n} \|\bar{\omega}_n(s)\|_{\dot{B}_{\infty, \infty}^{-1}}.
\end{aligned}$$

It remains to bound  $\sup_{j \in \mathbb{Z}} 2^{-j} \|\partial_m R(v_n, \dot{\Delta}_j \bar{\omega}_n)\|_{L^\infty}$ . Again we break  $v_n$  into a low-frequency and high-frequency term. We first estimate the high-frequency term. We reintroduce the sum over  $m$  and utilize the divergence-free property of  $v$  to put the partial derivative  $\partial_m$  on  $\bar{\omega}_n$ . We then apply Bernstein's Lemma to conclude that for any fixed  $j \in \mathbb{Z}$

$$\begin{aligned}
& \sum_m \|R((Id - S_0)v_n^m, \dot{\Delta}_j \partial_m \bar{\omega}_n)\|_{L^\infty} \\
& \leq C \sum_{|k-l| \leq 1} 2^{l-k} \|\Delta_k \nabla v_n\|_{L^\infty} \|\dot{\Delta}_j \Delta_l \bar{\omega}_n\|_{L^\infty} \leq C \|\nabla v_n\|_{L^\infty} \|\dot{\Delta}_j \bar{\omega}_n\|_{L^\infty}.
\end{aligned}
\tag{5.20}$$

The second inequality above follows because for fixed  $j \geq 0$ , we are summing only over  $l$  satisfying  $|l - j| \leq 1$ , while for fixed  $j < 0$ , we are only considering  $l$  satisfying  $-1 \leq l \leq 1$ . We now bound  $\|\nabla v_n\|_{L^\infty}$  as

in (5.16), we multiply (5.20) by  $2^{-j}$ , and we take the supremum over  $j \in \mathbb{Z}$ , which yields

$$\sup_{j \in \mathbb{Z}} 2^{-j} \sum_m \|\partial_m R(v_n, \dot{\Delta}_j \bar{\omega}_n)(s)\|_{L^\infty} \leq C e^{C_1 s} n \|\bar{\omega}_n(s)\|_{\dot{B}_{\infty, \infty}^{-1}}.$$

For the low-frequency term, we again use the divergence-free condition on  $v_n$  to write

$$\begin{aligned} & \sup_{j \in \mathbb{Z}} 2^{-j} \sum_m \|\partial_m R(S_0 v_n^m, \dot{\Delta}_j \bar{\omega}_n)\|_{L^\infty} \\ & \leq \sup_{j \in \mathbb{Z}} 2^{-j} \sum_{|k-l| \leq 1} \|\Delta_k S_0 v_n\|_{L^\infty} 2^{2j} \|\dot{\Delta}_j \Delta_l \bar{v}_n\|_{L^\infty} \leq C \|v\|_{L^\infty} \|\bar{v}_n\|_{L^\infty}. \end{aligned}$$

To get the first inequality, we bounded  $\|\dot{\Delta}_j \partial_m \bar{\omega}_n\|_{L^\infty}$  with  $\|\dot{\Delta}_j \partial_m \nabla \bar{v}_n\|_{L^\infty}$  and applied Bernstein's Lemma. The second inequality follows from the observation that we are only considering  $k \leq 1$ , and therefore, by properties of our partition of unity, we are only considering  $j \leq 3$ . As with previous terms, we use (3.6) to bound  $\|v\|_{L^\infty}$  and we use (5.6) to bound  $\|\bar{v}_n\|_{L^\infty}$ . We conclude that

$$\begin{aligned} & \sup_{j \in \mathbb{Z}} 2^{-j} \sum_m \|\partial_m R(S_0 v_n^m, \dot{\Delta}_j \bar{\omega}_n)(s)\|_{L^\infty} \\ & \leq C e^{C_1 s} 2^{-n\alpha} + C e^{C_1 s} n \|\bar{\omega}_n(s)\|_{\dot{B}_{\infty, \infty}^{-1}}. \end{aligned}$$

This completes the proof of Lemma 10.  $\square$

Bounding  $\|\bar{\omega}_n(s)\|_{\dot{B}_{\infty, \infty}^{-1}}$  with  $\|\bar{v}_n(s)\|_{\dot{B}_{\infty, \infty}^0}$  in (5.11), we have

$$\sup_{j \in \mathbb{Z}} 2^{-j} \|[\dot{\Delta}_j, v_n \cdot \nabla] \bar{\omega}_n(s)\|_{L^\infty} \leq C e^{C_1 s} 2^{-n\alpha} + C n e^{C_1 s} \|\bar{v}_n(s)\|_{\dot{B}_{\infty, \infty}^0}.$$

We combine this estimate with the estimates established for the other terms on the right hand side of (5.4), which yields

$$\begin{aligned} & \|\bar{v}_n(t)\|_{\dot{B}_{\infty, \infty}^0} \leq C \|\bar{\omega}_n(t)\|_{\dot{B}_{\infty, \infty}^{-1}} \\ (5.21) \quad & \leq C e^{C_1 t} 2^{-n\alpha} + \int_0^t C n e^{C_1 s} \|\bar{v}_n(s)\|_{\dot{B}_{\infty, \infty}^0} ds, \end{aligned}$$

for fixed  $\alpha \in (0, 1)$ . This completes the proof of Lemma 8.

*Acknowledgements.* The author would like to thank the referee for many thoughtful suggestions on revisions for this paper. This material is based upon work supported by the Center for Nonlinear Analysis (CNA) under the National Science Foundation Grant No. DMS 0635983.

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