SOLUTIONS TO THE 2D EULER EQUATIONS WITH VELOCITY UNBOUNDED AT INFINITY

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Abstract. We prove existence of solutions to the two-dimensional Euler equations with vorticity bounded and with velocity locally bounded but growing at infinity at a rate slower than a power of the logarithmic function. We place no integrability conditions on the initial vorticity. This result improves upon a result of Serfati which gives existence of a solution to the two-dimensional Euler equations with bounded velocity and vorticity.

1. Introduction

We consider the Euler equations governing incompressible inviscid fluid flow in the plane, given by

$$(E) \quad \begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p \\ \text{div } u = 0 \\ u|_{t=0} = u^0. \end{cases}$$

In this paper, we investigate the existence of solutions to (E) for which vorticity $\omega(u) = \partial_1 u_2 - \partial_2 u_1$ is bounded and velocity may grow at infinity, but at a rate slower than $\log_2^{1/4}|x|$.

The three-dimensionsal Euler equations with nondecaying velocity are considered in [6], where Constantin proves that there exists a solution to (E) with nondecaying velocity which blows up in finite time. In two dimensions, existence and uniqueness of solutions to (E) with $(u,\omega)\in L^\infty(\mathbb{R}^2)\times L^\infty(\mathbb{R}^2)$ and without any integrability conditions on u or ω is established by Serfati [9]. Properties of Serfati solutions are further investigated in [1], where the authors extend the existence and uniqueness results of Serfati to an exterior domain. Building on results from [9], Taniuchi [10] proves existence of solutions to (E) with velocity bounded and vorticity belonging to the space Y^0_{ul}, which contains *bmo* and allows for unbounded vorticity (without placing any integrability conditions on u or ω). In [11], Taniuchi-Yoneda-Young establish uniqueness of solutions in a subset of Taniuchi's existence class by modifying methods from [12].

Solutions to the two-dimensional Euler equations for which velocity grows at infinity are considered in [2]. The authors show that if initial velocity obeys the estimate

$$|u^0(x)| \leq C(1+|x|^\alpha) \tag{1.1}$$

for some $\alpha \in [0,1)$ and the initial vorticity ω^0 belongs to $L^p \cap L^\infty(\mathbb{R}^2)$ for $p < 2/\alpha$, then the velocity satisfies (1.1) at all positive times. Similar to [2], Brunelli [4] assumes that

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the initial velocity satisfies (1.1) with $\alpha = 1/2$ and that ω^0 is bounded and satisfies

$$\int_{\mathbb{R}^2} \frac{\omega^0(y)}{|x-y|}\,dy < \infty$$

for some $x \in \mathbb{R}^2$. He proceeds to show that under these assumptions, the growth rate of velocity is preserved at later times.

We remark that, in regards to the Navier-Stokes equations, short time existence of solutions in two and three dimensions with velocity bounded and nondecaying is shown in [7]. In [8], the authors show that in dimension two, the solution of [7] can be extended globally in time.

In this paper we consider solutions to (E) for which vorticity is bounded and potentially nondecaying and velocity may grow at infinity more slowly than a power of the logarithmic function. We prove the following theorem.

Theorem 1. *Let u^0 be such that gu^0 and $\omega^0 = \omega(u^0)$ belong to $L^\infty(\mathbb{R}^2)$, where $g(x) = \log_2^{-1/4}(2+|x|)$ for all $x \in \mathbb{R}^2$. There exists a weak solution u to (E) on $[0,\infty)$ with*

$$gu \in L^\infty_{loc}([0,\infty), L^\infty(\mathbb{R}^2)), \text{ and}$$
$$\omega \in L^\infty([0,\infty), L^\infty(\mathbb{R}^2)).$$

The proof of Theorem 1 consists of three steps: (i) Using the initial data u^0, we construct a sequence of smooth solutions (u_n) to the Euler equations which lie in our existence class and which converge uniformly on compact subsets of $\mathbb{R}^2$. (ii) We establish an upper bound on the L^∞-norms of the sequence (gu_n) which is independent of n. (iii) We pass to the limit and use the uniform bound from step (ii) to show that the limit u is a solution to (E) in our existence class with initial data u^0.

To establish the uniform bound in step (ii), we let u be a smooth solution to (E), fix $N \le -1$, and write gu as a sum of two terms:

$$gu(t,x) = gS_N u(t,x) + g(Id - S_N)u(t,x), \tag{1.2}$$

where $S_N u = \chi_N * u$, $\chi_N = 2^{2N}\chi(2^N\cdot)$, and χ is a radial Schwartz function which integrates to one. One can easily estimate the L^∞-norm of the second term of (1.2) using membership of ω to L^∞. The first term is more delicate. The main obstacle lies in estimating the pressure, specifically terms of the form

$$g(x) \sum_{i,j=1,2} \nabla S_N R_i R_j (u_i u_j), \tag{1.3}$$

where R_k denotes the Riesz operator. Since the Riesz operators are not bounded on $L^\infty(\mathbb{R}^2)$, we write (1.3) as a convolution and apply the Riesz operators to the function $\nabla\chi_N$. We are then able to bound (1.3) by

$$g(x)\|gu\|^2_{L^\infty} \int_{\mathbb{R}^2} |R_i R_j \nabla \chi_N(y)| (1/g^2)(x-y)\,dy, \tag{1.4}$$

which can be estimated using decay of $R_i R_j \nabla\chi_N(y)$. The appropriate choice of N will allow us to apply Osgood's Lemma, which will yield the desired uniform bound.

The strategy used to prove Theorem 1 can also be used to establish a weaker version of the result of [2], which allows for sublinear growth of velocity if the vorticity belongs to $L^p \cap L^\infty(\mathbb{R}^2)$ for some $p < \infty$. Specifically, setting $g(x) = (1+|x|)^{-\alpha}$ for some $\alpha \in [0, \frac{1}{3}]$, we can show that if gu^0 belongs to $L^\infty(\mathbb{R}^2)$ and ω^0 belongs to $L^p \cap L^\infty(\mathbb{R}^2)$ for $p \leq 1/\alpha$, then there exists a solution to (E) with

$$gu \in L^\infty_{loc}([0,\infty), L^\infty(\mathbb{R}^2)), \text{ and}$$
$$\omega \in L^\infty([0,\infty), L^p \cap L^\infty(\mathbb{R}^2)).$$

We omit the details.

We have no proof of uniqueness of solutions in our existence class at this time, but plan to address this question in a future publication.

The paper is organized as follows. In Section 2 we state some definitions and lemmas which will be used throughout the remainder of the paper. In Section 3, we establish an priori estimate which implies the appropriate uniform bound on a sequence of smooth solutions to (E). In Section 4, we construct our sequence of smooth solutions and show that this sequence converges locally uniformly in $\mathbb{R}^2$, thereby allowing us to pass to the limit and obtain existence of a weak solution to (E) which possesses the desired properties.

2. Definitions and Preliminary Lemmas

We begin with the definition of a weak solution to the two-dimensional Euler equations.

Definition. A vector field $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is called a weak solution to (E) on $[0,T)$ with initial data $u^0 : \mathbb{R}^2 \to \mathbb{R}^2$ if u satisfies the following:

(i) $u \in L^2_{loc}([0,T) \times \mathbb{R}^2)$,

(ii) $\int_0^T \int_{\mathbb{R}^2} \left\{ -u \cdot \frac{d}{dt}\phi - u_k u_l \frac{\partial}{\partial x_l}\phi_k \right\} dx\, dt = \int_{\mathbb{R}^2} u^0 \cdot \phi(0)\, dx$

for all $\phi \in C_0^\infty([0,T) \times \mathbb{R}^2)$ with div $\phi = 0$,

(iii) div $u = 0$ in $\mathcal{D}'(\mathbb{R}^2)$.

We now define the Littlewood-Paley operators, which will be particularly useful in Section 4. There exists two radial functions $\hat{\chi} \in S(\mathbb{R}^d)$ and $\hat{\varphi} \in S(\mathbb{R}^d)$ with supp $\hat{\chi} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq \frac{5}{6}\}$ and supp $\hat{\varphi} \subset \{\xi \in \mathbb{R}^d : \frac{3}{5} \leq |\xi| \leq \frac{5}{3}\}$ such that, if for every $j \in \mathbb{Z}$ we set $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j}\xi)$, then

$$\hat{\chi}(\xi) + \sum_{j \geq 0} \hat{\phi}_j(\xi) = 1 \text{ for all } \xi \in \mathbb{R}^d, \text{ and}$$
$$\sum_{j \in \mathbb{Z}} \hat{\phi}_j(\xi) = 1 \text{ for all } \xi \in \mathbb{R}^d \backslash \{0\}.$$

For $n \in \mathbb{Z}$ define $\hat{\chi}_n \in S(\mathbb{R}^d)$ by the equality

$$\hat{\chi}_n(\xi) = 1 - \sum_{j > n} \hat{\varphi}_j(\xi)$$

for all $\xi \in \mathbb{R}^d$, and for $f \in S'(\mathbb{R}^d)$ define the operator S_n by

$$S_n f = \chi_n * f.$$

For $f \in S'(\mathbb{R}^d)$ and $j \in \mathbb{Z}$, define the Littlewood-Paley operators Δ_j by

$$\Delta_j f = \varphi_j * f.$$

We will also need the paraproduct decomposition introduced by J.-M. Bony in [3]. We recall the definition of the paraproduct and remainder used in this decomposition.

Definition. Define the paraproduct of two functions f and g by

$$T_f g = \sum_{\substack{i,j \\ i\le j-2}} \Delta_i f \Delta_j g = \sum_{j=1}^{\infty} S_{j-2} f \Delta_j g.$$

We use $R(f,g)$ to denote the remainder. $R(f,g)$ is given by the following bilinear operator:

$$R(f,g) = \sum_{\substack{i,j \\ |i-j|\le 1}} \Delta_i f \Delta_j g.$$

Bony's decomposition then gives

$$fg = T_f g + T_g f + R(f,g).$$

We now define the Besov spaces.

Definition. Let $s \in \mathbb{R}$, $(p,q) \in [1,\infty] \times [1,\infty)$. We define the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^d)$ to be the space of tempered distributions f on $\mathbb{R}^d$ such that

$$||f||_{B^s_{p,q}} := ||\chi * f||_{L^p} + \left(\sum_{j=0}^{\infty} 2^{jqs} ||\Delta_j f||^q_{L^p}\right)^{\frac{1}{q}} < \infty.$$

When $q = \infty$, write

$$||f||_{B^s_{p,\infty}} := ||\chi * f||_{L^p} + \sup_{j\ge 0} 2^{js} ||\Delta_j f||_{L^p}.$$

It is well known (see [5]) that when $s > 0$ is not an integer, $B^s_{\infty,\infty}(\mathbb{R}^d)$ corresponds to the Holder space $C^s(\mathbb{R}^d)$.

We will make use of Bernstein's Lemma in what follows. We refer the reader to [5], chapter 2, for a proof of the lemma.

Lemma 1. *(Bernstein's Lemma) Let r_1 and r_2 satisfy $0 < r_1 < r_2 < \infty$, and let p and q satisfy $1 \le p \le q \le \infty$. There exists a positive constant C such that for every integer k , if u belongs to $L^p(\mathbb{R}^d)$, and supp $\hat{u} \subset B(0, r_1\lambda)$, then*

$$\sup_{|\alpha|=k} ||\partial^\alpha u||_{L^q} \le C^k \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} ||u||_{L^p}. \tag{2.1}$$

Furthermore, if supp $\hat{u} \subset C(0, r_1\lambda, r_2\lambda)$, then

$$C^{-k}\lambda^k ||u||_{L^p} \le \sup_{|\alpha|=k} ||\partial^\alpha u||_{L^p} \le C^k \lambda^k ||u||_{L^p}. \tag{2.2}$$

Finally, when estimating (1.4), we will need sufficient decay of $R_i R_j \nabla\chi(x)$. This decay is a consequence of the following theorem, which can be found in [13].

Theorem 2. *Assume f is a function on $\mathbb{R}^d$ which satisfies:*
1) $|f(x)| \le C(1+|x|)^{-d-N+\epsilon}$,
2) $|D^\alpha f(x)| \le C(1+|x|)^{-d-N-1+\epsilon}$ for all $|\alpha| = 1$, and

3) $\int x^\beta f(x)\,dx = 0$ *for all* $|\beta| < N$
for some fixed $\epsilon \in [0,1)$ *and some positive integer* N*. Then for each* i*,* $1 \le i \le d$*,*

$$|R_i f(x)| \le C(1+|x|)^{-d-N+\epsilon+\delta}$$

for every δ *satisfying* $0 < \delta < 1-\epsilon$*.*

We will apply Theorem 2 to $f(x) = \nabla\chi(x)$, where χ is as above. In particular, we have the following corollary to Theorem 2.

Corollary 2. *Assume* χ *belongs to* $\mathcal{S}(\mathbb{R}^2)$*. Given* $\epsilon > 0$*, there exists* $C > 0$ *such that for each* i*,* j *with* $1 \le i$*,* $j \le 2$*,* $|R_i R_j \nabla\chi(x)|$ *satisfies*

$$|R_i R_j \nabla\chi(x)| \le C(1+|x|)^{-3+\epsilon}.$$

Proof. Note that since $\nabla\chi$ is a Schwartz function which integrates to zero on $\mathbb{R}^2$, $\nabla\chi$ satisfies the conditions of Theorem 2 with $N = 1$ and $\epsilon = 0$. We conclude from Theorem 2 that

$$|R_i \nabla\chi(x)| \le C(1+|x|)^{-3+\delta} \tag{2.3}$$

for all $\delta \in (0,1)$. Now observe that for $|\alpha| = 2$, $D^\alpha\chi$ is a Schwartz function which satisfies condition 3 of Theorem 2 for $N = 2$. Therefore, $D^\alpha\chi$ satisfies the conditions of Theorem 2 with $N = 2$ and $\epsilon = 0$, hence

$$|R_i D^\alpha \chi(x)| \le C(1+|x|)^{-4+\delta} \tag{2.4}$$

for all $\delta \in (0,1)$. Finally, note that $R_i\nabla\chi$ also integrates to zero on $\mathbb{R}^2$ (for a proof of this, see Theorem 3.4 of [13]). The moment condition on $R_i\nabla\chi$, combined with (2.3) and (2.4), imply that $R_i\nabla\chi$ satisfies the conditions of Theorem 2 with $N = 1$ and $\epsilon = \delta$. Applying Theorem 2 to $R_i\nabla\chi$ then gives the estimate

$$|R_j R_i \nabla\chi(x)| \le C(1+|x|)^{-3+\delta+\tilde\epsilon}$$

for all $\tilde\epsilon$ satisfying $0 < \tilde\epsilon < 1-\delta$. This completes the proof of the theorem. □

3. A Priori Estimates

In Section 4, we prove existence of a weak solution to (E) with bounded vorticity and velocity which grows at infinity at a rate slower than $\log_2^{1/4}|x|$. We prove the theorem by considering a sequence of global-in-time smooth velocity solutions and passing to the limit in the appropriate norm. The following theorem gives a uniform bound on the L^∞-norm of a sequence of weighted smooth velocity solutions. This bound will allow us to conclude that the limit of the sequence of smooth velocities is a weak solution to the Euler equations and satisfies the desired properties. We prove the following.

Theorem 3. *Let* $h : \mathbb{R}^2 \to \mathbb{R}$ *be defined by* $h(x) = \log_2^{1/4}(2+|x|)$*, and set* $g(x) = \frac{1}{h(x)}$*. Assume* u^0 *lies in* $C_0^\infty(\mathbb{R}^2)$*. Then the solution* u *to (E) with initial data* u^0 *satisfies the estimate*

$$\|g(\cdot)u(t,\cdot)\|_{L^\infty} \le e^{C_0 e^{C_0 t}},$$

where C_0 *depends on* $\|gu^0\|_{L^\infty}$ *and* $\|\omega^0\|_{L^\infty}$*.*

Proof. Set $v(t,x) = g(x)u(t,x)$ for all $t \geq 0$ and for all $x \in \mathbb{R}^2$. We must estimate $\|v(t)\|_{L^\infty}$. For a fixed integer $N \leq -1$ (to be chosen later), write

$$|v(t,x)| \leq |g(x)S_N u(t,x)| + |g(x)(Id - S_N)u(t,x)|. \tag{3.1}$$

We consider $|g(x)S_N u(t,x)|$. We apply the operator S_N to (E) and multiply the resulting equation by $g(x)$. Integrating in time yields the inequality

$$|g(x)S_N u(t,x)| \leq |g(x)S_N u(0,x)| + \int_0^t g(x)|\nabla S_N P(u \otimes u)(s,x)|\, ds, \tag{3.2}$$

where P denotes the Helmholtz projection operator with ij-component given by $\delta_{ij} + R_i R_j$, and $R_l = (-\Delta)^{-1/2}\partial_l$ is the Riesz operator.

We first estimate $g(x)|\nabla S_N P(u \otimes u)(s,x)|$. We fix and suppress the time variable s in the calculations that follow. We temporarily drop the factor of g and write

$$\begin{aligned} |\nabla S_N P(u \otimes u)(x)| &= |\nabla S_N P(h^2(v \otimes v))(x)| \\ &= \left| \int_{\mathbb{R}^2} P\nabla \chi_N(y) h^2(x-y) v \otimes v(x-y)\, dy \right| \\ &\leq \|v\|_{L^\infty}^2 \int_{\mathbb{R}^2} |P\nabla\chi_N(y)||h^2(x-y)|\, dy. \end{aligned} \tag{3.3}$$

We now use the fact that the Riesz transforms commute with dilations and apply a change of variables to write

$$\begin{aligned} \int_{\mathbb{R}^2} |P\nabla\chi_N(y)||h^2(x-y)|\, dy &\leq 2^{3N} \int_{\mathbb{R}^2} |(P\nabla\chi)(2^N y)||h^2(x-y)|\, dy \\ &\leq 2^N \int_{\mathbb{R}^2} |P\nabla\chi(z)||h^2(x - 2^{-N}z)|\, dz \\ &\leq 2^N \int_{\mathbb{R}^2} |P\nabla\chi(z)| \log_2^{1/2}(2 + |x| + 2^{-N}|z|)\, dz \\ &\leq C2^N \int_{\mathbb{R}^2} |P\nabla\chi(z)| \log_2^{1/2}(2 + |x|)\, dz \\ &\quad + C2^N \int_{\mathbb{R}^2} |P\nabla\chi(z)| \log_2^{1/2}(2 + 2^{-N}|z|)\, dz = I + II, \end{aligned} \tag{3.4}$$

where we used concavity of $\log_2^{1/2}(2 + |\cdot|)$ to get the last inequality above. To estimate I, we observe that

$$I \leq Ch^2(x)2^N \int_{\mathbb{R}^2} |P\nabla\chi(z)|\, dz. \tag{3.5}$$

The integral in (3.5) converges by Corollary 2. We conclude that

$$I \leq Ch^2(x)2^N.$$

We now estimate II. Since $N \leq -1$, we can write

$$\begin{aligned} II &\le C2^N \int_{\mathbb{R}^2} |P\nabla\chi(z)| \log_2^{1/2}(2^{-N}(2+|z|))\, dz \\ &\le C2^N \int_{\mathbb{R}^2} |P\nabla\chi(z)|(-N+\log_2(2+|z|))^{1/2}\, dz \\ &\le C2^N \int_{\mathbb{R}^2} |P\nabla\chi(z)|(\sqrt{-N}+\log_2^{1/2}(2+|z|))\, dz \\ &\le C\sqrt{-N}2^N + C2^N \int_{\mathbb{R}^2} |P\nabla\chi(z)| \log_2^{1/2}(2+|z|)\, dz \le C\sqrt{-N}2^N, \end{aligned}$$

where we again used Corollary 2 to get the last inequality. Substituting the estimates for I and II into (3.4), and substituting the resulting estimate into (3.3) yields

$$g(x)|\nabla S_N P(u\otimes u)(x)| \le C\|v\|_{L^\infty}^2 h(x)2^N + Cg(x)\|v\|_{L^\infty}^2 2^N\sqrt{-N}. \tag{3.6}$$

To complete the bound for $|g(x)S_N u(t,x)|$, it remains to estimate $|g(x)S_N u^0(x)|$. For this term, we use arguments similar to those above and write

$$\begin{aligned} g(x)|S_N u^0(x)| &= g(x)|S_N(hv^0)(x)| \le g(x)\int_{\mathbb{R}^2} |\chi_N(y)||(hv^0)(x-y)|\, dy \\ &\le g(x)\|v^0\|_{L^\infty} \int_{\mathbb{R}^2} |\chi_N(y)|(h(x)+h(y))\, dy \\ &\le C\|v^0\|_{L^\infty} + g(x)\|v^0\|_{L^\infty} \int_{\mathbb{R}^2} |\chi(y)| h(2^{-N}y)\, dy \\ &\le C\|v^0\|_{L^\infty} + Cg(x)\|v^0\|_{L^\infty}(-N)^{1/4}. \end{aligned} \tag{3.7}$$

We substitute (3.7) and (3.6) into (3.2). This gives

$$\begin{aligned} |g(x)S_N u(t,x)| &\le C\|v^0\|_{L^\infty} + Cg(x)\|v^0\|_{L^\infty}(-N)^{1/4} \\ &\quad + \int_0^t \left(\|v(s)\|_{L^\infty}^2 h(x)2^N + g(x)\|v(s)\|_{L^\infty}^2 2^N\sqrt{-N}\right) ds. \end{aligned} \tag{3.8}$$

It remains to estimate the high frequency term $g(x)|(Id-S_N)u(t,x)|$. In this case, keeping in mind that $N\le -1$, we use Bernstein's Lemma to write

$$\begin{aligned} g(x)|(Id-S_N)u(t,x)| &\le g(x)\left(\sum_{j=N}^{0} \|\Delta_j u(t)\|_{L^\infty} + \sum_{j>0} \|\Delta_j u(t)\|_{L^\infty}\right) \\ &\le g(x)\left(\sum_{j=N}^{0} 2^{-j}\|\Delta_j \omega(t)\|_{L^\infty} + C\|\omega(t)\|_{L^\infty}\right) \le Cg(x)2^{-N}\|\omega^0\|_{L^\infty}, \end{aligned} \tag{3.9}$$

where we used conservation of L^∞-norm of vorticity for smooth solutions to get the last inequality. We now insert our estimates for (3.8) and (3.9) into (3.1). After bounding $(-N)^{1/4}$ above by 2^{-N}, we conclude that

$$
\begin{aligned}
|v(t,x)| &\le C\|v^0\|_{L^\infty} + Cg(x)2^{-N}\left(\|\omega^0\|_{L^\infty} + \|v^0\|_{L^\infty}\right) \\
&\quad + \int_0^t \left(\|v(s)\|_{L^\infty}^2 h(x) 2^N + g(x)\|v(s)\|_{L^\infty}^2 2^N\sqrt{-N}\right) ds \\
&= C\|v^0\|_{L^\infty} + Cg(x)2^{-N}\left(\|\omega^0\|_{L^\infty} + \|v^0\|_{L^\infty}\right) \\
&\quad + \left(h(x)2^N + g(x)2^N\sqrt{-N}\right)\int_0^t \|v(s)\|_{L^\infty}^2\, ds.
\end{aligned} \tag{3.10}
$$

Finally, set

$$
N = -\log_2\left(h(x)\left(4 + \int_0^t \|v(s)\|_{L^\infty}^2\right)^{1/2}\right). \tag{3.11}
$$

Substituting (3.11) into (3.10) and simplifying gives

$$
\begin{aligned}
|v(t,x)| &\le C_0\left(4 + \int_0^t \|v(s)\|_{L^\infty}^2\right)^{1/2} + \frac{C\log_2^{1/2}\left(\left(4 + \int_0^t \|v(s)\|_{L^\infty}^2\right)^{1/2}\right)\int_0^t \|v\|_{L^\infty}^2\, ds}{\left(4 + \int_0^t \|v(s)\|_{L^\infty}^2\right)^{1/2}} \\
&\le C_0\left(4 + \int_0^t \|v(s)\|_{L^\infty}^2\right)^{1/2}\log_2^{1/2}\left(\left(4 + \int_0^t \|v(s)\|_{L^\infty}^2\right)^{1/2}\right),
\end{aligned}
$$

where we used the fact that $\log_2^{1/2}\left(\left(4 + \int_0^t \|v(s)\|_{L^\infty}^2\right)^{1/2}\right) \ge 1$ in our simplification. Note that C_0 depends on $\|\omega^0\|_{L^\infty}$ and $\|v^0\|_{L^\infty}$. Squaring both sides, we get

$$
|v(t,x)|^2 \le C_0\left(4 + \int_0^t \|v(s)\|_{L^\infty}^2\right)\log_2\left(4 + \int_0^t \|v(s)\|_{L^\infty}^2\right).
$$

To complete the proof of the estimate on $\|v(t)\|_{L^\infty}$, we apply Osgood's Lemma. Set $\delta(t) = \int_0^t \|v(s)\|_{L^\infty}^2$, so that

$$
|v(t,x)|^2 \le C_0\left(4 + \delta(t)\right)\log_2\left(4 + \delta(t)\right). \tag{3.12}
$$

Integrating both sides in the time variable gives

$$
\delta(t) = \int_0^t \|v(s)\|_{L^\infty}^2 \le C_0\int_0^t \left(4 + \delta(s)\right)\log_2\left(4 + \delta(s)\right)\, ds.
$$

We apply Osgood's Lemma to obtain the following inequality for any $t \le T$:

$$
\log_2(\log_2(4 + \delta(t))) \le C + C_0 t,
$$

where C is an absolute constant and C_0 depends on $\|v^0\|_{L^\infty}$ and $\|\omega^0\|_{L^\infty}$. Taking the exponential of both sides twice gives

$$
\delta(t) \le e^{C_0 e^{C_0 t}}. \tag{3.13}
$$

We substitute (3.13) into (3.12), completing the proof. □

4. Proof of Theorem 1

To prove Theorem 1, we construct a sequence (u_n) of smooth solutions to (E) for which there exist $C_0 > 0$ and $C_1 > 0$ such that $\|gu_n\|_{L^\infty([0,T];L^\infty)} \leq C_0$ and $\|\omega_n\|_{L^\infty([0,\infty);L^\infty)} \leq C_1$ for all n. In view of Theorem 3, we must construct (u_n) so that

$$\|gu_n^0\|_{L^\infty} \leq C \text{ and } \|\omega_n^0\|_{L^\infty} \leq C \tag{4.1}$$

for all n, where C depends on $\|gu^0\|_{L^\infty}$ and $\|\omega^0\|_{L^\infty}$. To construct our sequence of approximating solutions, we follow the approach of [1]. Specifically, we construct a sequence of smooth, compactly supported initial velocities (u_n^0) by smoothing and cutting the stream function ψ satisfying $u^0 = \nabla^\perp\psi$. These initial velocities will generate a sequence of smooth velocity solutions (u_n). After establishing uniform estimates on $\|gu_n\|_{L^\infty([0,T];L^\infty)}$ and $\|\omega_n\|_{L^\infty([0,\infty);L^\infty)}$, we proceed to show that this sequence is uniformly bounded and equicontinuous in the appropriate Holder and Besov norms (locally in space), which will allow us to apply an Arzela-Ascoli type of argument. This strategy is similar to that employed in [10].

Let η be a smooth function which is identically 1 on $B_1(0)$ and which vanishes outside of $B_2(0)$. Define $\eta_n(x) = \eta(x/n)$. Set $u_n^0 = \nabla^\perp(\eta_n S_n\psi)$ and $\omega_n^0 = \text{curl } u_n^0$. We must show that (u_n^0) and (ω_n^0) satisfy (4.1). We prove the following lemma.

Lemma 3. *Let g, u_n^0, and ω_n^0 be as above. There exists a constant $C > 0$, depending only on $\|gu^0\|_{L^\infty}$ and $\|\omega^0\|_{L^\infty}$, such that for all n,*

$$\|gu_n^0\|_{L^\infty} \leq C \text{ and } \|\omega_n^0\|_{L^\infty} \leq C.$$

Proof. Before estimating $\|gu_n^0\|_{L^\infty}$, we make a few observations. First, note that the stream function ψ, for which $u^0 = \nabla^\perp\psi$, satisfies

$$\psi(x) = -\int_0^x (u^0)^\perp \cdot \, ds$$

for each $x \in \mathbb{R}^2$. From this and the pointwise estimate $|u^0(x)| \leq Ch(x)$, it follows that for every $x \in \mathbb{R}^2$,

$$|\psi(x)| \leq |x|h(x).$$

Using properties of h, we conclude that for each $n \in \mathbb{N}$,

$$\begin{aligned} |S_n\psi(x)| &\leq \int_{\mathbb{R}^2} |\chi_n(y)||x-y|h(x-y)\,dy \\ &\leq \int_{\mathbb{R}^2} |\chi(y)|(|x| + 2^{-n}|y|)(h(x) + h(2^{-n}y))\,dy \\ &\leq C(|x|+1)h(x). \end{aligned} \tag{4.2}$$

Also, note that

$$
\begin{aligned}
|S_n u^0(x)| &= \left| \int_{\mathbb{R}^2} \chi_n(y) h(x-y)(gu^0)(x-y)\, dy \right| \\
&\leq \|gu^0\|_{L^\infty} \int_{\mathbb{R}^2} |\chi(y) h(x - 2^{-n} y)|\, dy \\
&\leq C\|gu^0\|_{L^\infty} \int_{\mathbb{R}^2} |\chi(y)|(h(x) + h(2^{-n}y))\, dy \\
&\leq C\|gu^0\|_{L^\infty} h(x) + C\|gu^0\|_{L^\infty} \leq C\|gu^0\|_{L^\infty} h(x).
\end{aligned}
\tag{4.3}
$$

We are now in a position to bound $\|gu_n^0\|_{L^\infty}$. Using (4.2) and (4.3), we write

$$
\begin{aligned}
|(gu_n^0)(x)| &= \left| g(x)\nabla^\perp(\eta_n S_n \psi)(x) \right| \\
&\leq \left| g(x)\eta_n(x) S_n \nabla^\perp \psi(x) \right| + \left| g(x) S_n\psi(x) \nabla^\perp \eta_n(x) \right| \\
&= \left| g(x)\eta_n(x) S_n u^0(x) \right| + \left| (1+|x|)\nabla^\perp \eta_n(x) \right| \\
&\leq C\|gu^0\|_{L^\infty} + C(1+n)n^{-1} \leq C\|gu^0\|_{L^\infty} + C.
\end{aligned}
$$

To estimate $\|\omega_n^0\|_{L^\infty}$, we observe that

$$
\begin{aligned}
|\omega_n^0(x)| &= |S_n\psi(x)\Delta\eta_n(x) + \eta_n(x)\Delta S_n\psi(x) + 2\nabla\eta_n(x)\cdot\nabla S_n\psi(x)| \\
&\leq C(1+n)\log^{\frac{1}{4}}(2+2n)n^{-2} + C\|\omega^0\|_{L^\infty} + C\|gu^0\|_{L^\infty} n^{-1}\log^{\frac{1}{4}}(2+2n) \\
&\leq C + C\|\omega^0\|_{L^\infty} + C\|gu^0\|_{L^\infty},
\end{aligned}
$$

where we again used (4.2) and (4.3) to get the second to last inequality. This completes the proof of Lemma 3. □

In what follows, we consider the sequence of smooth solutions (u_n) to (E) which are generated by the sequence of initial velocities (u_n^0). We set $\omega_n = \text{curl } u_n$.

Set $v_n = gu_n$, and for fixed $R > 0$, set $\phi = \eta_R$. We consider the sequence (ϕv_n). Our goal is to prove that for any $\delta > 0$ and $T > 0$, (ϕv_n) is uniformly bounded in $C^{1-\delta}(\mathbb{R}^2)$ on $[0,T]$ and equicontinuous in $B^0_{\infty,\infty}(\mathbb{R}^2)$ on $[0,T]$. We can then complete the proof of Theorem 1 by showing that a subsequence of (u_n) converges locally uniformly to a weak solution u of (E).

We first show that for each fixed $\delta > 0$, (ϕv_n) is uniformly bounded in $C^{1-\delta}(\mathbb{R}^2)$ on $[0,T]$. We will in fact show that the sequence (ϕv_n) is uniformly bounded in the log-Lipschitz norm, which will imply uniform boundedness in the $C^{1-\delta}$-norm. We prove the following proposition.

Proposition 4. *Let ϕ, v_n, u_n, and ω_n be as above. There exists a constant $C > 0$, independent of n, such that*

$$\|\phi v_n\|_{LL} \leq C(\|v_n\|_{L^\infty} + \|\omega_n\|_{L^\infty}).$$

Proof. We fix and suppress the time variable t. By Bernstein's Lemma, for x, $y \in \mathbb{R}^2$ and $N \geq 0$,

$$\begin{aligned}
|\phi v_n(x) - \phi v_n(y)| &\leq \sum_{j=-1}^{N} \|\Delta_j \nabla(\phi v_n)\|_{L^\infty} |x-y| + 2\sum_{j>N} \|\Delta_j(\phi v_n)\|_{L^\infty} \\
&\leq \|v_n\|_{L^\infty}|x-y| + \sum_{j=0}^{N} \|\Delta_j \nabla(\phi v_n)\|_{L^\infty}|x-y| + 2\sum_{j>N} 2^{-j}\|\Delta_j \nabla(\phi v_n)\|_{L^\infty} \\
&\leq \|v_n\|_{L^\infty}|x-y| + \sum_{j=0}^{N} \left(\|\omega(\phi v_n)\|_{L^\infty} + \|\mathrm{div}(\phi v_n)\|_{L^\infty}\right)|x-y| \\
&\quad + C2^{-N}\left(\|\omega(\phi v_n)\|_{L^\infty} + \|\mathrm{div}(\phi v_n)\|_{L^\infty}\right).
\end{aligned} \tag{4.4}$$

To get the last inequality above, we used the Biot-Savart law for vector fields with non-zero divergence, given by

$$w(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \left(\frac{(x-y)^\perp}{|x-y|^2}(\omega(w))(y) + \frac{x-y}{|x-y|^2}\ \mathrm{div}\ w(y) \right) dy,$$

combined with the fact that for Calderon-Zygmund operators T, $\Delta_j T$ is bounded on L^∞ when $j \geq 0$. Now note that

$$\begin{aligned}
\omega(\phi v_n) &= \nabla\phi \cdot v_n^\perp + \phi\omega(v_n) = \nabla\phi \cdot v_n^\perp + \phi(\nabla g \cdot u_n^\perp + g\omega(u_n)) \\
&= \nabla\phi \cdot v_n^\perp + \phi k \cdot v_n^\perp + \phi g \omega(u_n),
\end{aligned}$$

where k is the bounded vector field satisfying $\nabla g \cdot u_n^\perp = k \cdot g u_n^\perp = k \cdot v_n^\perp$. We conclude that

$$\|\omega(\phi v_n)\|_{L^\infty} \leq C\|v_n\|_{L^\infty} + C\|\omega(u_n)\|_{L^\infty}. \tag{4.5}$$

To estimate $\|\mathrm{div}(\phi v_n)\|_{L^\infty}$, we observe that

$$\begin{aligned}
\mathrm{div}(\phi v_n) &= \nabla\phi \cdot v_n + \phi\ \mathrm{div}\ v_n = \nabla\phi \cdot v_n + \phi\nabla g \cdot u_n \\
&= \nabla\phi \cdot v_n + \phi k \cdot v_n,
\end{aligned}$$

so

$$\|\mathrm{div}(\phi v_n)\|_{L^\infty} \leq C\|v_n\|_{L^\infty}. \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4) yields

$$\begin{aligned}
|\phi v_n(x) - \phi v_n(y)| &\leq \|v_n\|_{L^\infty}|x-y| + CN(\|v_n\|_{L^\infty} + \|\omega_n\|_{L^\infty})|x-y| \\
&\quad + C2^{-N}(\|v_n\|_{L^\infty} + \|\omega_n\|_{L^\infty}).
\end{aligned}$$

Set $N = -\log_2|x-y|$ if $|x-y| \leq 1$, and set $N = 0$ otherwise. We conclude that

$$\|\phi v_n\|_{LL} \leq C(\|v_n\|_{L^\infty} + \|\omega_n\|_{L^\infty}).$$

This completes the proof. □

We reintroduce the time variable and apply the bounds

$$\|v_n(t)\|_{L^\infty} \leq e^{C_0 e^{C_0 t}}, \|\omega_n(t)\|_{L^\infty} \leq C\|\omega_n^0\|_{L^\infty},$$

where C_0 depends on $\|v_n^0\|_{L^\infty}$ and $\|\omega_n^0\|_{L^\infty}$. Using Lemma 3, one can conclude that (ϕv_n) is uniformly bounded in the log-Lipschitz norm on $[0,T]$.

To simplify the presentation in what follows, we use the notation Δ_{-1} to denote the Littlewood-Paley operator corresponding to convolution with the smooth bump function χ; that is, for $f \in \mathcal{S}'$,

$$\Delta_{-1} f = \chi * f.$$

This deviates from the definition of Δ_{-1} given in Section 2.

We now show that (ϕv_n) is equicontinuous in the $B^0_{\infty,\infty}$-norm on $[0,T]$. Specifically, we claim that there exists a constant $C > 0$, independent of n, such that

$$\|\phi v_n(t) - \phi v_n(s)\|_{B^0_{\infty,\infty}} \leq C|t-s| \tag{4.7}$$

for all s, $t \in [0,T]$. We show that (4.7) holds in the proof of Proposition 8 below. Before we prove Proposition 8, we establish a series of lemmas (Lemma 5, Lemma 6, and Lemma 7). To motivate the need for these lemmas, observe that by the definition of the $B^0_{\infty,\infty}$-norm, the proof of Proposition 8 will require that we estimate terms of the form

$$\|\Delta_j(\phi g(u_n(t) - u_n(s)))\|_{L^\infty} \tag{4.8}$$

for $j \geq -1$. We will use Bony's paraproduct decomposition, and will therefore have to examine the products

$$\|\Delta_k(\phi g)\Delta_{k'}(u_n(t) - u_n(s))\|_{L^\infty}. \tag{4.9}$$

These terms are addressed for the case k, $k' \geq 0$ in the following lemma.

Lemma 5. *Let k, $k' \geq 0$. Assume ϕ, g, u_n, v_n, and ω_n are as above. For fixed $T > 0$ there exists a constant C independent of n such that*

$$\|\Delta_k(\phi g)\Delta_{k'}(u_n(t) - u_n(s))\|_{L^\infty} \leq C 2^{-k}|t-s| \sup_{\tau\in[s,t]} \|v_n(\tau)\|_{L^\infty}\|\omega_n(\tau)\|_{L^\infty}$$

for all s, $t \in [0,T]$.

Proof. To simplify notation, we suppress the index n in what follows. The Biot-Savart law implies that

$$\begin{aligned} u(t) - u(s) = \nabla^\perp\Delta^{-1}(\omega(t) - \omega(s)) &= \int_s^t \nabla^\perp\Delta^{-1}(u\cdot\nabla\omega)(\tau)\,d\tau \\ &= \int_s^t \nabla^\perp\Delta^{-1}\nabla\cdot(u\omega)(\tau)\,d\tau. \end{aligned}$$

An application of the Littlewood-Paley operator Δ_k with $k \geq 0$ gives

$$\begin{aligned} \Delta_k(u(t) - u(s)) &= \int_s^t \Delta_k\nabla^\perp\Delta^{-1}\nabla\cdot(u\omega)(\tau)\,d\tau \\ &\leq |t-s| \sup_{\tau\in[s,t]} \left|\Delta_k\nabla^\perp\Delta^{-1}\nabla\cdot(u\omega)(\tau)\right|. \end{aligned} \tag{4.10}$$

We suppress dependence on τ and estimate $\Delta_k\nabla^\perp\Delta^{-1}\nabla\cdot(u\omega)$. Note that $\Delta_k\nabla^\perp\Delta^{-1}\nabla\cdot(u\omega)$ is a vector field where each component is a sum of terms of the form

$$\Delta_k R_i R_j u_l \omega \tag{4.11}$$

for some $1 \leq l, i, j \leq 2$. For each of these terms, we use concavity of h to write

$$
\begin{aligned}
|\Delta_k R_i R_j u_l \omega(x)| &= \left| \int (R_i R_j \phi_k)(y)(h v_l \omega)(x-y)\, dy \right| \\
&\leq \|v\|_{L^\infty} \|\omega\|_{L^\infty} \int |R_i R_j \phi_k(y)| \log_2^{1/4}(2+|x|+|y|)\, dy \qquad (4.12)\\
&\leq \|v\|_{L^\infty} \|\omega\|_{L^\infty} \int |R_i R_j \phi_k(y)| (h(x)+h(y))\, dy.
\end{aligned}
$$

Since $\hat{\varphi}$ is supported away from the origin, Corollary 2 applies with φ in place of $\nabla\chi$. We use this observation to simplify (4.12) even further. By arguments similar to those in Section 3, we write

$$
\begin{aligned}
|\Delta_k R_i R_j u_l \omega(x)| &\leq \|v\|_{L^\infty} \|\omega\|_{L^\infty} \int |R_i R_j \phi_k(y)| (h(x)+h(y))\, dy \\
&\leq C\|v\|_{L^\infty}\|\omega\|_{L^\infty} h(x) + \|v\|_{L^\infty}\|\omega\|_{L^\infty} \int |R_i R_j \phi(y)| h(2^{-k}y)\, dy \\
&\leq C\|v\|_{L^\infty}\|\omega\|_{L^\infty} h(x) + \|v\|_{L^\infty}\|\omega\|_{L^\infty} \int |R_i R_j \phi(y)| h(y)\, dy \\
&\leq C\|v\|_{L^\infty}\|\omega\|_{L^\infty} h(x).
\end{aligned}
$$

Reintroducing $\Delta_k(\phi g)(x)$, we conclude that

$$
|\Delta_k(\phi g) \Delta_{k'} R_i R_j u_l \omega(x)| \leq C |h(x) \Delta_k(\phi g)(x)| \|v\|_{L^\infty} \|\omega\|_{L^\infty}. \qquad (4.13)
$$

By a change of variables,

$$
\begin{aligned}
|h(x)\Delta_k(\phi g)(x)| &\leq |\Delta_k \phi(x)| + |[\Delta_k, h](\phi g)(x)| \\
&\leq |\Delta_k \phi(x)| + \left| \int \varphi_k(y)(h(x)-h(x-y))(\phi g)(x-y)\, dy \right| \\
&\leq 2^{-k}\|\Delta_k \nabla\phi\|_{L^\infty} + \int |\varphi(z)||h(x)-h(x-2^{-k}z)||(\phi g)(x-2^{-k}z)|\, dz \qquad (4.14)\\
&\leq 2^{-k}\|\nabla\phi\|_{L^\infty} + 2^{-k}\|\nabla h\|_{L^\infty}\|\phi g\|_{L^\infty} \int |\varphi(z)||z|\, dz \leq C2^{-k}.
\end{aligned}
$$

The estimates (4.14) and (4.13) imply that

$$
|\Delta_k(\phi g)\Delta_{k'} R_i R_j (u_l \omega)(x)| \leq C 2^{-k} \|v\|_{L^\infty}\|\omega\|_{L^\infty}.
$$

This estimate, along with (4.10), gives

$$
|\Delta_k(\phi g)\Delta_{k'}(u(t)-u(s))(x)| \leq C 2^{-k} |t-s| \sup_{\tau\in[s,t]} \|v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{L^\infty} \qquad (4.15)
$$

for k, $k' \geq 0$. This completes the proof. □

By an argument similar to that in the proof of Lemma 5, one can show the following.

Lemma 6. *Let $k \geq 2$. Assume ϕ, g, u_n, v_n, and ω_n are as above. For fixed $T > 0$ there exists a constant C independent of n such that*

$$
|S_{k-2}(\phi g)\Delta_k(u(t)-u(s))(x)| \leq C|t-s| \sup_{\tau\in[s,t]} \|v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{L^\infty} \qquad (4.16)
$$

for all $s,\ t \in [0,T]$.

Proof. The proof of Lemma 6 is identical to the proof of Lemma 5 (with $S_{k-2}(\phi g)$ in place of $\Delta_k(\phi g)$), until we reach (4.14), at which time we must estimate $|h(x)S_{k-2}(\phi g)(x)|$. A commutator estimate similar to that used in (4.14) implies that

$$|h(x)S_{k-2}(\phi g)(x)| \leq \|\phi\|_{L^\infty} + 2^{-k}\|\nabla h\|_{L^\infty}\|\phi g\|_{L^\infty}\int |\chi(z)||z|\,dz \leq C.$$

Then (4.16) follows from applying this estimate to the remainder of the proof of Lemma 5. $\square$

When applying Bony's paraproduct decomposition to (4.8), we will also need to estimate terms of the form $|\Delta_k(\phi g)S_{k-2}(u(t)-u(s))(x)|$ for $k \geq 2$. We have the following lemma.

Lemma 7. *Let* $k \geq 2$. *Assume* ϕ, g, u_n, v_n, *and* ω_n *are as above. For fixed* $T > 0$ *there exists a constant* C *independent of* n *such that*

$$|\Delta_k(\phi g)S_{k-2}(u_n(t)-u_n(s))(x)| \leq C|t-s|\sup_{\tau\in[s,t]}\|v_n(\tau)\|^2_{L^\infty} \tag{4.17}$$

for all $s,\ t \in [0,T]$.

Proof. The proof is very similar to the proof of Theorem 3. Again, to simplify notation, we suppress the index n in what follows. Observe that

$$\begin{aligned} |\Delta_k(\phi g)S_{k-2}(u(t)-u(s))(x)| &\leq |\Delta_k(\phi g)(x)|\int_s^t |S_{k-2}P(u\cdot\nabla u)(\tau)|\,d\tau \\ &\leq C2^k|\Delta_k(\phi g)(x)||t-s|\sup_{\tau\in[s,t]}\|v(\tau)\|^2_{L^\infty}\int_{\mathbb{R}^2}|P\nabla\chi(z)|h^2(x-2^{-k}z)\,dz \end{aligned} \tag{4.18}$$

by arguments identical to those in (3.3) and (3.4). Keeping in mind that $k \geq 2$, by properties of h^2 and Corollary 2,

$$\begin{aligned} |\Delta_k(\phi g)(x)|\int_{\mathbb{R}^2}|P\nabla\chi(z)|h^2(x-2^{-k}z)\,dz &\leq |\Delta_k(\phi g)(x)|h^2(x)\int_{\mathbb{R}^2}|P\nabla\chi(z)|\,dz \\ &+ |\Delta_k(\phi g)(x)|\int_{\mathbb{R}^2}|P\nabla\chi(z)|h^2(z)\,dz \leq C|\Delta_k(\phi g)(x)|h^2(x). \end{aligned} \tag{4.19}$$

Moreover, since $k \geq 2$, by an argument identical to (4.14),

$$|\Delta_k(\phi g)(x)|h^2(x) \leq C2^{-k}.$$

Substituting this estimate into (4.19) and substituting the resulting estimate into (4.18) gives (4.17), completing the proof. $\square$

We are now in a position to prove the following proposition, which implies equicontinuity of (ϕv_n) in $B^0_{\infty,\infty}$ on $[0,T]$.

Proposition 8. *Assume* ϕ, v_n, *and* ω_n *are as above. Then for fixed* $T > 0$ *there exists a constant* C *independent of* n *such that*

$$\|\phi v_n(t)-\phi v_n(s)\|_{B^0_{\infty,\infty}} \leq C\sup_{\tau\in[s,t]}(\|v_n(\tau)\|_{L^\infty}\|\omega_n(\tau)\|_{L^\infty}+\|v_n(\tau)\|^2_{L^\infty})|t-s|$$

for all $s,\ t \in [0,T]$.

Proof. To simplify notation, we suppress the index n and use Bony's paraproduct decomposition to write, for $j \geq -1$,

(4.20)

$$\Delta_j(\phi(v(t)-v(s))) = \Delta_j((\phi g)(u(t)-u(s))) = \Delta_j\left(\sum_{|k-j|\leq 3} S_{k-2}(\phi g)\Delta_k(u(t)-u(s))\right) + \Delta_j\left(\sum_{|k-j|\leq 3} \Delta_k(\phi g)S_{k-2}(u(t)-u(s))\right) + \Delta_j\left(\sum_{\substack{|k-k'|\leq 1,\\ \max\{k,k'\}\geq j-3}} \Delta_k(\phi g)\Delta_{k'}(u(t)-u(s))\right).$$

The estimates from Lemma 5, Lemma 6, and Lemma 7 imply that for fixed $j \geq 4$,

$$\begin{aligned}|\Delta_j(\phi(v(t)-v(s)))(x)| &\leq \sum_{|k-j|\leq 3} \|S_{k-2}(\phi g)\Delta_k(u(t)-u(s))\|_{L^\infty} \\ &+ \sum_{|k-j|\leq 3} \|\Delta_k(\phi g)S_{k-2}(u(t)-u(s))\|_{L^\infty} + \sum_{\substack{|k-k'|\leq 1,\\ \max\{k,k'\}\geq j-3}} \|\Delta_k(\phi g)\Delta_{k'}(u(t)-u(s))\|_{L^\infty} \\ &\leq \sum_{|k-j|\leq 3} C|t-s| \sup_{\tau\in[s,t]} \|v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{L^\infty} + \sum_{|k-j|\leq 3} C|t-s| \sup_{\tau\in[s,t]} \|v(\tau)\|_{L^\infty}^2 \\ &\quad + \sum_{\substack{|k-k'|\leq 1,\\ \max\{k,k'\}\geq j-3}} C2^{-k}|t-s| \sup_{\tau\in[s,t]} \|v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{L^\infty} \\ &\quad \leq C|t-s| \sup_{\tau\in[s,t]} (\|v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{L^\infty} + \|v(\tau)\|_{L^\infty}^2).\end{aligned}$$

For the case $j \leq 3$, the L^∞-norm of first two terms on the right hand side of the second equality in (4.20) are estimated in exactly the same way as in the case $j \geq 4$. However, the estimate of the third term differs when $j \leq 3$, because we must consider $|\Delta_{-1}(\phi g)\Delta_{-1}(u(t)-u(s))(x)|$, $|\Delta_{-1}(\phi g)\Delta_0(u(t)-u(s))(x)|$, and $|\Delta_0(\phi g)\Delta_{-1}(u(t)-u(s))(x)|$. An argument virtually identical to the proof of Lemma 7 yields the two estimates

$$|\Delta_{-1}(\phi g)\Delta_{-1}(u(t)-u(s))(x)| \leq C|t-s| \sup_{\tau\in[s,t]} \|v(\tau)\|_{L^\infty}^2, \tag{4.21}$$

and

$$|\Delta_0(\phi g)\Delta_{-1}(u(t)-u(s))(x)| \leq C|t-s| \sup_{\tau\in[s,t]} \|v(\tau)\|_{L^\infty}^2. \tag{4.22}$$

(Note that we will not have the canceling factors of 2^k and 2^{-k}.) Moreover, an argument virtually identical to the proof of Lemma 6 gives

$$|\Delta_{-1}(\phi g)\Delta_0(u(t)-u(s))(x)| \leq C|t-s| \sup_{\tau\in[s,t]} \|v(\tau)\|_{L^\infty}\|\omega(\tau)\|_{L^\infty}. \tag{4.23}$$

When $j \leq 3$, (4.21), (4.22), and (4.23) give a bound for three of the terms in the third sum on the right hand side of the second equality in (4.20). For the other terms in the sum, both k, $k' \geq 0$, so we can invoke Lemma 5. Therefore, if $j \leq 3$ is fixed, then

$$|\Delta_j(\phi(v(t) - \phi v(s)))(x)| \leq C|t-s| \sup_{\tau \in [s,t]} (\|v(\tau)\|_{L^\infty} \|\omega(\tau)\|_{L^\infty} + \|v(\tau)\|_{L^\infty}^2).$$

We conclude that

$$\sup_{j \geq -1} \|\Delta_j(\phi(v(t) - \phi v(s)))\|_{L^\infty} \leq C|t-s| \sup_{\tau \in [s,t]} (\|v(\tau)\|_{L^\infty} \|\omega(\tau)\|_{L^\infty} + \|v(\tau)\|_{L^\infty}^2),$$

proving the lemma. □

We can now complete the proof of Theorem 1. We claim that (ϕv_n) is Cauchy (up to a subsequence) in $L^\infty([0,T]; B^0_{\infty,\infty}(\mathbb{R}^2))$. By Proposition 8, Theorem 3, and Lemma 3, given $\epsilon > 0$, there exists $\delta > 0$ such that for all n,

$$\|\phi v_n(t) - \phi v_n(s)\|_{B^0_{\infty,\infty}} < \epsilon/3 \tag{4.24}$$

whenever s, $t \in [0,T]$ satisfy $|t-s| < \delta$. Given this δ, construct a partition of $[0,T]$, $0 = t_0 < t_1 < .... < t_N = T$, such that $t_{i+1} - t_i < \delta$. Equiboundedness of (ϕv_n) in $C^{1-\delta}(\mathbb{R}^2)$ on $[0,T]$ (which gives equicontinuity of $(\phi v_n(t))$ for each $t \in [0,T]$) implies via the Arzela-Ascoli Theorem that there exists a subsequence of (ϕv_n), which we relabel (ϕv_n), such that for each t_i in our partition, $(\phi v_n(t_i))$ is Cauchy in the $L^\infty(\mathbb{R}^2)$-norm. Let N be such that for all m, $n \geq N$ and for all t_i in our partition,

$$\|\phi v_n(t_i) - \phi v_m(t_i)\|_{L^\infty} < \epsilon/3. \tag{4.25}$$

By (4.24) and (4.25), for all pairs m, $n \geq N$ and for each $t \in [0,T]$, there exists t_i such that

$$\begin{aligned} \|\phi v_n(t) - \phi v_m(t)\|_{B^0_{\infty,\infty}} &\leq \|\phi v_n(t) - \phi v_n(t_i)\|_{B^0_{\infty,\infty}} \\ &+ \|\phi v_n(t_i) - \phi v_m(t_i)\|_{B^0_{\infty,\infty}} + \|\phi v_m(t_i) - \phi v_m(t)\|_{B^0_{\infty,\infty}} < \epsilon, \end{aligned}$$

where we used the embedding $L^\infty \hookrightarrow B^0_{\infty,\infty}$. We conclude that (ϕv_n) is Cauchy in $L^\infty([0,T]; B^0_{\infty,\infty}(\mathbb{R}^2))$. Now note that for every integer $M \geq -1$,

$$\begin{aligned} \|f\|_{L^\infty} \leq \|f\|_{B^0_{\infty,1}} &\leq (M+1)\|f\|_{B^0_{\infty,\infty}} + \|f\|_{B^{1-\delta}_{\infty,\infty}} \sum_{j \geq M} 2^{-(1-\delta)j} \\ &\leq (M+1)\|f\|_{B^0_{\infty,\infty}} + \|f\|_{B^{1-\delta}_{\infty,\infty}} C 2^{-(1-\delta)M}. \end{aligned}$$

We infer that

$$\|f\|_{L^\infty} \leq \rho \|f\|_{B^{1-\delta}_{\infty,\infty}} + C(\delta,\rho)\|f\|_{B^0_{\infty,\infty}} \text{ for all } \rho > 0. \tag{4.26}$$

Assume $\|\phi v_n\|_{L^\infty([0,T]; B^{1-\delta}_{\infty,\infty})} \leq K$ for all n. Given $\epsilon > 0$, choose $\rho = \epsilon/4K$, . Given this choice of ρ, choose N such that for all m,$n \geq N$,

$$\|\phi v_n - \phi v_m\|_{L^\infty([0,T]; B^0_{\infty,\infty})} < \frac{\epsilon}{2C(\delta,\rho)}.$$

By (4.26), for all m,$n \geq N$,

$$\|\phi v_n - \phi v_m\|_{L^\infty([0,T]; L^\infty)} < \epsilon.$$

Thus (ϕv_n) converges in $L^\infty([0,T]; L^\infty(\mathbb{R}^2))$. This implies that (ϕu_n) also converges in $L^\infty([0,T]; L^\infty(\mathbb{R}^2))$ (just choose N such that $\|\phi v_n - \phi v_m\|_{L^\infty([0,T]; L^\infty)} < \epsilon/\|h\|_{L^\infty(B_R(0))}$, where the support of ϕ is contained in $B_R(0)$).

We claim that (ϕu_n) converges to a function ϕu in $L^\infty([0,T];L^\infty(\mathbb{R}^2))$, where u is a weak solution to (E). For fixed $t\in[0,T]$ (suppressed to simplify notation), observe that, up to subsequences, (v_n) converges weak-* in $L^\infty(\mathbb{R}^2)$ to some v. It follows that (u_n) converges to $u=hv$ in $\mathcal{S}'(\mathbb{R}^2)$, as, for any Schwartz function ψ,

$$\int (u_n-u)\psi=\int (hv_n-hv)\psi=\int (v_n-v)h\psi\to 0 \text{ as } n\to\infty,$$

since $h\psi$ belongs to $L^1(\mathbb{R}^2)$. This implies (ϕu_n) converges to (ϕu) in $\mathcal{S}'(\mathbb{R}^2)$, so ϕu_n converges to ϕu in $L^\infty([0,T];L^\infty(\mathbb{R}^2))$.

We conclude that for any $R>0$, there exists a sequence (u_n) converging to u in $L^\infty([0,T];L^\infty(B_R(0)))$. Recall that given $R>0$, this sequence (u_n) is a subsequence (which depends on R) of the original sequence of smooth velocities. By a standard diagonalization argument, we can find one sequence (which we relabel (u_n)) that converges to u in $L^\infty([0,T];L^\infty(B_R(0)))$ for every $R>0$. This convergence is sufficient to show that u satisfies Definition 2.

To see that the weak solution (u,ω) satisfies the conditions of Theorem 1, recall that for fixed $t\in[0,T]$,

$$v_n\to v=gu \text{ weak-* in } L^\infty(\mathbb{R}^2).$$

Therefore, for each $t\in[0,T]$,

$$\|g(\cdot)u(t,\cdot)\|_{L^\infty}\le e^{C_0e^{C_0t}},$$

where C_0 depends on $\|gu^0\|_{L^\infty}$ and $\|\omega^0\|_{L^\infty}$. Moreover, since $\|\omega_n(t)\|_{L^\infty}\le\|\omega^0\|_{L^\infty}+\|gu^0\|_{L^\infty}$ for all n, up to subsequences, ω_n converges weak-* in $L^\infty(\mathbb{R}^2)$. Since (u_n) converges to u in $\mathcal{S}'(\mathbb{R}^2)$, (ω_n) converges to $\omega=\nabla\times u$ in $\mathcal{S}'(\mathbb{R}^2)$, so ω_n converges weak-* to ω in $L^\infty(\mathbb{R}^2)$. We conclude that for each $t\in[0,T]$,

$$\|\omega(t)\|_{L^\infty}\le\|\omega^0\|_{L^\infty}+\|gu^0\|_{L^\infty}.$$

This completes the proof of Theorem 1.

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