# VANISHING VISCOSITY IN THE PLANE FOR NONDECAYING VELOCITY AND VORTICITY II

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ABSTRACT. We consider solutions to the two-dimensional incompressible Navier-Stokes and Euler equations for which velocity and vorticity are bounded in the plane. We show that for every $T > 0$, the Navier-Stokes velocity converges in $L^\infty([0,T]; L^\infty(\mathbb{R}^2))$ to the Euler velocity generated from the same initial data as viscosity approaches 0. This result improves upon the work of [7], where we prove that the vanishing viscosity limit holds on a sufficiently short time interval, and upon [8], where we assume decay of the velocity vector field at infinity.

## 1. INTRODUCTION

In this paper, we study the vanishing viscosity limit of solutions to the two-dimensional incompressible Navier-Stokes equations. Recall that the Navier-Stokes equations modeling incompressible viscous fluid flow on $\mathbb{R}^n$ are given by

$$(NS) \quad \begin{cases} \partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu = -\nabla p_\nu \\ \text{div } u_\nu = 0 \\ u_\nu|_{t=0} = u_\nu^0. \end{cases}$$

When $\nu = 0$, the Navier-Stokes equations reduce to the Euler equations modeling incompressible inviscid fluid flow on $\mathbb{R}^n$:

$$(E) \quad \begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p \\ \text{div } u = 0 \\ u|_{t=0} = u^0. \end{cases}$$

We consider solutions to $(NS)$ and $(E)$ with bounded velocity and vorticity which do not necessarily decay at infinity. We show that such solutions to $(NS)$ converge to solutions of $(E)$ with the same initial data in the $L^\infty$-norm, where convergence is uniform over any finite time interval. This result builds upon and is a continuation of work

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in [7] and [8]. For this reason, we will often refer to [7] and [8] for background information and useful estimates.

The existence and uniqueness of solutions to $(NS)$ without any decay assumptions on the initial velocity is considered by Giga, Inui, and Matsui in [9]. The authors establish the short-time existence and uniqueness of mild solutions $v_\nu$ to $(NS)$ in the space $C([0,T_0];BUC(\mathbb{R}^n))$ when initial velocity is in $BUC(\mathbb{R}^n)$, $n \geq 2$. Here $BUC(\mathbb{R}^n)$ denotes the space of bounded uniformly continuous functions on $\mathbb{R}^n$. In [10], Giga, Matsui, and Sawada prove that when $n = 2$, the unique solution can be extended globally in time. Existence and uniqueness of solutions to $(E)$ with bounded velocity and vorticity with $n = 2$ is due to Serfati [14]. We briefly discuss these results in Section 2.

In this paper we prove that solutions $u_\nu$ to $(NS)$ of [10] converge uniformly on $\mathbb{R}^2$ to Serfati solutions to $(E)$ as viscosity approaches 0, where convergence is uniform over any finite time interval (see Theorem 3). To establish the result, we apply Littlewood-Paley theory and Bony's paradifferential calculus [2] and follow the general strategy of [7] and [8]. Specifically, we consider low and high frequencies of the difference between the the solutions to $(NS)$ and $(E)$ separately. We first show that for fixed $t$ and for any positive integer $n$,

$$\|u_\nu(t) - u(t)\|_{L^\infty} \leq n\|u_\nu(t) - u(t)\|_{B^0_{\infty,\infty}} + 2^{-n}\|\omega_\nu(t) - \omega(t)\|_{L^\infty}. \tag{1.1}$$

(See [7] for a definition of the Besov space $B^0_{\infty,\infty}$.) Letting $n$ be a function of $\nu$ so that as $\nu$ approaches 0, $n$ approaches infinity, we show that the right hand side of (1.1) approaches 0 and $n$ approaches infinity. Since the second term on the right hand side of (1.1) can be bounded above by $2^{-n}(\|\omega_\nu(t)\|_{L^\infty} + \|\omega(t)\|_{L^\infty})$, we have essentially reduced the problem to proving that the vanishing viscosity limit holds in the $B^0_{\infty,\infty}$-norm. Since $L^\infty$ embeds continuously into $B^0_{\infty,\infty}$, we expect this problem to be easier than proving that the vanishing viscosity limit holds in the $L^\infty$-norm; however we must establish a rate of convergence sufficiently fast to combat the growth of the factor of $n$ in front of the Besov norm.

Working in the Besov space $B^0_{\infty,\infty}$ has several advantages over working in $L^\infty$. Recall that for two-dimensional fluids we can express the Euler velocity gradient in terms of its vorticity by the relation $\nabla u = \nabla\nabla^\perp\Delta^{-1}\omega$. We can also express the Euler pressure in terms of velocity by the equality $p(t) = \sum_{i,j=1}^2 R_iR_ju_iu_j(t)$, where $R_i$ denotes the Riesz operator (similar relations hold for the Navier-Stokes velocity, vorticity, and pressure). The main mathematical obstacle when studying solutions to fluid equations in $L^\infty$ is the lack of boundedness

of the Calderon-Zygmund operators $\nabla\nabla^{\perp}\Delta^{-1}$ and $R_jR_i$ on $L^\infty$. However, if we let $\Delta_j$ denote the Littlewood-Paley operator which projects in frequency space onto an annulus with inner and outer radius of order $2^j$, then for any $j \geq 0$, $f \in \mathcal{S}'$, and Calderon-Zygmund operator $A$, we have

$$\|\Delta_j A f\|_{L^\infty} \leq \|\Delta_j f\|_{L^\infty}. \tag{1.2}$$

Therefore, when proving estimates in the $B^0_{\infty,\infty}$-norm, we can localize the frequencies of $(NS)$ and $(E)$ by applying the Littlewood-Paley operator $\Delta_j$ to the equations. We can then estimate the difference $\Delta_j(u_\nu - u)$ in the $L^\infty$-norm using (1.2). The presence of the Littlewood-Paley operator thus facilitates estimates for velocity gradients and pressure terms.

In [7], we prove that when $u, u_\nu, \omega$, and $\omega_\nu$ belong to $L^\infty_{loc}(\mathbb{R}^+; L^\infty(\mathbb{R}^2))$, there exists $T > 0$ such that

$$\|u_\nu - u\|_{L^\infty([0,T];L^\infty(\mathbb{R}^2))} \to 0 \text{ as } \nu \to 0. \tag{1.3}$$

To show (1.3), we reduce the problem to showing that the vanishing viscosity limit holds in the homogeneous $\dot{B}^0_{\infty,\infty}$-norm, but we are only able to show convergence in this norm for short time. In this paper, we show that (1.3) holds for every $T > 0$ by showing that the vanishing viscosity limit holds in the inhomogeneous $B^0_{\infty,\infty}$-norm on *any* finite time interval $[0, T]$.

We remark that this improvement of our previous result is not a consequence of using the inhomgeneous norm in place of the homogeneous norm. In fact, we are able to prove the same convergence result regardless of which Besov norm we use (the proof using the inhomogeneous norm is cleaner). Rather, in this paper we are able to improve upon the results in [7] because we change our approach when estimating the commutator resulting from an application of the Littlewood-Paley operator to the nonlinear terms in $(NS)$ and $(E)$. Our approach here is similar to those in [16], [1], and [15]. As a result of our methods, we are able to prove the estimate

$$\begin{aligned} \|(u_\nu - u)(t)\|_{B^0_{\infty,\infty}} &\leq C(T)2^{-n\alpha} \\ &+ \int_0^t C\left(2^{-p} + p\|(u_\nu - u)(s)\|_{B^0_{\infty,\infty}}\right) \end{aligned} \tag{1.4}$$

for any $p \in [2, \infty)$. By choosing $p$ as a logarithmic function of $\|u_\nu - u\|_{B^0_{\infty,\infty}}$, we are able to apply Osgood's Lemma, yielding a rate of convergence. In [7], our methods only allow us to prove an estimate similar to (1.4) with $n$ in place of $p$. Since $n$ is a function of viscosity, we must apply Gronwall's Lemma and introduce a factor of $e^{nt}$ on the right

hand side, which prevents us from proving that the inviscid limit holds on any finite time interval.

The paper is organized as follows. In Section 2, we review properties of nondecaying solutions to the fluid equations. In Section 3 and Section 4, we state and prove the main result; we devote Section 4 entirely to showing that the vanishing viscosity limit holds in the $B^0_{\infty,\infty}$-norm.

For background information on Littlewood-Paley theory, Bony's paraproduct decomposition, Besov spaces, and technical lemmas used throughout the paper, we refer the reader to Section 2 of [7].

## 2. Existence and Uniqueness of Nondecaying Solutions to the Fluid Equations

In this section, we briefly summarize what is known about nondecaying solutions to $(NS)$ and $(E)$. We begin with the mild solutions to $(NS)$ established in [9]. By a mild solution to $(NS)$, we mean a solution $u_\nu$ of the integral equation

$$u_\nu(t,x) = e^{t\nu\Delta}u_\nu^0 - \int_0^t e^{(t-s)\nu\Delta}\mathbf{P}(u_\nu\cdot\nabla u_\nu)(s)ds. \tag{2.1}$$

In (2.1), $e^{\tau\nu\Delta}$ denotes convolution with the Gauss kernel; that is, for $f\in S'$, $e^{\tau\nu\Delta}f = G_{\tau\nu}*f$, where $G_{\tau\nu}(x) = \frac{1}{4\pi\tau\nu}\exp\{\frac{-|x|^2}{4\tau\nu}\}$. Also, $\mathbf{P}$ denotes the Helmholtz projection operator with $ij$-component given by $\delta_{ij}+R_iR_j$, where $R_l = (-\Delta)^{-\frac{1}{2}}\partial_l$ is the Riesz operator. In [9], Giga, Inui, and Matsui prove the following result regarding mild solutions in $\mathbb{R}^n$, $n\geq 2$.

**Theorem 1.** *Let $BUC$ denote the space of bounded, uniformly continuous functions, and assume ${u_\nu}^0$ belongs to $BUC(\mathbb{R}^n)$ for fixed $n\geq 2$. There exists a $T_0>0$ and a unique solution to (2.1) in the space $C([0,T_0];BUC(\mathbb{R}^n))$ with initial data ${u_\nu}^0$. Moreover, if we assume div ${u_\nu}^0 = 0$, and if we define $p_\nu(t) = \sum_{i,j=1}^2 R_iR_ju_{\nu i}u_{\nu j}(t)$ for each $t\in[0,T_0]$, then $u_\nu$ belongs to $C^\infty([0,T_0]\times\mathbb{R}^n)$ and solves $(NS)$.*

*Remark* 2.2. For the main theorem of this paper, we assume that $\omega^0$ is bounded on $\mathbb{R}^2$, which implies that $u_\nu^0$ belongs to $C^\alpha(\mathbb{R}^2)$ for every $\alpha<1$ and is therefore in $BUC(\mathbb{R}^2)$ (see, for example, Lemma 4 of [7]).

In [10], Giga, Matsui, and Sawada show that when $n=2$, the solution to $(NS)$ established in Theorem 1 can be extended to a global-in-time smooth solution. Moreover, in [13], Sawada and Taniuchi show that if $u_\nu^0$ and $\omega_\nu^0$ belong to $L^\infty(\mathbb{R}^2)$, then the following exponential estimate holds:

$$||u_\nu(t)||_{L^\infty}\leq C||u_\nu^0||_{L^\infty}e^{Ct||\omega_\nu^0||_{L^\infty}}. \tag{2.3}$$

For ideal incompressible fluids, Serfati proves the following existence and uniqueness result in [14].

**Theorem 2.** *Let $u^0$ and $\omega^0$ belong to $L^\infty(\mathbb{R}^2)$, and let $c \in \mathbb{R}$. For every $T > 0$ there exists a unique solution $(u, p)$ to $(E)$ in the space $L^\infty([0,T]; L^\infty(\mathbb{R}^2)) \times L^\infty([0,T]; C(\mathbb{R}^2))$ with $\omega \in L^\infty([0,T]; L^\infty(\mathbb{R}^2))$, $p(0) = c$, and with $\frac{p(t,x)}{|x|} \to 0$ as $|x| \to \infty$.*

Serfati also proved an estimate analogous to (2.3) for his solutions:

$$||u(t)||_{L^\infty} \leq C||u^0||_{L^\infty} e^{C_1||\omega^0||_{L^\infty} t}. \tag{2.4}$$

Finally, we recall that we have a uniform bound in time on the $L^\infty$-norms of the vorticities corresponding to the solutions of $(NS)$ and $(E)$. For fixed $\nu \geq 0$, we have that

$$||\omega_\nu(t)||_{L^\infty} \leq ||\omega_\nu^0||_{L^\infty} \tag{2.5}$$

for all $t \geq 0$. One can prove this bound by applying the maximum principle to the vorticity formulations of $(NS)$ and $(E)$. We refer the reader to Lemma 3.1 of [13] for a detailed proof.

## 3. Statement and Proof of the Main Result

We are now prepared to state the main theorem.

**Theorem 3.** *Let $u_\nu$ be the unique solution to $(NS)$ and $u$ the unique solution to $(E)$, both with initial data $u^0$ and $\omega^0$ belonging to $L^\infty(\mathbb{R}^2)$, and with $p_\nu$ and $p$ satisfying the conditions of Theorems 1 and 2, respectively. Then there exists a constant $M$, depending only on $\|u_\nu\|_{L^\infty}$, $\|u\|_{L^\infty}$, $\|\omega_\nu\|_{L^\infty}$,and $\|\omega\|_{L^\infty}$, such that the following estimate holds for any fixed $T > 0$ and for any $\alpha \in (0,1)$:*

$$||u_\nu - u||_{L^\infty([0,T];L^\infty(\mathbb{R}^2))} \leq C_1(2 - \log(\sqrt{\nu})^{\alpha e^{-C_2}})(\sqrt{\nu})^{\alpha e^{-C_2}}. \tag{3.1}$$

*Proof.* Throughout the proof of Theorem 3, we let $M$ denote a constant, dependent on $T$, which satisfies

$$M \geq 1 + \sup_{t\in[0,T]} (\|u_\nu(t)\|_{L^\infty} + \|u(t)\|_{L^\infty} + \|\omega_\nu(t)\|_{L^\infty} + \|\omega(t)\|_{L^\infty}). \tag{3.2}$$

We note that the value of $M$ will change throughout the proof but will always satisfy (3.2). The existence results in Section 2 imply that $M$ will be finite for any $T > 0$.

Let $u$ be the unique Serfati solution to $(E)$, and let $u_\nu$ be the unique solution to $(NS)$ given by [10]. We fix $n$ to be a positive integer, and we fix $T > 0$. We will eventually choose $n = -\frac{1}{2}\log_2 \nu$ so that, as $\nu$ approaches 0, $n$ approaches infinity.

We begin with the following inequality:

$$\begin{aligned}\|u_\nu - u\|_{L^\infty([0,T];L^\infty)} &\leq \sum_{j=-1}^{n} \|\Delta_j(u_\nu - u)\|_{L^\infty([0,T];L^\infty)} \\ &+ \sum_{j=n+1}^{\infty} \|\Delta_j(u_\nu - u)\|_{L^\infty([0,T];L^\infty)}.\end{aligned} \tag{3.3}$$

We can estimate the second term on the right hand side of (3.3) using Bernstein's Lemma and the estimate

$$\|\Delta_j \nabla u\|_{L^\infty} \leq \|\Delta_j \omega\|_{L^\infty} \text{ for } j \geq 0. \tag{3.4}$$

(Both (3.4) and Bernstein's Lemma can be found in Section 2 of [7].) We obtain the inequality

$$\begin{aligned}&\sum_{j=n+1}^{\infty} \|\Delta_j(u_\nu - u)\|_{L^\infty([0,T];L^\infty)} \\ &\leq \sum_{j=n+1}^{\infty} 2^{-j} \|\Delta_j(\nabla u_\nu - \nabla u)\|_{L^\infty([0,T];L^\infty)} \leq M 2^{-n}.\end{aligned} \tag{3.5}$$

To estimate the first term on the right hand side of (3.3), we use the definition of $B^0_{\infty,\infty}$ to observe that

$$\sum_{j=-1}^{n} \|\Delta_j(u_\nu - u)\|_{L^\infty([0,T];L^\infty)} \leq Cn \|u_\nu - u\|_{L^\infty([0,T];B^0_{\infty,\infty})}. \tag{3.6}$$

After substituting (3.6) and (3.5) into (3.3), we conclude that

$$\|u_\nu - u\|_{L^\infty([0,T];L^\infty)} \leq Cn \|u_\nu - u\|_{L^\infty([0,T];B^0_{\infty,\infty})} + M 2^{-n}. \tag{3.7}$$

We must estimate the difference of $u_\nu$ and $u$ in the $B^0_{\infty,\infty}$-norm. We temporarily assume that the following estimate holds for all $\alpha \in (0,1)$:

$$||u_\nu - u||_{L^\infty([0,T];B^0_{\infty,\infty})} \leq M(T+1)(2^{-n\alpha})^{e^{-MT}}. \tag{3.8}$$

Assuming that (3.8) holds, we see from (3.7) and (3.8) that

$$\|u_\nu - u\|_{L^\infty([0,T];L^\infty)} \leq M(T+1)(2^{-n\alpha})^{e^{-MT}}.$$

The estimate (3.1) follows after setting $\nu = 2^{-2n}$. Therefore, to complete the proof of Theorem 3, it remains to prove (3.8).

## 4. Proof of (3.8)

We begin with some notation. We let $u_n = S_n u$, $\omega_n = S_n \omega(u)$, $\bar{u}_n = u_\nu - u_n$, and $\bar{\omega}_n = \omega_\nu - \omega_n$. Throughout most of the proof of (3.8), the time $t$ is fixed and suppressed in the calculations.

Fix $p \in (1, \infty)$ (to be chosen later). We apply Bernstein's Lemma and (3.4) to establish the estimate

$$\begin{aligned} \|u_\nu - u\|_{B^0_{\infty,\infty}} \leq & \sup_{-1\leq l\leq 2} \|\Delta_{-1}(u_\nu - u)\|_{L^\infty} + \sup_{3\leq l\leq p} 2^{-l}\|\Delta_l(\omega_\nu - \omega)\|_{L^\infty} \\ & + \sup_{l>p} 2^{-l}\|\Delta_l(\omega_\nu - \omega)\|_{L^\infty}. \end{aligned} \tag{4.1}$$

The separation of frequencies at $l = 2$ will simplify estimates in what follows.

We will first consider the difference $\sup_{3\leq l\leq p} 2^{-l}\|\Delta_l(\omega_\nu - \omega)\|_{L^\infty}$. We will eventually need to estimate the viscosity term $\nu\|\Delta\omega\|_{L^\infty}$. To facilitate this estimate, we smooth out the Euler vorticity and write

$$\begin{aligned} \sup_{3\leq l\leq p} 2^{-l}\|\Delta_l(\omega_\nu - \omega)\|_{L^\infty} &\leq \sup_{3\leq l\leq p} 2^{-l}\|\Delta_l\bar{\omega}_n\|_{L^\infty} \\ &\quad + \sup_{3\leq l\leq p} 2^{-l}\|\Delta_l(\omega_n - \omega)\|_{L^\infty} \\ &\leq \sup_{3\leq l\leq p} 2^{-l}\|\Delta_l\bar{\omega}_n\|_{L^\infty} + \sup_{l\geq n} 2^{-l}\|\Delta_l(\omega_n - \omega)\|_{L^\infty} \\ &\leq \sup_{3\leq l\leq p} 2^{-l}\|\Delta_l\bar{\omega}_n\|_{L^\infty} + M2^{-n}, \end{aligned} \tag{4.2}$$

where we used properties of the Fourier support of $\omega_n$ to get the second inequality. We now estimate $\sup_{3\leq l\leq p} 2^{-l}\|\Delta_l\bar{\omega}_n\|_{L^\infty}$. We note that $\omega_\nu$ and $\omega_n$ satisfy the following two equations:

$$\partial_t\omega_\nu + u_\nu \cdot \nabla\omega_\nu - \nu\Delta\omega_\nu = 0, \tag{4.3}$$

and

$$\partial_t\omega_n + u_n \cdot \nabla\omega_n = \nabla \cdot \tau_n(u, \omega), \tag{4.4}$$

where

$$\tau_n(u, \omega) = (u - u_n)(\omega - \omega_n) - r_n(u, \omega),$$

and

$$r_n(u, \omega) = \int \check{\psi}(y)(u(x - 2^{-n}y) - u(x))(\omega(x - 2^{-n}y) - \omega(x))dy.$$

The equation (4.4) was utilized by Constantin and Wu in [6] and by Constantin, E, and Titi in a proof of Onsager's conjecture in [5]. We

subtract (4.4) from (4.3) and, for fixed $l$, we apply the Littlewood-Paley operator $\Delta_l$ to the difference of the two equations. After adding $(S_{l-2}u_\nu)\cdot\nabla\Delta_l\bar{\omega}_n$ to both sides of the resulting equation, we obtain

(4.5)
$$\partial_t\Delta_l\bar{\omega}_n + (S_{l-2}u_\nu)\cdot\nabla\Delta_l\bar{\omega}_n - \nu\Delta\Delta_l\bar{\omega}_n = (S_{l-2}u_\nu)\cdot\nabla\Delta_l\bar{\omega}_n - \Delta_l(u_\nu\cdot\nabla\bar{\omega}_n) - \Delta_l(\bar{u}_n\cdot\nabla\omega_n) + \nu\Delta\Delta_l\omega_n - \Delta_l\nabla\cdot\tau_n(u,\omega).$$

Borrowing notation from [15], we define

$$\begin{aligned} I^{l,k} &= (S_{l-2}u_\nu^k)\partial_k\Delta_l\bar{\omega}_n - \partial_k\Delta_l(u_\nu^k\bar{\omega}_n), \\ J^{l,k} &= -\partial_k\Delta_l(\bar{u}_n^k\omega_n). \end{aligned} \tag{4.6}$$

From (4.5), we see that

$$\begin{aligned} &\partial_t\Delta_l\bar{\omega}_n + (S_{l-2}u_\nu)\cdot\nabla\Delta_l\bar{\omega}_n - \nu\Delta\Delta_l\bar{\omega}_n \\ &\qquad = \sum_{k=1}^{2}(I^{l,k}+J^{l,k}) + \nu\Delta\Delta_l\omega_n - \Delta_l\nabla\cdot\tau_n(u,\omega). \end{aligned} \tag{4.7}$$

Since $S_{l-2}u_\nu$ belongs to $L^1_{loc}(\mathbb{R}^+;Lip(\mathbb{R}^2))$ and is divergence-free, we can apply the following lemma for the transport diffusion equation from [11].

**Lemma 1.** *Let $p\in[1,\infty]$, and let $u$ be a divergence-free vector field belonging to $L^1_{loc}(\mathbb{R}^+;Lip(\mathbb{R}^d))$. Moreover, assume the function $f$ belongs to $L^1_{loc}(\mathbb{R}^+;L^p(\mathbb{R}^d))$ and the function $a^0$ belongs to $L^p(\mathbb{R}^d)$. Then any solution $a$ to the problem*

$$\begin{cases} \partial_t a + u\cdot\nabla a - \nu\Delta a = f, \\ a|_{t=0} = a^0 \end{cases}$$

*satisfies the following estimate:*

$$||a(t)||_{L^p} \le ||a^0||_{L^p} + \int_0^t ||f(s)||_{L^p}ds.$$

An application of Lemma 1 to (4.7) yields

(4.8)
$$\|\Delta_l\bar{\omega}_n(t)\|_{L^\infty} \le \|\Delta_l\bar{\omega}_n(0)\|_{L^\infty} + \int_0^t\left(\sum_{k=1}^{2}(\|I^{l,k}(s)\|_{L^\infty} + \|J^{l,k}(s)\|_{L^\infty}\right)ds$$
$$+ \int_0^t(\nu\|\Delta\Delta_l\omega_n(s)\|_{L^\infty} + \|\Delta_l\nabla\cdot\tau_n(u,\omega)(s)\|_{L^\infty})\,ds.$$

Our goal is to establish an upper bound for $\sup_{3\le l\le p}\|\Delta_l\bar{\omega_n}(t)\|_{L^\infty}$. In what follows, we will estimate each term on the right hand side of (4.8), multiply by $2^{-l}$, and take the supremum over $l$ satisfying $3\le$

$l \leq p$. Estimates for the last two terms on the right hand side of (4.8) follow from work in [7]. Indeed, in [7] we use boundedness of the Euler vorticity and membership of the Euler velocity to $C^{\alpha}(\mathbb{R}^2)$ for any $\alpha \in (0,1)$ to show that for such $\alpha$,

$$\sup_{l\geq 0} 2^{-l}\|\Delta_l \nabla \cdot \tau_n(u,\omega)\|_{L^\infty} \leq \|\nabla \cdot \tau_n(u,\omega)\|_{L^\infty} \leq M2^{-n\alpha}. \tag{4.9}$$

We also showed in [7], using Bernstein's Lemma and properties of the Fourier support of $\omega_n$, that

$$\sup_{l\geq 0} 2^{-l}\nu\|\Delta_l \Delta\omega_n\|_{L^\infty} \leq 2^n\nu\|\omega_n\|_{L^\infty} \leq M2^{-n}, \tag{4.10}$$

where we set $\nu = 2^{-2n}$. To estimate the initial data, we utilize the Fourier support of $\omega_n^0 = S_n\omega^0$ to write

$$\sup_{3\leq l\leq p} 2^{-l}\|\Delta_l \bar{\omega}_n(0)\|_{L^\infty} \leq \sup_{l\geq n} 2^{-l}\|\Delta_l \bar{\omega}_n(0)\|_{L^\infty} \leq M2^{-n}. \tag{4.11}$$

Multiplying (4.8) by $2^{-l}$, taking the supremum of (4.8) over $l$ satisfying $3 \leq l \leq p$, and applying the estimates (4.9), (4.10), and (4.11) gives

$$\begin{aligned} \sup_{3\leq l\leq p} 2^{-l}\|\Delta_l \bar{\omega}_n(t)\|_{L^\infty} &\leq M(t+1)2^{-n\alpha} \\ &+ \sup_{3\leq l\leq p} 2^{-l}\int_0^t \left(\sum_{k=1}^{2}(\|I^{l,k}(s)\|_{L^\infty} + \|J^{l,k}(s)\|_{L^\infty}\right) ds. \end{aligned} \tag{4.12}$$

It remains to estimate $I^{l,k}$ and $J^{l,k}$. We begin with $J^{l,k}$. We again borrow notation from [15] and use Bony's paraproduct decomposition to write

$$\begin{aligned} J^{l,k} &= -\partial_k\Delta_l \sum_{|j-l|\leq 3, j\geq 1} S_{j-2}\bar{u}_n^k \Delta_j\omega_n \\ &- \partial_k\Delta_l \sum_{|j-l|\leq 3, j\geq 1} \Delta_j\bar{u}_n^k S_{j-2}\omega_n \\ &- \partial_k\Delta_l \sum_{|j-j'|\leq 1, \max\{j,j'\}\geq l-3} \Delta_j\bar{u}_n^k \Delta_{j'}\omega_n \\ &= J_1^{l,k} + J_2^{l,k} + J_3^{l,k}. \end{aligned} \tag{4.13}$$

We estimate $J_1^{l,k}$. Several applications of Bernstein's Lemma give

$$\begin{aligned} \|J_1^{l,k}\|_{L^\infty} &\leq 2^l \sum_{|j-l|\leq 3, j\geq 1} \|S_{j-2}\bar{u}_n\|_{L^\infty}\|\Delta_j\omega_n\|_{L^\infty} \\ &\leq 2^l \sum_{|j-l|\leq 3, j\geq 1} \|\Delta_j\omega\|_{L^\infty} \sum_{k\leq j-2} \|\Delta_k\bar{u}_n\|_{L^\infty}. \end{aligned} \tag{4.14}$$

Multiplying by $2^{-l}$ and taking the supremum over $l$ satisfying $3 \leq l \leq p$, we conclude that

$$\sup_{3\leq l\leq p} 2^{-l}\|J_1^{l,k}\|_{L^\infty} \leq Mp\|\bar{u}_n\|_{B^0_{\infty,\infty}}. \tag{4.15}$$

We now estimate $J_2^{l,k}$. We write

$$\begin{aligned}\|J_2^{l,k}\|_{L^\infty} &\leq 2^l \sum_{|j-l|\leq 3, j\geq 1} \|\Delta_l(\Delta_j\bar{u}_n S_{j-2}\omega_n)\|_{L^\infty} \\ &\leq 2^l \sum_{|j-l|\leq 3, j\geq 1} \|\Delta_j\bar{u}_n\|_{L^\infty}\|S_{j-2}\omega_n\|_{L^\infty}\end{aligned} \tag{4.16}$$

so that

$$\sup_{3\leq l\leq p} 2^{-l}\|J_2^{l,k}\|_{L^\infty} \leq M\|\bar{u}_n\|_{B^0_{\infty,\infty}}. \tag{4.17}$$

To estimate $J_3^{l,k}$, we use properties of Littlewood-Paley operators to observe that

$$\begin{aligned}\|J_3^{l,k}\|_{L^\infty} &\leq 2^l \sum_{|j-j'|\leq 1, \max\{j,j'\}\geq l-3} \|\Delta_j\bar{u}_n\|_{L^\infty}\|\Delta_{j'}\omega_n\|_{L^\infty} \\ &\leq C2^l \sum_{j\geq l-3} \|\Delta_j\bar{u}_n\|_{L^\infty}\|\Delta_j\omega_n\|_{L^\infty} \leq C2^l\|\omega\|_{L^\infty}\|\bar{u}_n\|_{B^0_{\infty,1}}.\end{aligned} \tag{4.18}$$

We estimate the $B^0_{\infty,1}$-norm of $\bar{u}_n$ as follows. We bound the low frequencies using the definition of $B^0_{\infty,\infty}$ and we estimate the high frequencies using Bernstein's lemma, (3.4), and boundedness of vorticity. We have the series of estimates

$$\begin{aligned}\|\bar{u}_n\|_{B^0_{\infty,1}} &\leq \sum_{j=-1}^{p} \|\Delta_j\bar{u}_n\|_{L^\infty} + \sum_{j>p} 2^{-j}\|\Delta_j\bar{\omega}_n\|_{L^\infty} \\ &\leq Cp\|\bar{u}_n\|_{B^0_{\infty,\infty}} + M2^{-p}.\end{aligned} \tag{4.19}$$

Substituting this estimate into (4.18), multiplying by $2^{-l}$ and taking the supremum over $l$ between 3 and $p$ yields the estimate

$$\sup_{3\leq l\leq p} 2^{-l}\|J_3^{l,k}\|_{L^\infty} \leq M\left(2^{-p} + p\|\bar{u}_n\|_{B^0_{\infty,\infty}}\right). \tag{4.20}$$

Combining the estimates for (4.15), (4.17), and (4.20), we conclude that

$$\sup_{3\leq l\leq p} 2^{-l}\sum_{k=1}^{2}\|J^{l,k}\|_{L^\infty} \leq M(2^{-p} + p\|\bar{u}_n\|_{B^0_{\infty,\infty}}). \tag{4.21}$$

We now estimate $I^{l,k}$ for $l$ satisfying $3 \leq l \leq p$. We apply Theorem 6.1 of [16] to write

$$\begin{aligned}\sum_{k=1}^{2} \|I^{l,k}\|_{L^\infty} &\leq C \sum_{|j-l|\leq 3} \|S_{j-2}\nabla\bar{\omega}_n\|_{L^\infty}\|\Delta_j u_\nu\|_{L^\infty} \\ &+ \sum_{|j-l|\leq 3} \|S_{j-2}\nabla u_\nu\|_{L^\infty}\|\Delta_j\bar{\omega}_n\|_{L^\infty} \\ &+ C2^l \sum_{j\geq l-3, |j-j'|\leq 1} 2^{-j}\|\Delta_j\nabla u_\nu\|_{L^\infty}\|\Delta_{j'}\bar{\omega}_n\|_{L^\infty} \\ &= X_1^l + X_2^l + X_3^l.\end{aligned}$$

To estimate $X_1^l$, keeping in mind that $l \geq 3$, we use Bernstein's Lemma and (3.4) to write

$$\sum_{|j-l|\leq 3} \|S_{j-2}\nabla\bar{\omega}_n\|_{L^\infty}\|\Delta_j u_\nu\|_{L^\infty} \leq C2^l \sum_{|j-l|\leq 3} \|S_{j-2}\bar{u}_n\|_{L^\infty}\|\Delta_j\omega_\nu\|_{L^\infty}.$$

The remainder of the estimate for $X_1^l$ is identical to that for $J_1^{l,k}$. Multiplying by $2^{-l}$ and taking the supremum over $l$ between 3 and $p$, we conclude that

$$\sup_{3\leq l\leq p} 2^{-l}X_1^l \leq Mp\|\bar{u}_n\|_{B^0_{\infty,\infty}}. \tag{4.22}$$

To estimate $X_2^l$ for $3 \leq l \leq p$, we again apply Bernstein's Lemma and (3.4) to write

$$\begin{aligned}X_2^l &= \sum_{|j-l|\leq 3} \|S_{j-2}\nabla u_\nu\|_{L^\infty}\|\Delta_j\bar{\omega}_n\|_{L^\infty} \\ &\leq C2^l \sum_{|j-l|\leq 3} (\|u_\nu\|_{L^\infty} + (j-1)\|\omega_\nu\|_{L^\infty})\|\Delta_j\bar{u}_n\|_{L^\infty} \\ &\leq Ml2^l \sum_{|j-l|\leq 3} \|\Delta_j\bar{u}_n\|_{L^\infty}.\end{aligned} \tag{4.23}$$

To get the first inequality above, we bounded the term $\|S_{j-2}\nabla u_\nu\|_{L^\infty}$ above by the sum resulting from the $S_{j-2}$ operator. We then applied (3.4). After multiplying (4.23) by $2^{-l}$ and taking the supremum over $l$ satisfying $3 \leq l \leq p$, we find that

$$\sup_{3\leq l\leq p} 2^{-l}X_2^l \leq Mp\|\bar{u}_n\|_{B^0_{\infty,\infty}}. \tag{4.24}$$

The estimate for $X_3^l$ is similar to that for $J_3^{l,k}$. For $l$ satisfying $3 \leq l \leq p$, we write

$$\begin{aligned} X_3^l &= C2^l \sum_{j \geq l-3, |j-j'| \leq 1} 2^{-j} \|\Delta_j \nabla u_\nu\|_{L^\infty} \|\Delta_{j'} \bar{\omega}_n\|_{L^\infty} \\ &\leq C2^l \sum_{j \geq l-3} \|\Delta_j \omega_\nu\|_{L^\infty} \|\Delta_j \bar{u}_n\|_{L^\infty}, \end{aligned} \tag{4.25}$$

where we used Bernstein's Lemma and (3.4) to get the last inequality. We now use the same argument as that in (4.18) and (4.19) to conclude that

$$\sup_{3 \leq l \leq p} 2^{-l} X_3^l \leq M \left( 2^{-p} + p \|\bar{u}_n\|_{B^0_{\infty,\infty}} \right). \tag{4.26}$$

Combining the above estimates for $X_1^l$, $X_2^l$, an $X_3^l$, we have

$$\sup_{3 \leq l \leq p} 2^{-l} \sum_{k=1}^{2} \|I^{l,k}\|_{L^\infty} \leq M \left( 2^{-p} + p \|\bar{u}_n\|_{B^0_{\infty,\infty}} \right). \tag{4.27}$$

Applying the estimates (4.21) and (4.27) to (4.12), we conclude that

$$\sup_{3 \leq l \leq p} 2^{-l} \|\Delta_l \bar{\omega}_n(t)\|_{L^\infty} \leq C(t+1)2^{-n\alpha} + M \int_0^t \left( 2^{-p} + p \|W(s)\|_{\dot{B}^0_{\infty,\infty}} \right) ds \tag{4.28}$$

for any $\alpha \in (0,1)$. We substitute (4.28) into (4.2). This gives

$$\begin{aligned} \sup_{3 \leq l \leq p} 2^{-l} \|\Delta_l (\omega_\nu - \omega)(t)\|_{L^\infty} &\leq C(t+1)2^{-n\alpha} \\ &+ M \int_0^t \left( 2^{-p} + p \|\bar{u}_n(s)\|_{\dot{B}^0_{\infty,\infty}} \right) ds. \end{aligned} \tag{4.29}$$

Inspection of (4.1) reveals that we must still estimate $\sup_{-1 \leq l \leq 2} \|\Delta_l (u_\nu - u)(t)\|_{L^\infty}$ and $\sup_{l>p} 2^{-l} \|\Delta_l (\omega_\nu - \omega)(t)\|_{L^\infty}$. These two terms are more straightforward. We estimate the term $\sup_{l>p} 2^{-l} \|\Delta_l (\omega_\nu - \omega)(t)\|_{L^\infty}$ by observing that

$$\sup_{l>p} 2^{-l} \|\Delta_l (\omega_\nu - \omega)(t)\|_{L^\infty} \leq M 2^{-p}. \tag{4.30}$$

To estimate $\sup_{-1 \leq l \leq 2} \|\Delta_l (u_\nu - u)(t)\|_{L^\infty}$, we use the velocity formulation. Setting $\bar{p} = p_\nu - p$ and $\bar{u} = u_\nu - u$, we subtract $(E)$ from $(NS)$. This gives

$$\partial_t \bar{u} + u_\nu \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u - \nu \Delta \bar{u} = -\nabla \bar{p} + \nu \Delta u_\nu. \tag{4.31}$$

We apply $\Delta_l$ to (4.31) for $-1 \leq l \leq 2$. This gives

$$\begin{aligned}\partial_t \Delta_l \bar{u} + (\Delta_l u_\nu) \cdot \nabla \Delta_l \bar{u} - \nu \Delta_l \Delta \bar{u} = (\Delta_l u_\nu) \cdot \nabla \Delta_l \bar{u} \\ - \Delta_l (u_\nu \cdot \nabla \bar{u}) - \Delta_l (\bar{u} \cdot \nabla u) - \Delta_l \nabla \bar{p} + \nu \Delta_l \Delta u_\nu.\end{aligned} \tag{4.32}$$

Again by Lemma 1, we have

(4.33)

$$\begin{aligned}\|\Delta_l \bar{u}(t)\|_{L^\infty} \leq \int_0^t (\|(\Delta_l u_\nu \cdot \nabla \Delta_l \bar{u})(s)\|_{L^\infty} + \|\Delta_l (u_\nu \cdot \nabla \bar{u})(s)\|_{L^\infty} \\ + \|\Delta_l (\bar{u} \cdot \nabla u)(s)\|_{L^\infty} + \|\Delta_l \nabla \bar{p}(s)\|_{L^\infty} + \nu \|\Delta_l \Delta u_\nu(s)\|_{L^\infty})\, ds.\end{aligned}$$

We have the following straightforward estimates, all which follow from Bernstein's Lemma and the divergence-free property of the velocity:

$$\begin{aligned}&\|(\Delta_l u_\nu) \cdot \nabla \Delta_l \bar{u}\|_{L^\infty} \leq C \|u_\nu\|_{L^\infty} 2^l \|\Delta_l \bar{u}\|_{L^\infty} \leq M 2^l \|\bar{u}\|_{L^\infty}, \\ &\|\Delta_l (u_\nu \cdot \nabla \bar{u})\|_{L^\infty} \leq C 2^l \|u_\nu\|_{L^\infty} \|\bar{u}\|_{L^\infty} \leq M 2^l \|\bar{u}\|_{L^\infty}, \\ &\|\Delta_l (\bar{u} \cdot \nabla u)\|_{L^\infty} \leq 2^l \|\bar{u}\|_{L^\infty} \|u\|_{L^\infty} \leq M 2^l \|\bar{u}\|_{L^\infty}, \\ &\nu \|\Delta_l \Delta u_\nu\|_{L^\infty} \leq C \nu 2^{2l} \|u_\nu\|_{L^\infty} \leq M \nu 2^{2l}.\end{aligned} \tag{4.34}$$

To estimate the pressure, we follow an argument in [15]. For $0 \leq l \leq 2$,

$$\begin{aligned}\|\Delta_l \nabla \bar{p}\|_{L^\infty} &= \left\| \sum_{i,i'=1}^{2} R_i R_{i'} \nabla \Delta_l (\bar{u}^i u^{i'} + u_\nu^i \bar{u}^{i'}) \right\|_{L^\infty} \\ &\leq \|R_i R_{i'} \nabla \check{\varphi}_l\|_{L^1} \|\bar{u}^i u^{i'} + u_\nu^i \bar{u}^{i'}\|_{L^\infty} \leq M 2^l \|\bar{u}\|_{L^\infty},\end{aligned} \tag{4.35}$$

where we applied the series of estimates $\|R_i R_{i'} \nabla \check{\varphi}_l\|_{L^1} \leq \|R_i R_{i'} \nabla \check{\varphi}_l\|_{\mathcal{H}^1} \leq \|\nabla \check{\varphi}_l\|_{\mathcal{H}^1} \leq C 2^l$ to get the last inequality. For the case $l = -1$, we apply the same series of estimates as in (4.35) with $\check{\psi}$ in place of $\check{\varphi}_l$.

Substituting the estimates (4.34) and (4.35) into (4.33) and taking the supremum over $-1 \leq l \leq 2$ yields

$$\sup_{-1 \leq l \leq 2} \|\Delta_l \bar{u}(t)\|_{L^\infty} \leq M \int_0^t (\|\bar{u}\|_{L^\infty} + 2^{-2n}), \tag{4.36}$$

where we used the equality $\nu = 2^{-2n}$. We now apply the embedding $B^0_{\infty,1} \hookrightarrow L^\infty$, along with (4.19) to conclude that

$$\sup_{-1 \leq l \leq 2} \|\Delta_l \bar{u}(t)\|_{L^\infty} \leq M t 2^{-2n} + M \int_0^t (p \|\bar{u}(s)\|_{B^0_{\infty,\infty}} + 2^{-p})\, ds. \tag{4.37}$$

We substitute the estimates (4.37), (4.29), and (4.30) into (4.1). We conclude that

$$\begin{aligned}\sup_{l\geq -1}\|\Delta_l\bar{u}(t)\|_{L^\infty} &\leq M(T+1)2^{-n\alpha}\\ &\quad + M2^{-p} + \int_0^t M\left(2^{-p} + p\|\bar{u}(s)\|_{B^0_{\infty,\infty}}\right)ds.\end{aligned} \tag{4.38}$$

To complete the proof of (3.8), we will apply Osgood's Lemma to (4.38). We first note that by the embedding $L^\infty \hookrightarrow B^0_{\infty,\infty}$,

$$\|\bar{u}(t)\|_{B^0_{\infty,\infty}} \leq \|\bar{u}(t)\|_{L^\infty} \leq \|u_\nu(t)\|_{L^\infty} + \|u(t)\|_{L^\infty} \leq M$$

for all $t \in [0,T]$. For each $t \in [0,T]$, set

$$\delta(t) = \frac{\int_0^t \|\bar{u}(s)\|_{B^0_{\infty,\infty}}}{MT} \leq 1, \tag{4.39}$$

and set $p = 2 - \log\delta(t)$. Then (4.38) reduces to

$$\begin{aligned}\|\bar{u}(t)\|_{B^0_{\infty,\infty}} &\leq M(T+1)2^{-n\alpha}\\ &\quad + M(T+1)\delta(t) + M^2T(2-\log_2\delta(t))\delta(t).\end{aligned} \tag{4.40}$$

Integrating both sides over $[0,t]$ and dividing both sides by $MT$ yields the inequality

$$\delta(t) \leq (T+1)2^{-n\alpha} + \left(\frac{T+1}{T} + M\right)\int_0^t (2-\log_2\delta(s))\delta(s)\,ds. \tag{4.41}$$

We are now in a position to use Osgood's Lemma. A proof of the lemma can be found in [3].

**Lemma 2.** *(Osgood's Lemma) Let $\rho$ be a positive borelian function, and let $\gamma$ be a locally integrable positive function. Assume that for some strictly positive number $\beta$, the function $\rho$ satisfies*

$$\rho(t) \leq \beta + \int_{t_0}^t \gamma(s)\mu(\rho(s))\,ds.$$

*Then*

$$-\phi(\rho(t)) + \phi(\beta) \leq \int_{t_0}^t \gamma(s)\,ds,$$

*where $\phi(x) = \int_x^1 \frac{1}{\mu(r)}\,dr$.*

We set

$$\mu(r) = r(2 - \log r),$$
$$\rho(t) = \delta(t),$$
$$\beta = (T+1)2^{-n\alpha}, \text{ and}$$
$$\gamma(t) = \frac{T+1}{T} + M := C_0(M,T),$$

and we apply Osgood's Lemma to obtain following inequality for any $t \leq T$:

$$-\log(2 - \log \delta(t)) + \log(2 - \log((T+1)2^{-n\alpha})) \leq C_0(M,T)t.$$

Taking the exponential twice gives

$$\delta(t) \leq e^{2-2e^{-C_0(M,T)t}}((T+1)2^{-n\alpha})^{e^{-C_0(M,T)t}}. \tag{4.42}$$

The inequality (3.8) follows after substituting (4.42) into (4.40) and letting $\nu = 2^{-2n}$. □

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