UNIQUENESS FOR ACTIVE SCALAR EQUATIONS IN A ZYGMUND SPACE

ELAINE COZZI

ABSTRACT. We consider a class of active scalar equations which includes, for example, the 2D Euler equations, the 2D Navier-Stokes equations, and various aggregation equations including the Keller-Segel model. For this class of equations, we establish uniqueness of solutions in the Zygmund space C^0_*. This result improves upon that in [1], where the authors show uniqueness of solutions in BMO. As a corollary of our methods, we establish the uniform in space vanishing viscosity limit of Holder continuous solutions to the aggregation equation with Newtonian potential.

1. INTRODUCTION

In this paper we investigate uniqueness of solutions for a class of active scalar equations of the form

$$\partial_t \rho + \nabla \cdot (\rho V \rho) = \nu \Delta \rho, \tag{1.1}$$

where $\nu \geq 0$ is a fixed constant, V is, broadly speakly, a linear smoothing operator of order one, and $\rho : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ is the unknown scalar-valued function of space and time variables. Examples of PDE in this class include the two-dimensional Euler equations or Navier-Stokes equations, where ρ denotes the scalar vorticity, and some aggregation equations, including the Keller-Segel model [3, 4] and the inviscid aggregation equation with Newtonian potential [8, 5], where ρ typically denotes the density of some population.

Our goal is to show uniqueness of weak solutions to (1.1) in the Zygmund space $C^0_*(\mathbb{R}^d)$, with $d = 2$ or 3. This is, to our knowledge, the strongest uniqueness result for this class of equations, improving upon a recent result of Azzam and Bedrossian [1], who establish uniqueness in the space $BMO(\mathbb{R}^d)$. To prove our result, we use Littlewood-Paley theory and Bony's paraproduct decomposition to estimate the difference of two solutions to (1.1) in a homogeneous Besov space $\dot{B}^{-1}_{q,\infty}$, with $q < \infty$. Our methods are similar to those in [1] in that our uniqueness proof utilizes energy methods in a homogeneous space (in [1] the authors use the space $\dot{H}^{-1}$). Unlike [1], however, we make use of Littlewood-Paley operators to prove estimates which are localized in Fourier space, allowing for a sharper result. Our techniques are motivated by those in [11], where Vishik applies Littlewood-Paley methods to prove uniqueness of solutions to the Euler equations in a Besov type space which contains C^0_*.

As a corollary of our uniqueness theorem, in Section 4 we establish the uniform-in-space vanishing viscosity limit of the aggregation equation with Newtonian potential for weak solutions in a Holder space C^α, $\alpha > 0$. This result is an improvement of a result in [8], in which the authors establish the vanishing viscosity limit for strong solutions in C^α with $\alpha > 1$. Our strategy is to first apply methods from the uniqueness proof to establish a $\dot{B}^{-1}_{q,\infty}$ estimate for the difference of the solutions of the viscous and inviscid equations, respectively. We then use interpolation and Holder regularity of the solutions to derive an estimate in the L^∞-norm.

1991 *Mathematics Subject Classification.* Primary 76D05, 76C99.

The paper is organized as follows: in Section 2, we introduce the Littlewood-Paley operators and useful function spaces. We also state a few useful lemmas. We then introduce properties of solutions to the aggregation equation with Newtonian potential. In Section 3, we prove uniqueness of solutions to (1.1) in C^0_*. In Section 4, we apply estimates from Section 3 to establish the vanishing viscosity limit of solutions to the aggregation equation with Newtonian potential.

2. Background and Preliminary Lemmas

2.1. **Littlewood-Paley Operators and Function Spaces.** We first define the Littlewood-Paley operators. We let $\varphi \in S(\mathbb{R}^d)$ satisfy supp $\varphi \subset \{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, and for every $j \in \mathbb{Z}$ we let $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ (so $\check{\varphi}_j(x) = 2^{jd}\check{\varphi}(2^j x)$). Observe that, if $|j - j'| \geq 2$, then supp $\varphi_j \cap$ supp $\varphi_{j'} = \emptyset$. We define $\psi_n \in S(\mathbb{R}^d)$ by the equality

$$\psi_n(\xi) = 1 - \sum_{j>n} \varphi_j(\xi)$$

for all $\xi \in \mathbb{R}^d$. For $f \in S'(\mathbb{R}^d)$ and $j \in \mathbb{Z}$, we define the inhomogeneous Littlewood-Paley operators Δ_j by

$$\Delta_j f = \begin{cases} 0, & j < -1, \\ \check{\psi}_{-1} * f, & j = -1, \\ \check{\varphi}_j * f, & j > -1, \end{cases}$$

and for all $j \in \mathbb{Z}$, we define the homogeneous Littlewood-Paley operators $\dot{\Delta}_j$ by

$$\dot{\Delta}_j f = \check{\varphi}_j * f.$$

Note that $\dot{\Delta}_j f = \Delta_j f$ when $j \geq 0$.

Finally, for $f \in S'(\mathbb{R}^d)$ we define the operator $S_n f$ by

$$S_n f = \check{\psi}_n * f = \sum_{j=-\infty}^{n} \Delta_j f.$$

It is well known that for all $f \in S'(\mathbb{R}^d)$, $S_n f$ converges to f in the sense of distributions (see, for example, [7]).

In the proof of the main theorem we use the paraproduct decomposition introduced by J.-M. Bony in [6]. We recall the definition of the paraproduct and remainder used in this decomposition.

Definition 2.1. *Define the paraproduct of two functions f and g by*

$$T_f g = \sum_{\substack{i,j \\ i \leq j-2}} \Delta_i f \Delta_j g = \sum_{j=1}^{\infty} S_{j-2} f \Delta_j g.$$

We use $R(f,g)$ to denote the remainder. $R(f,g)$ is given by the following bilinear operator:

$$R(f,g) = \sum_{\substack{i,j \\ |i-j| \leq 1}} \Delta_i f \Delta_j g.$$

Bony's decomposition then gives

$$fg = T_f g + T_g f + R(f,g).$$

We now define the homogeneous and inhomogeneous Besov spaces.

Definition 2.2. *Let $s \in \mathbb{R}$, $(p,q) \in [1,\infty] \times [1,\infty)$. The inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^d)$ is defined to be the space of tempered distributions f on $\mathbb{R}^d$ such that*

$$||f||_{B^s_{p,q}} := \left(\sum_{j=-1}^{\infty} 2^{jqs} ||\Delta_j f||^q_{L^p} \right)^{\frac{1}{q}} < \infty.$$

When $q = \infty$, write

$$||f||_{B^s_{p,\infty}} := \sup_{j \geq -1} 2^{js} ||\Delta_j f||_{L^p}.$$

Let $s \in \mathbb{R}$, $(p,q) \in [1,\infty] \times [1,\infty)$. The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ is defined to be the space of tempered distributions f on $\mathbb{R}^d$ such that

$$||f||_{\dot{B}^s_{p,q}} := \left(\sum_{j=-\infty}^{\infty} 2^{jqs} ||\dot{\Delta}_j f||^q_{L^p} \right)^{\frac{1}{q}} < \infty.$$

When $q = \infty$, write

$$||f||_{\dot{B}^s_{p,\infty}} := \sup_{j \in \mathbb{Z}} 2^{js} ||\dot{\Delta}_j f||_{L^p}.$$

We also define the Zygmund spaces.

Definition 2.3. *Let $s \in \mathbb{R}$. The Zygmund space $C^s_*(\mathbb{R}^d)$ is the set of all tempered distributions f on $\mathbb{R}^d$ such that*

$$||f||_{C^s_*} := \sup_{j \geq -1} 2^{js} ||\Delta_j f||_{L^\infty} < \infty.$$

It is well-known that $C^s_*(\mathbb{R}^d)$ coincides with the classical Holder space $C^s(\mathbb{R}^d)$ when s is not an integer and $s > 0$.

2.2. **Useful Lemmas.** We will make frequent use of Bernstein's Lemma. We refer the reader to [7], chapter 2, for a proof of the lemma.

Lemma 2.4. *(Bernstein's Lemma) Let r_1 and r_2 satisfy $0 < r_1 < r_2 < \infty$, and let p and q satisfy $1 \leq p \leq q \leq \infty$. There exists a positive constant C such that for every integer k, if u belongs to $L^p(\mathbb{R}^d)$, and supp $\hat{u} \subset B(0, r_1\lambda)$, then*

$$\sup_{|\alpha|=k} ||\partial^\alpha u||_{L^q} \leq C^k \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} ||u||_{L^p}. \tag{2.1}$$

Furthermore, if supp $\hat{u} \subset C(0, r_1\lambda, r_2\lambda)$, then

$$C^{-k}\lambda^k ||u||_{L^p} \leq \sup_{|\alpha|=k} ||\partial^\alpha u||_{L^p} \leq C^k \lambda^k ||u||_{L^p}. \tag{2.2}$$

We also make use of the following positivity lemma. A proof of the lemma can be found in [9].

Lemma 2.5. *Assume f satisfies $f, \Delta f \in L^p(\mathbb{R}^d)$ for some $p \in [2,\infty)$. Then*

$$-\int_{\mathbb{R}^d} |f|^{p-2} f \Delta f \, dx \geq 0.$$

Finally, Osgood's Lemma will be useful in what follows. A proof of the lemma can be found in [7].

Lemma 2.6. *(Osgood's Lemma) Let ρ be a positive borelian function, let γ be a locally integrable positive function, and let μ be a continuous, increasing function. Assume that for some number $\beta \geq 0$, the function ρ satisfies*

$$\rho(t) \leq \beta + \int_{t_0}^{t} \gamma(s)\mu(\rho(s))\, ds.$$

If $\beta > 0$, then

$$-\phi(\rho(t)) + \phi(\beta) \leq \int_{t_0}^{t} \gamma(s)\, ds,$$

where $\phi(x) = \int_x^1 \frac{1}{\mu(r)}\, dr$. If $\beta = 0$, and μ satisfies

$$\int_0^1 \frac{dr}{\mu(r)} = +\infty,$$

then ρ is identically zero.

2.3. **Properties of the smoothing operator V.** In what follows, we assume that V is a linear operator satisfying the following properties:

P1) For each $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^d)$, $\|\nabla V f\|_{L^p} \leq C(p)\|f\|_{L^p}$.

P2) V commutes with the Littlewood-Paley operators, and for all $f \in S'(\mathbb{R}^d)$, there exists $C > 0$ such that for all $j \in \mathbb{Z}$, $\|\nabla V \dot{\Delta}_j f\|_{L^\infty} \leq C\|\dot{\Delta}_j f\|_{L^\infty}$.

P3) Given p and q with $1 < p < q < \infty$ and $1 + \frac{1}{q} = \frac{d-1}{d} + \frac{1}{p}$, for all $f \in L^p(\mathbb{R}^d)$, there exists $C > 0$ such that $\|V f\|_{L^q} \leq C\|f\|_{L^p}$.

Note that P1 is motivated by the boundedness of Calderon-Zygmund operators on L^p, $p \in (1, \infty)$, while P3 is motivated by the Hardy-Littlewood-Sobolev inequality.

2.4. **Definition of weak solution to (1.1).** We will use the following definition of a weak solution to (1.1) in Section 3.

Definition 2.7. *We say that $\rho : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is a weak solution to (1.1) on $[0, T]$ if ρ belongs to $L^2(0, T; L^2(\mathbb{R}^d))$ and if for all $\phi \in C_c^\infty([0, T] \times \mathbb{R}^d)$,*

$$\int_0^T \int_{\mathbb{R}^d} \rho(\phi_t + V\rho \cdot \nabla\phi + \nu\Delta\phi)\, dx\, dt.$$

Equivalently, to say that ρ is a weak solution to (1.1) means that (1.1) holds in the sense of distributions.

2.5. **The Aggregation Equation with Newtonian Potential.** The aggregation equation with Newtonian potential is given by

$$(AG_\nu) \qquad \begin{cases} \partial_t \rho^\nu + \nabla \cdot (\rho^\nu v^\nu) = \nu\Delta\rho^\nu, \\ v^\nu = -\nabla\Phi * \rho^\nu, \\ \rho^\nu(0) = \rho_0. \end{cases}$$

Here Φ denotes the Newtonian potential, ρ^ν is the density, and v^ν is the velocity. The system (AG_1) represents a limiting case of the Keller-Segel equation modeling chemotaxis. Note also that (AG_ν) is a special case of (ASE) with $V\rho^\nu = -\nabla\Phi * \rho^\nu$.

In Section 4, we establish the vanishing viscosity limit of C^α solutions to (AG_ν) for any fixed $\alpha > 0$. The vanishing viscosity limit for (AG_ν) is addressed in [8], where the authors establish the limit under the assumption that the solutions belong to C^α for $\alpha > 1$.

The solutions under consideration in Section 4 are weak solutions, satisfying Definition 2.7 above. We will assume in addition that the limiting solution ρ^0 to (AG_0) is a so-called Lagrangian solution. Properties of Lagrangian solutions to (AG_0) are discussed at length in Section 5 of [8]. We define Lagrangian solutions here and state a few properties that will be useful in Section 4. We refer the reader to [5] and Section 5 of [8] for further details.

Definition 2.8. *Fix $T > 0$. Let $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ with $X(t,\cdot)$ a homeomorphism for all $t \in [0,T]$ and let $\rho_0 \in L^\infty(\mathbb{R}^d)$. Define $\rho^0 \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ by*

$$\rho^0(t,x) = \frac{\rho_0(X^{-1}(t,x))}{1 - t\rho_0(X^{-1}(t,x))} \tag{2.3}$$

and let $v^0 := -\nabla\Phi * \rho^0$. Here, X^{-1} is defined by $X^{-1}(t, X(t,x)) = x$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$. Then (X, ρ^0, v^0) (or more simply ρ^0) is a Lagrangian solution to (AG_0) with initial density ρ_0 if X is the flow map for v^0; that is, if

$$X(t,x) = x + \int_0^t v^0(s, X(s,x))\,ds$$

for all $t \geq 0$, $x \in \mathbb{R}^d$.

We remark that a Lagrangian solution ρ^0 to (AG_0) is also a weak solution to (AG_0); see, for example, Theorem 5.4 of [8] and its proof.

The short-time existence of C^α Lagrangian solutions to (AG_0) in dimensions 2 and 3 with compactly supported initial density was first established in [5], with an alternate proof given in [8]. It follows from the equality (2.3) that, for each $t \in \left[0, \frac{1}{\|\rho_0\|_{L^\infty}}\right)$, Lagrangian solutions satisfy

$$\|\rho^0(t)\|_{L^\infty} \leq \frac{\|\rho_0\|_{L^\infty}}{1 - \|\rho_0\|_{L^\infty} t}, \tag{2.4}$$

and for all $q \in [1,\infty)$,

$$\|\rho^0(t)\|_{L^q} \leq \|\rho_0\|_{L^q}(1 - \|\rho_0\|_{L^\infty} t)^{\frac{1}{q}-1}. \tag{2.5}$$

Moreover, if the support of the initial density ρ_0 is contained in a ball of radius R_0, then for each fixed $t \in [0,T)$, $\rho^0(t)$ has compactly support in $\mathbb{R}^d$, and the support of $\rho^0(t)$ is contained in a ball of radius

$$R(t) = R_0 + \frac{C}{1 - T\|\rho_0\|_{L^\infty}}(\|\rho_0\|_{L^1} + \|\rho_0\|_{L^\infty})t. \tag{2.6}$$

Finally, for C^α Lagrangian solutions to (AG_0), $\alpha \in [0,1)$, the following inequality holds for each $t \in [0,T)$:

$$\|\nabla X(t)\|_{L^\infty}, \|\nabla X^{-1}(t)\|_{L^\infty} \leq C(T, \|\rho_0\|_{L^1}, \|\rho_0\|_{C^\alpha}). \tag{2.7}$$

If, in addition, ρ_0 is in $C^1(\mathbb{R}^d)$, then by the quotient rule, for each $t \in [0,T)$,

$$\|\nabla\rho^0(t)\|_{L^\infty} \leq (1 + Ct\|\rho_0\|_{L^\infty})\|(1 - t\rho_0)^{-1}\|_{L^\infty}^2 \|\nabla\rho_0\|_{L^\infty}\|\nabla X^{-1}(t)\|_{L^\infty}. \tag{2.8}$$

3. Statement and Proof of Uniqueness

In this section we prove the following theorem.

Theorem 3.1. *Let $T > 0$ and let p_0 and p_1 satisfy $1 < p_0 < d < p_1 < \infty$. Assume ρ^1 and ρ^2 are weak solutions of (1.1) in $C([0,T]; L^{p_0} \cap L^{p_1} \cap C^0_*(\mathbb{R}^d))$ such that $\rho^1_0 = \rho^2_0$. Then $\rho^1(t) = \rho^2(t)$ for every $t \in [0,T]$.*

Proof. To prove Theorem 3.1, we estimate the difference between ρ^1 and ρ^2 in the homogeneous Besov space $\dot{B}^{-1}_{q,\infty}$ for q sufficiently large.

Assume that ρ^1 and ρ^2 solve (1.1). Let $\bar{\rho} = \rho^1 - \rho^2$. Then $\bar{\rho}$ satisfies

$$\partial_t \bar{\rho} + \nabla \cdot (\bar{\rho} V \rho_1) + \nabla \cdot (\rho_2 V \bar{\rho}) = \nu \Delta \bar{\rho}. \tag{3.1}$$

For fixed $l \in \mathbb{Z}$, we apply the $\dot{\Delta}_l$ operator to (3.1). This gives

$$\partial_t \dot{\Delta}_l \bar{\rho} - \nu \Delta \dot{\Delta}_l \bar{\rho} = -(\dot{\Delta}_l \nabla \cdot (\bar{\rho} V \rho_1) + \dot{\Delta}_l \nabla \cdot (\rho_2 V \bar{\rho})) := -(A_l + B_l). \tag{3.2}$$

We must manipulate $A_l = \dot{\Delta}_l \nabla \cdot (\bar{\rho} V \rho_1)$ further. By the product rule,

$$\begin{aligned}
A_l = \dot{\Delta}_l \nabla \cdot (\bar{\rho} V \rho_1) &= \nabla \cdot [\dot{\Delta}_l (V\rho_1 \bar{\rho}) - S_{l-2} V \rho_1 \dot{\Delta}_l \bar{\rho}] + \nabla \cdot (S_{l-2} V \rho_1 \dot{\Delta}_l \bar{\rho}) \\
&= R_l(V\rho_1, \bar{\rho}) + \nabla \cdot (S_{l-2} V \rho_1 \dot{\Delta}_l \ \bar{\rho}) \\
&= R_l(V\rho_1, \bar{\rho}) + \nabla \dot{\Delta}_l \ \bar{\rho} \cdot S_{l-2} V \rho_1 + \dot{\Delta}_l \ \bar{\rho} \nabla \cdot S_{l-2} V \rho_1 \\
&= R_l(V\rho_1, \bar{\rho}) + A^1_l + A^2_l,
\end{aligned}$$

where

$$\begin{aligned}
&R_l(V\rho_1, \bar{\rho}) = \nabla \cdot [\dot{\Delta}_l (V\rho_1 \bar{\rho}) - S_{l-2} V \rho_1 \dot{\Delta}_l \bar{\rho}], \\
&A^1_l = \nabla \dot{\Delta}_l \ \bar{\rho} \cdot S_{l-2} V \rho_1, \text{ and} \\
&A^2_l = \dot{\Delta}_l \ \bar{\rho} \nabla \cdot S_{l-2} V \rho_1.
\end{aligned}$$

In order to utilize Property P3 of V in what follows, we assume s satisfies $1 + \frac{1}{s} = \frac{d-1}{d} + \frac{1}{p_0}$, and we fix $q \in [2, \infty)$ sufficiently large to justify the calculations. We multiply (3.2) by $\dot{\Delta}_l \bar{\rho} |\dot{\Delta}_l \bar{\rho}|^{q-2}$ and integrate to obtain

$$\begin{aligned}
\frac{1}{q} \frac{d}{dt} \int_{\mathbb{R}^n} \|\dot{\Delta}_l \bar{\rho}\|^q_{L^q} - \nu \int_{\mathbb{R}^d} \dot{\Delta}_l \bar{\rho} |\dot{\Delta}_l \bar{\rho}|^{q-2} \Delta \dot{\Delta}_l \bar{\rho} \\
= - \int_{\mathbb{R}^d} \dot{\Delta}_l \bar{\rho} |\dot{\Delta}_l \bar{\rho}|^{q-2} (R_l(V\rho_1, \bar{\rho}) + A^1_l + A^2_l + B_l).
\end{aligned} \tag{3.3}$$

It follows from Lemma 2.5 that

$$-\nu \int_{\mathbb{R}^d} \dot{\Delta}_l \bar{\rho} |\dot{\Delta}_l \bar{\rho}|^{q-2} \Delta \dot{\Delta}_l \bar{\rho} \geq 0.$$

Moreover, it follows from integration by parts that

$$\int_{\mathbb{R}^d} \dot{\Delta}_l \bar{\rho} |\dot{\Delta}_l \bar{\rho}|^{q-2} A^1_l \, dx \leq C \int_{\mathbb{R}^d} |S_{l-2} \nabla V \rho_1| |\dot{\Delta}_l \bar{\rho}|^q \, dx \leq C \|S_{l-2} \nabla V \rho_1\|_{L^\infty} \|\dot{\Delta}_l \bar{\rho}\|^q_{L^q}.$$

Then (3.3) reduces to

$$\begin{aligned}
\frac{d}{dt} \|\dot{\Delta}_l \bar{\rho}\|^q_{L^q} \leq q(\|\dot{\Delta}_l \bar{\rho}\|^{q-1}_{L^q} \|R_l(V\rho_1, \bar{\rho})\|_{L^q} + \|\dot{\Delta}_l \bar{\rho}\|^{q-1}_{L^q} \|A^2_l\|_{L^q} \\
+ \|\dot{\Delta}_l \bar{\rho}\|^{q-1}_{L^q} \|B_l\|_{L^q} + C\|S_{l-2} \nabla V \rho_1\|_{L^\infty} \|\dot{\Delta}_l \bar{\rho}\|^q_{L^q}).
\end{aligned} \tag{3.4}$$

After taking the time derivative of the left hand side and dividing through by $q\|\dot{\Delta}_l\bar{\rho}\|_{L^q}^{q-1}$, we conclude that

$$\frac{d}{dt}\|\dot{\Delta}_l\bar{\rho}\|_{L^q} \le \|R_l(V\rho_1,\bar{\rho})\|_{L^q} + \|A_l^2\|_{L^q} + \|B_l\|_{L^q} + C\|S_{l-2}\nabla V\rho_1\|_{L^\infty}\|\dot{\Delta}_l\bar{\rho}\|_{L^q}. \tag{3.5}$$

We now estimate the terms on the right hand side. In what follows, we assume $p \ge 2$ is fixed (to be chosen later).

We begin with $\|S_{l-2}\nabla V\rho_1\|_{L^\infty}\|\dot{\Delta}_l\bar{\rho}\|_{L^q}$. First note that $S_{l-2}\nabla V\rho_1 = 0$ when $l \le 0$, so we may assume that $l \ge 1$. For the case $l \ge 1$, Bernstein's Lemma and property P2 of V imply that

$$\begin{aligned} \|S_{l-2}\nabla V\rho_1\|_{L^\infty} &\le \|\Delta_{-1}\nabla V\rho_1\|_{L^\infty} + \sum_{k=0}^{l-2}\|\Delta_k\nabla V\rho_1\|_{L^\infty} \\ &\le C\|\Delta_{-1}\nabla V\rho_1\|_{L^{p_1}} + \sum_{k=0}^{l-2}\|\Delta_k\rho_1\|_{L^\infty} \le C\|\rho_1\|_{L^{p_1}} + (l-1)\|\rho_1\|_{C^0_*}. \end{aligned} \tag{3.6}$$

This gives

$$\begin{aligned} \sup_{-\infty<l\le p} 2^{-l}\|S_{l-2}\nabla V\rho_1\|_{L^\infty}\|\dot{\Delta}_l\bar{\rho}\|_{L^q} &\le C(\|\rho_1\|_{L^{p_1}} + (p-1)\|\rho_1\|_{C^0_*})\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}} \\ &\le C(p-1)(\|\rho_1\|_{L^{p_1}} + \|\rho_1\|_{C^0_*})\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}}. \end{aligned} \tag{3.7}$$

We now estimate $\|B_l\|_{L^q} = \|\dot{\Delta}_l\nabla\cdot(\rho_2 V\bar{\rho})\|_{L^q}$. Following [10], we define

$$J^{l,k} := \partial_k\dot{\Delta}_l\left((V\bar{\rho})^k\rho_2\right),$$

and we use Bony's paraproduct decomposition to write

$$\begin{aligned} J^{l,k} &= \partial_k\dot{\Delta}_l\sum_{|j-l|\le 3, j\ge 1} S_{j-2}(V\bar{\rho})^k\Delta_j\rho_2 + \partial_k\dot{\Delta}_l\sum_{|j-l|\le 3, j\ge 1}\Delta_j(V\bar{\rho})^k S_{j-2}\rho_2 \\ &\quad + \partial_k\dot{\Delta}_l\sum_{|j-j'|\le 1, \max\{j,j'\}\ge l-3}\Delta_j(V\bar{\rho})^k\Delta_{j'}\rho_2 \\ &:= J_1^{l,k} + J_2^{l,k} + J_3^{l,k}. \end{aligned}$$

We first estimate $J_1^{l,k}$ in the L^q-norm. By Bernstein's Lemma and properties P1 and P2 of V,

$$\begin{aligned} \|J_1^{l,k}\|_{L^q} &\le 2^l\sum_{|j-l|\le 3, j\ge 1}\|S_{j-2}V\bar{\rho}\|_{L^q}\|\Delta_j\rho_2\|_{L^\infty} \le 2^l\sum_{|j-l|\le 3, j\ge 1}\|\rho_2\|_{C^0_*}\sum_{k\le j-2}\|\Delta_k V\bar{\rho}\|_{L^q} \\ &\le C2^l\|\rho_2\|_{C^0_*}\sum_{k\le l+1}\|\Delta_k V\bar{\rho}\|_{L^q} \le C2^l\|\rho_2\|_{C^0_*}\left(\|\Delta_{-1}V\bar{\rho}\|_{L^q} + \sum_{k=0}^{l+1}\|\Delta_k V\bar{\rho}\|_{L^q}\right) \\ &\le C2^l\|\rho_2\|_{C^0_*}\left(\|\Delta_{-1}V\bar{\rho}\|_{L^q} + \sum_{k=0}^{l+1}2^{-k}\|\Delta_k\bar{\rho}\|_{L^q}\right) \\ &\le C2^l\|\rho_2\|_{C^0_*}\left(\|\Delta_{-1}V\bar{\rho}\|_{L^q} + (l+2)\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}}\right). \end{aligned}$$

Multiplying by 2^{-l} and taking the supremum over $l \le p$ gives

$$\sup_{-\infty<l\le p} 2^{-l}\|J_1^{l,k}\|_{L^q} \le C\|\rho_2\|_{C^0_*}\left(\|\Delta_{-1}V\bar{\rho}\|_{L^q} + (p+2)\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}}\right).$$

We now estimate $J_2^{l,k}$ in the L^q-norm. Again by Bernstein's Lemma and properties P1 and P2 of V,

$$\begin{aligned}\|J_2^{l,k}\|_{L^q} &\leq 2^l \sum_{|j-l|\leq 3, j\geq 1} \|S_{j-2}\rho_2\|_{L^\infty}\|\Delta_j V\bar{\rho}\|_{L^q} \leq 2^l \sum_{|j-l|\leq 3, j\geq 1} j\|\rho_2\|_{C^0_*}\|\Delta_j V\bar{\rho}\|_{L^q} \\ &\leq 2^l \sum_{|j-l|\leq 3, j\geq 1} j\|\rho_2\|_{C^0_*}2^{-j}\|\Delta_j \nabla V\bar{\rho}\|_{L^q} \leq C2^l \max\{l+3,1\}\|\rho_2\|_{C^0_*}\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}}.\end{aligned}$$

Multiplying by 2^{-l} and taking the supremum over $l \leq p$ gives

$$\sup_{-\infty<l\leq p} 2^{-l}\|J_2^{l,k}\|_{L^q} \leq C\|\rho_2\|_{C^0_*}(p+3)\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}}.$$

Finally, we estimate $J_3^{l,k}$. By Bernstein's Lemma and properties of Littlewood-Paley operators,

$$\|J_3^{l,k}\|_{L^q} \leq C2^l \sum_{j\geq \max\{l-3,-1\}} \|\Delta_j\rho_2\|_{L^\infty}\|\Delta_j V\bar{\rho}\|_{L^q} \leq C2^l\|\rho_2\|_{C^0_*}\|V\bar{\rho}\|_{B^0_{q,1}}. \tag{3.8}$$

By Bernstein's Lemma and property P1 of V,

$$\begin{aligned}\|V\bar{\rho}\|_{B^0_{q,1}} &\leq \|\Delta_{-1}V\bar{\rho}\|_{L^q} + \sum_{j=0}^{p-1}\|\Delta_j V\bar{\rho}\|_{L^q} + \sum_{j\geq p}\|\Delta_j V\bar{\rho}\|_{L^q} \\ &\leq \|\Delta_{-1}V\bar{\rho}\|_{L^q} + \sum_{j=0}^{p-1}2^{-j}\|\Delta_j \nabla V\bar{\rho}\|_{L^q} + \sum_{j\geq p}2^{jd\left(\frac{1}{p_1}-\frac{1}{q}\right)}2^{-j}\|\Delta_j\nabla V\bar{\rho}\|_{L^{p_1}} \\ &\leq \|\Delta_{-1}V\bar{\rho}\|_{L^q} + p\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}} + 2^{-\frac{pd}{q}}\|\bar{\rho}\|_{L^{p_1}},\end{aligned}$$

where we used that $p_1 > d$ to get the last inequality. Substituting this estimate into (3.8), multiplying by 2^{-l}, and taking the supremum over $l \leq p$ gives

$$\sup_{-\infty<l\leq p} 2^{-l}\|J_3^{l,k}\|_{L^q} \leq C\|\rho_2\|_{C^0_*}\left(\|\Delta_{-1}V\bar{\rho}\|_{L^q} + p\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}} + 2^{-\frac{pd}{q}}\|\bar{\rho}\|_{L^{p_1}}\right). \tag{3.9}$$

Combining the estimates for $J_1^{l,k}$, $J_2^{l,k}$, and $J_3^{l,k}$, we conclude that

$$\begin{aligned}\sup_{-\infty<l\leq p} 2^{-l}\|B_l\|_{L^q} &= \sup_{-\infty<l\leq p} 2^{-l}\|\dot{\Delta}_l\nabla\cdot(\rho_2 V\bar{\rho})\|_{L^q} \\ &\leq \sup_{-\infty<l\leq p}\sum_{k=1}^{d} 2^{-l}(\|J_1^{l,k}\|_{L^q} + \|J_2^{l,k}\|_{L^q} + \|J_3^{l,k}\|_{L^q}) \\ &\leq C\|\rho_2\|_{C^0_*}\left(\|\Delta_{-1}V\bar{\rho}\|_{L^q} + (p+3)\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}} + 2^{-\frac{pd}{q}}\|\bar{\rho}\|_{L^{p_1}}\right).\end{aligned} \tag{3.10}$$

We now estimate $R_l(V\rho_1, \bar{\rho}) = \nabla\cdot(\dot{\Delta}_l(V\rho_1\bar{\rho}) - S_{l-2}V\rho_1\dot{\Delta}_l\bar{\rho})$. We follow techniques used in [11] and [2]. Specifically, we write

$$R_l = R_l^1 + R_l^2 + R_l^3 + R_l^4, \tag{3.11}$$

where

$$\begin{aligned} R_l^1 &= \nabla \cdot (\dot{\Delta}_l T_{\bar{\rho}} V\rho_1), \\ R_l^2 &= -\nabla \cdot [T_{V\rho_1}, \dot{\Delta}_l]\bar{\rho}, \\ R_l^3 &= \nabla \cdot (T_{V\rho_1 - S_{l-2}V\rho_1}\dot{\Delta}_l)\bar{\rho}, \\ R_l^4 &= \nabla \cdot \Big\{ \dot{\Delta}_l R(V\rho_1, \bar{\rho}) - R(S_{l-2}V\rho_1, \dot{\Delta}_l\bar{\rho}) \Big\}. \end{aligned} \tag{3.12}$$

By estimating each of the four terms individually, we will show that

$$\begin{aligned} \sup_{-\infty < l \le p} 2^{-l}\|R_l\|_{L^q} &\le C(\|\rho_1\|_{C^0_*} + \|\rho_1\|_{L^{p_1}} + \|\Delta_{-1}V\rho_1\|_{L^\infty}) \\ &\times (\|\Delta_{-1}\bar{\rho}\|_{L^q} + (p+3)\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}} + 2^{-\frac{pd}{q}}\|\bar{\rho}\|_{L^{p_1}}). \end{aligned} \tag{3.13}$$

We begin with R_l^1. By Bernstein's Lemma and property P2 of V,

$$\begin{aligned} \|R_l^1\|_{L^q} &\le 2^l \sum_{|j-l|\le 3, j\ge 1} \|S_{j-2}\bar{\rho}\|_{L^q}\|\Delta_j V\rho_1\|_{L^\infty} \\ &\le 2^l \sum_{|j-l|\le 3, j\ge 1} \sum_{k\le j-2} 2^k 2^{-k}\|\Delta_k\bar{\rho}\|_{L^q} 2^{-j}\|\nabla\Delta_j V\rho_1\|_{L^\infty} \\ &\le 2^l \sum_{|j-l|\le 3, j\ge 1} 2^j \sum_{k\le j-2} 2^{-k}\|\Delta_k\bar{\rho}\|_{L^q} 2^{-j}\|\Delta_j\rho_1\|_{L^\infty} \\ &\le C2^l\|\rho_1\|_{C^0_*}(\|\Delta_{-1}\bar{\rho}\|_{L^q} + \max\{1, l+2\}\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}}). \end{aligned} \tag{3.14}$$

Multiplying by 2^{-l} and taking the supremum over $l \le p$ gives

$$\sup_{-\infty < l \le p} 2^{-l}\|R_l^1\|_{L^2} \le C\|\rho_1\|_{C^0_*}(\|\Delta_{-1}\bar{\rho}\|_{L^q} + (p+2)\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}}). \tag{3.15}$$

We now estimate R_l^2. Writing out the commutator and using properties of Littlewood-Paley operators gives

$$\begin{aligned} R_l^2 &= -\nabla \cdot [T_{V\rho_1}, \dot{\Delta}_l]\bar{\rho} = -\nabla \cdot \Big(T_{V\rho_1}\dot{\Delta}_l\bar{\rho} - \dot{\Delta}_l(T_{V\rho_1}\bar{\rho}) \Big) \\ &= -\nabla \cdot \left(\sum_{j\ge 1, |j-l|\le 3} S_{j-2}V\rho_1\Delta_j\dot{\Delta}_l\bar{\rho} - \dot{\Delta}_l \sum_{j\ge 1, |j-l|\le 3} S_{j-2}V\rho_1\Delta_j\bar{\rho} \right). \end{aligned}$$

Applying the L^q-norm, Bernstein's Lemma, and property P2 of V gives

$$\begin{aligned}
\|R_l^2\|_{L^q} &\le 2^l \left\| \sum_{j\ge 1, |j-l|\le 3} \Big(S_{j-2}V\rho_1 \Delta_j \dot{\Delta}_l \bar{\rho} - \dot{\Delta}_l (S_{j-2}V\rho_1 \Delta_j \bar{\rho}) \Big) \right\|_{L^q} \\
&= 2^l \left\| \sum_{j\ge 1, |j-l|\le 3} \int_{\mathbb{R}^d} \check{\varphi}_l(y)(S_{j-2}V\rho_1(x-y) - S_{j-2}V\rho_1(x))\Delta_j\bar{\rho}(x-y)\, dy \right\|_{L^q_x} \\
&\le 2^l \sum_{j\ge 1, |j-l|\le 3} \int_{\mathbb{R}^d} \|\check{\varphi}_l(y)(S_{j-2}V\rho_1(x-y) - S_{j-2}V\rho_1(x))\Delta_j\bar{\rho}(x-y)\|_{L^q_x}\, dy \\
&= 2^l \sum_{j\ge 1, |j-l|\le 3} \|S_{j-2}\nabla V\rho_1\|_{L^\infty} \|\Delta_j\bar{\rho}\|_{L^q} \int_{\mathbb{R}^d} |2^{ld}\check{\varphi}(2^l y)||y|\, dy \\
&\le C \sum_{j\ge 1, |j-l|\le 3} \|S_{j-2}\nabla V\rho_1\|_{L^\infty} \|\Delta_j\bar{\rho}\|_{L^q}, \\
&\le C(\|\Delta_{-1}\nabla V\rho_1\|_{L^\infty} + \max\{1, l+2\}\|\rho_1\|_{C^*_0}) \sup_{j\ge 1, |j-l|\le 3} \|\Delta_j\bar{\rho}\|_{L^q},
\end{aligned}$$

where we used the mean value theorem to get the last equality, and we applied a change of variables to get the second-to-last inequality. Multiplying the resulting estimate by 2^{-l} and taking the supremum over $l \le p$ gives

$$\sup_{-\infty < l \le p} 2^{-l}\|R_l^2\|_{L^q} \le C(\|\Delta_{-1}\nabla V\rho_1\|_{L^\infty} + (p+2)\|\rho_1\|_{C^*_0})\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}}.$$

We now estimate R_l^3. First note that by the product rule and properties of Littlewood-Paley operators,

$$\begin{aligned}
\|R_l^3\|_{L^q} &= \left\| \nabla \cdot \sum_{|j-l|\le 3, j\ge 1} S_{j-2}(V\rho_1 - S_{l-2}V\rho_1)\Delta_j \dot{\Delta}_l \bar{\rho} \right\|_{L^q} \\
&\le \sum_{|k-l|\le 3} \|\Delta_k \nabla V\rho_1\|_{L^\infty} \|\dot{\Delta}_l\bar{\rho}\|_{L^q} + \sum_{|k-l|\le 3} \|\Delta_k V\rho_1\|_{L^\infty} \|\dot{\Delta}_l \nabla\bar{\rho}\|_{L^q}.
\end{aligned} \tag{3.16}$$

We consider three cases separately: $l < -2$, $-2 \le l \le 2$, and $l > 2$. For the first case, R_l^3 is identically zero. For the second case, Bernstein's Lemma and properties P1 and P2 of V give

$$\begin{aligned}
\|R_l^3\|_{L^q} &\le \left(\|\Delta_{-1}\nabla V\rho_1\|_{L^\infty} + \sum_{|k-l|\le 3, k\ge 0} \|\Delta_k \nabla V\rho_1\|_{L^\infty} \right) \|\dot{\Delta}_l\bar{\rho}\|_{L^q} \\
&\quad + \left(\|\Delta_{-1}V\rho_1\|_{L^\infty} + \sum_{|k-l|\le 3, k\ge 0} \|\Delta_k V\rho_1\|_{L^\infty} \right) \|\dot{\Delta}_l\nabla\bar{\rho}\|_{L^q} \\
&\le \left(\|\Delta_{-1}\nabla V\rho_1\|_{L^\infty} + C\|\rho_1\|_{C^0_*}\right) \|\dot{\Delta}_l\bar{\rho}\|_{L^q} \\
&\quad + C\left(\|\Delta_{-1}V\rho_1\|_{L^\infty} + \sum_{|k-l|\le 3, k\ge 0} 2^{-k}\|\Delta_k \nabla V\rho_1\|_{L^\infty} \right) 2^l \|\dot{\Delta}_l\bar{\rho}\|_{L^q} \\
&\le C(\|\rho_1\|_{L^{p_1}} + \|\rho_1\|_{C^0_*})\|\dot{\Delta}_l\bar{\rho}\|_{L^q} + C(\|\Delta_{-1}V\rho_1\|_{L^\infty} + \|\rho_1\|_{C^0_*})\|\dot{\Delta}_l\bar{\rho}\|_{L^q}.
\end{aligned}$$

Finally, for the third case, we can again use Bernstein's Lemma and property P2 of V to write

$$\begin{aligned}\|R_l^3\|_{L^q} &\leq \sum_{|k-l|\leq 3, k\geq 0} \|\Delta_k \nabla V\rho_1\|_{L^\infty}\|\dot{\Delta}_l\bar{\rho}\|_{L^q} + \sum_{|k-l|\leq 3, k\geq 0} \|\Delta_k V\rho_1\|_{L^\infty}\|\dot{\Delta}_l\nabla\bar{\rho}\|_{L^q}\\ &\leq \sum_{|k-l|\leq 3, k\geq 0} \|\Delta_k \nabla V\rho_1\|_{L^\infty}\|\dot{\Delta}_l\bar{\rho}\|_{L^q} + \sum_{|k-l|\leq 3, k\geq 0} 2^{l-k}\|\Delta_k \nabla V\rho_1\|_{L^\infty}\|\dot{\Delta}_l\bar{\rho}\|_{L^q}\\ &\leq C\|\rho_1\|_{C^0_*}\|\dot{\Delta}_l\bar{\rho}\|_{L^q}.\end{aligned}$$

Combining the three cases above, multiplying by 2^{-l}, and taking the supremum over $l \leq p$ gives

$$\sup_{-\infty<l\leq p} 2^{-l}\|R_l^3\|_{L^q} \leq C(\|\Delta_{-1}V\rho_1\|_{L^\infty} + \|\rho_1\|_{L^{p_1}} + \|\rho_1\|_{C^0_*})\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}}.$$

Finally, we estimate R_l^4. We do this without utilizing the difference. We first write

$$\|R_l^4\|_{L^q} \leq \|\nabla\cdot\dot{\Delta}_l R(\bar{\rho}, V\rho_1)\|_{L^q} + \|\nabla\cdot R(S_{l-2}V\rho_1, \dot{\Delta}_l\bar{\rho})\|_{L^q} := \|R_l^{4,1}\|_{L^q} + \|R_l^{4,2}\|_{L^q}.$$

Then using an argument similar to that in [10], we apply Bernstein's Lemma and property P2 of V to deduce that

$$\begin{aligned}\|R_l^{4,1}\|_{L^q} &\leq C2^l \sum_{j\geq\max\{l-3,-1\}} \|\Delta_j V\rho_1\|_{L^\infty}\|\Delta_j\bar{\rho}\|_{L^q}\\ &\leq C2^l\left(\|\Delta_{-1}V\rho_1\|_{L^\infty}\|\Delta_{-1}\bar{\rho}\|_{L^q} + \sum_{j\geq 0} 2^{-j}\|\nabla\Delta_j V\rho_1\|_{L^\infty}\|\Delta_j\bar{\rho}\|_{L^q}\right)\\ &\leq C2^l\left(\|\Delta_{-1}V\rho_1\|_{L^\infty}\|\Delta_{-1}\bar{\rho}\|_{L^q} + \sum_{j\geq 0} 2^{-j}\|\Delta_j \rho_1\|_{L^\infty}\|\Delta_j\bar{\rho}\|_{L^q}\right)\\ &\leq C2^l\left(\|\rho_1\|_{C^0_*} + \|\Delta_{-1}V\rho_1\|_{L^\infty}\right)\|\bar{\rho}\|_{B^{-1}_{q,1}}.\end{aligned}$$

Multiplying by 2^{-l} and taking the supremum over $l \leq p$ gives

$$\sup_{-\infty<l\leq p} 2^{-l}\|R_l^{4,1}\|_{L^q} \leq C(\|\rho_1\|_{C^0_*} + \|\Delta_{-1}V\rho_1\|_{L^\infty})\|\bar{\rho}\|_{B^{-1}_{q,1}}. \tag{3.17}$$

By the definition of $\dot{B}^{-1}_{q,\infty}$ and Bernstein's Lemma,

$$\begin{aligned}\|\bar{\rho}\|_{B^{-1}_{q,1}} &\leq C\|\Delta_{-1}\bar{\rho}\|_{L^q} + \sum_{0\leq j<p} 2^{-j}\|\Delta_j\bar{\rho}\|_{L^q} + \sum_{j\geq p} 2^{-j}\|\Delta_j\bar{\rho}\|_{L^q}\\ &\leq C\|\Delta_{-1}\bar{\rho}\|_{L^q} + p\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}} + \sum_{j\geq p} 2^{-j}2^{jd\left(\frac{1}{p_1}-\frac{1}{q}\right)}\|\bar{\rho}\|_{L^{p_1}}\\ &\leq C\|\Delta_{-1}\bar{\rho}\|_{L^q} + p\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}} + 2^{-\frac{pd}{q}}\|\bar{\rho}\|_{L^{p_1}},\end{aligned}$$

where we used that $p_1 > d$ to get the last inequality. After substituting this estimate into (3.17), we conclude that

$$\sup_{-\infty<l\leq p} 2^{-l}\|R_l^{4,1}\|_{L^q} \leq C(\|\rho_1\|_{C^0_*} + \|\Delta_{-1}V\rho_1\|_{L^\infty})(\|\Delta_{-1}\bar{\rho}\|_{L^q} + p\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}} + 2^{-\frac{pd}{q}}\|\bar{\rho}\|_{L^{p_1}}).$$

We now estimate $\|R_l^{4,2}\|_{L^q}$. We apply the definition of the remainder term, the product rule, properties of Littlewood-Paley operators, Bernstein's Lemma, and property P2 of V to write

$$\begin{aligned}
\|R_l^{4,2}\|_{L^q} &\leq C\sum_{|j-l|\leq 1}\|\Delta_j S_{l-2}\nabla V\rho_1\|_{L^\infty}\|\Delta_j\dot{\Delta}_l\bar{\rho}\|_{L^q} + C\sum_{|j-l|\leq 1}\|\Delta_j S_{l-2}V\rho_1\|_{L^\infty}\|\Delta_j\dot{\Delta}_l\nabla\bar{\rho}\|_{L^q}\\
&\leq C\|\Delta_{-1}S_{l-2}\nabla V\rho_1\|_{L^\infty}\|\Delta_{-1}\dot{\Delta}_l\bar{\rho}\|_{L^q} + C\|\Delta_{-1}S_{l-2}V\rho_1\|_{L^\infty}\|\Delta_{-1}\dot{\Delta}_l\nabla\bar{\rho}\|_{L^q}\\
&+ C\sum_{|j-l|\leq 1, j\geq 0}(\|\Delta_j S_{l-2}\nabla V\rho_1\|_{L^\infty}\|\Delta_j\dot{\Delta}_l\bar{\rho}\|_{L^q} + \|\Delta_j S_{l-2}V\rho_1\|_{L^\infty}\|\Delta_j\dot{\Delta}_l\nabla\bar{\rho}\|_{L^q})\\
&\leq C\|\Delta_{-1}V\rho_1\|_{L^\infty}\|\dot{\Delta}_l\bar{\rho}\|_{L^q} + C\sum_{|j-l|\leq 1, j\geq 0}\|\Delta_j\nabla V\rho_1\|_{L^\infty}\|\dot{\Delta}_l\bar{\rho}\|_{L^q}\\
&\leq C\left(\|\Delta_{-1}V\rho_1\|_{L^\infty} + \sum_{|j-l|\leq 1, j\geq 0}\|\Delta_j\rho_1\|_{L^\infty}\right)\|\dot{\Delta}_l\bar{\rho}\|_{L^q}.
\end{aligned}$$

Multiplying by 2^{-l} and taking the supremum over $l \leq p$ gives

$$\sup_{-\infty<l\leq p} 2^{-l}\|R_l^{4,2}\|_{L^q} \leq C(\|\Delta_{-1}V\rho_1\|_{L^\infty} + \|\rho_1\|_{C^0_*})\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}}.$$

Combining the estimates for R_l^1 through R_l^4, we conclude that

$$\begin{aligned}
\sup_{-\infty<l\leq p} 2^{-l}\|R_l\|_{L^q} &\leq C(\|\rho_1\|_{C^0_*} + \|\rho_1\|_{L^{p_1}} + \|\Delta_{-1}V\rho_1\|_{L^\infty})\\
&\times(\|\Delta_{-1}\bar{\rho}\|_{L^q} + (p+3)\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}} + 2^{-\frac{pd}{q}}\|\bar{\rho}\|_{L^{p_1}}).
\end{aligned} \tag{3.18}$$

We now estimate $A_l^2 = \dot{\Delta}_l\bar{\rho}\nabla\cdot S_{l-2}V\rho_1$. Note that A_l^2 is identically zero when $l \leq 0$. For $l \geq 1$, write

$$\begin{aligned}
\|\dot{\Delta}_l\bar{\rho}\nabla\cdot S_{l-2}V\rho_1\|_{L^q} &\leq \|\dot{\Delta}_l\bar{\rho}\|_{L^q}\|\nabla\cdot S_{l-2}V\rho_1\|_{L^\infty}\\
&\leq \|\dot{\Delta}_l\bar{\rho}\|_{L^q}\sum_{k=-1}^{l-2}\|\nabla\cdot\Delta_k V\rho_1\|_{L^\infty}\\
&\leq \|\dot{\Delta}_l\bar{\rho}\|_{L^q}\left(\|\nabla\cdot\Delta_{-1}V\rho_1\|_{L^\infty} + \sum_{k=0}^{\max\{0,l-2\}}\|\nabla\cdot\Delta_k V\rho_1\|_{L^\infty}\right)\\
&\leq \|\dot{\Delta}_l\bar{\rho}\|_{L^q}(C\|\rho_1\|_{L^{p_1}} + (l-1)\|\rho_1\|_{C^0_*}),
\end{aligned}$$

where we used Bernstein's Lemma and properties P1 and P2 of V to get the last inequality. Multiplying by 2^{-l} and taking the supremum over $l \leq p$ gives

$$\begin{aligned}
\sup_{-\infty<l\leq p} 2^{-l}\|A_l^2\|_{L^q} &\leq C\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}}(\|\rho_1\|_{L^{p_1}} + (p-1)\|\rho_1\|_{C^0_*})\\
&\leq C(p-1)\|\bar{\rho}\|_{\dot{B}^{-1}_{q,\infty}}(\|\rho_1\|_{L^{p_1}} + \|\rho_1\|_{C^0_*}).
\end{aligned} \tag{3.19}$$

We now estimate the high frequencies. By Bernstein's Lemma,

$$\sup_{l\geq p} 2^{-l}\|\dot{\Delta}_l\bar{\rho}\|_{L^q} \leq \sup_{l\geq p} 2^{-l}2^{ld(\frac{1}{p_1}-\frac{1}{q})}\|\bar{\rho}\|_{L^{p_1}} \leq \sup_{l\geq p} 2^{-\frac{ld}{q}}\|\bar{\rho}\|_{L^{p_1}} \leq 2^{-\frac{pd}{q}}\|\bar{\rho}\|_{L^{p_1}} \tag{3.20}$$

since $p_1 > d$ by assumption.

We integrate (3.5) in time, multiply by 2^{-l}, take the supremum over $l \leq p$, and apply the estimates (3.7), (3.10), (3.18) and (3.19). Combining the resulting estimate with (3.20) gives

$$\|\bar\rho(t)\|_{\dot B^{-1}_{q,\infty}} \leq 2^{-\frac{pd}{q}}\|\bar\rho(t)\|_{L^{p_1}} + C\int_0^t (\|\rho_1(s)\|_{C^0_*} + \|\rho_2(s)\|_{C^0_*} + \|\rho_1(s)\|_{L^{p_1}} + \|\Delta_{-1}V\rho_1(s)\|_{L^\infty})$$
$$\times\, (\|\Delta_{-1}\bar\rho(s)\|_{L^q} + \|\Delta_{-1}V\bar\rho(s)\|_{L^q} + p\|\bar\rho(s)\|_{\dot B^{-1}_{q,\infty}} + 2^{-\frac{pd}{q}}\|\bar\rho(s)\|_{L^{p_1}})\, ds. \tag{3.21}$$

Before we apply Osgood's Lemma, we must estimate several of the terms under the time integral in (3.21). We first estimate $\|\Delta_{-1}V\bar\rho(s)\|_{L^q}$. Observe that, by Bernstein's Lemma and properties P1, P2, and P3 of V,

$$\begin{aligned} \|\Delta_{-1}V\bar\rho\|_{L^q} &\leq \|\check\psi_{-p} * V\bar\rho\|_{L^q} + \|\Delta_{-1}V\bar\rho - \check\psi_{-p} * V\bar\rho\|_{L^q} \\ &\leq \|\check\psi_{-p} * V\bar\rho\|_{L^q} + \sum_{k=-p}^{-1}\|\dot\Delta_k V\bar\rho\|_{L^q} \leq 2^{-pd(\frac{1}{s}-\frac{1}{q})}\|\check\psi_{-p} * V\bar\rho\|_{L^s} + Cp\|\bar\rho\|_{\dot B^{-1}_{q,\infty}} \\ &\leq 2^{-pd(\frac{1}{s}-\frac{1}{q})}\|\bar\rho\|_{L^{p_0}} + Cp\|\bar\rho\|_{\dot B^{-1}_{q,\infty}}. \end{aligned} \tag{3.22}$$

We must also estimate $\|\Delta_{-1}V\rho_1(s)\|_{L^\infty}$. By Bernstein's Lemma,

$$\|\Delta_{-1}V\rho_1(s)\|_{L^\infty} \leq C\|V\rho_1(s)\|_{L^s} \leq C\|\rho_1(s)\|_{L^{p_0}}, \tag{3.23}$$

where we again used property P3 of V.

Finally, we must estimate $\|\Delta_{-1}\bar\rho\|_{L^q}$. By Bernstein's Lemma and our choice of $s > p_0$,

$$\begin{aligned} \|\Delta_{-1}\bar\rho\|_{L^q} &\leq \|\check\psi_{-p} * \bar\rho\|_{L^q} + \|\Delta_{-1}\bar\rho - \check\psi_{-p} * \bar\rho\|_{L^q} \\ &\leq 2^{-pd\left(\frac{1}{p_0}-\frac{1}{q}\right)}\|\check\psi_{-p} * \bar\rho\|_{L^{p_0}} + \sum_{j=-p}^{-1} 2^j 2^{-j}\|\dot\Delta_j\bar\rho\|_{L^q} \\ &\leq 2^{-pd\left(\frac{1}{p_0}-\frac{1}{q}\right)}\|\bar\rho\|_{L^{p_0}} + C\|\bar\rho\|_{\dot B^{-1}_{q,\infty}} \leq 2^{-pd\left(\frac{1}{s}-\frac{1}{q}\right)}\|\bar\rho\|_{L^{p_0}} + C\|\bar\rho\|_{\dot B^{-1}_{q,\infty}}. \end{aligned} \tag{3.24}$$

Substituting (3.22), (3.23), and (3.24) into (3.21) gives

$$\|\bar\rho(t)\|_{\dot B^{-1}_{q,\infty}} \leq 2^{-\varepsilon_0 p}\|\bar\rho(t)\|_{L^{p_1}} + C\int_0^t (\|\rho_1(s)\|_{C^0_*} + \|\rho_2(s)\|_{C^0_*} + \|\rho_1(s)\|_{L^{p_1}} + \|\rho_1(s)\|_{L^{p_0}})$$
$$\times\, (p\|\bar\rho(s)\|_{\dot B^{-1}_{q,\infty}} + 2^{-\varepsilon_0 p}\|\bar\rho(s)\|_{L^{p_0}})\, ds \tag{3.25}$$

for sufficiently small $\varepsilon_0 = \varepsilon_0(s, q)$.

Observe that, for q sufficiently large, Bernstein's Lemma gives

$$\begin{aligned} \|\bar\rho\|_{\dot B^{-1}_{q,\infty}} &= \sup_{l\in\mathbb{Z}} 2^{-l}\|\dot\Delta_l\bar\rho\|_{L^q} \leq \sup_{l\leq 0} 2^{-l}\|\dot\Delta_l\bar\rho\|_{L^q} + \sup_{l>0} 2^{-l}\|\dot\Delta_l\bar\rho\|_{L^q} \\ &\leq \sup_{l\leq 0} 2^{-l}2^{ld\left(\frac{1}{p_0}-\frac{1}{q}\right)}\|\dot\Delta_l\bar\rho\|_{L^{p_0}} + \sup_{l>0} 2^{-l}2^{ld\left(\frac{1}{p_1}-\frac{1}{q}\right)}\|\dot\Delta_l\bar\rho\|_{L^{p_1}} \\ &\leq \|\bar\rho\|_{L^{p_0}} + \|\bar\rho\|_{L^{p_1}} \leq \|\rho_1\|_{L^{p_0}\cap L^{p_1}} + \|\rho_2\|_{L^{p_0}\cap L^{p_1}}. \end{aligned}$$

Set

$$M = \sup_{t\in[0,T]} (\|\rho_1(t)\|_{L^{p_0}\cap L^{p_1}} + \|\rho_2(t)\|_{L^{p_0}\cap L^{p_1}}),$$

and for each $t \in [0,T]$, set

$$\delta(t) = \frac{\int_0^t \|\bar{\rho}(s)\|_{\dot{B}^{-1}_{q,\infty}}\,ds}{MT} \leq 1.$$

Let $p = \frac{1}{\varepsilon_0}(2 - \ln\,\delta(t))$. Substituting this value of p into (3.25) gives

$$\|\bar{\rho}(t)\|_{\dot{B}^{-1}_{q,\infty}} \leq \left(\frac{CM^2(T+1)}{\varepsilon_0}\right)\delta(t)(2-\ln\,\delta(t)) = C(M,T,\varepsilon_0)\delta(t)(2-\ln\,\delta(t)).$$

Integrating both sides from 0 to t and dividing both sides by MT gives

$$\delta(t) \leq C(M,T,\varepsilon_0)\int_0^t \delta(s)(2-\ln\,\delta(s))\,ds. \tag{3.26}$$

We apply Osgood's Lemma with $\rho(t) = \delta(t)$, $\mu(r) = r(2-\ln\,r)$, and $\gamma(t) = C(M,T,\varepsilon_0)$ for each $t \in [0,T]$. This proves Theorem 3.1. □

4. The Vanishing Viscosity Limit for (AG_ν)

In this section we establish the vanishing viscosity limit for Holder continuous solutions to the aggregation equation with Newtonian potential. These solutions and their properties are discussed in Section 2. We prove the following theorem.

Theorem 4.1. *Let $T > 0$ and $\alpha > 0$ be fixed, and let $d = 2$ or 3. Let ρ^ν and ρ^0 be solutions to (AG_ν) and (AG_0), respectively, in $C([0,T];C^\alpha(\mathbb{R}^d))$, generated from the same compactly supported initial data $\rho_0 \in C^\alpha(\mathbb{R}^d)$. There exists $C > 0$ such that for ν sufficiently small and for any $t < T$ and $\beta < \alpha$,*

$$\|(\rho^\nu - \rho^0)(t)\|_{L^\infty} \leq C\nu^{\beta Ce^{-Ct}}.$$

Proof. We first apply Theorem 3.1 to show that ρ^ν converges to ρ^0 as $\nu \to 0$ in the $\dot{B}^{-1}_{q,\infty}$-norm for q sufficiently large. Specifically, we show that under the assumptions of Theorem 4.1,

$$\|(\rho^\nu - \rho^0)(t)\|_{\dot{B}^{-1}_{q,\infty}} \leq C\nu^{\beta e^{-Ct}}. \tag{4.1}$$

We then use interpolation and spatial regularity of ρ^ν and ρ^0 to complete the proof of Theorem 4.1.

Let $\chi \in C^\infty_c(\mathbb{R}^d)$ be a smooth bump function with $\chi(x) = 1$ for all $x \in B_1(0)$, and $\chi(x) = 0$ for all $x \in B_2(0)^c$. Set $\chi_n(x) = \chi(\frac{x}{n})$ for each n and each $x \in \mathbb{R}^d$, and set $\rho_{0,n} = \chi_n S_n \rho_0$. Assume ρ^ν_n and ρ^0_n are solutions to (AG_ν) and (AG_0), respectively, with initial data $\rho_{0,n}$. For fixed $t \in [0,T]$, write

$$\begin{aligned}\|(\rho^\nu - \rho^0)(t)\|_{\dot{B}^{-1}_{q,\infty}} &\leq \|(\rho^\nu - \rho^\nu_n)(t)\|_{\dot{B}^{-1}_{q,\infty}} + \|(\rho^\nu_n - \rho^0_n)(t)\|_{\dot{B}^{-1}_{q,\infty}} + \|(\rho^0_n - \rho^0)(t)\|_{\dot{B}^{-1}_{q,\infty}}\\ &:= \delta_1(t) + \delta_2(t) + \delta_3(t).\end{aligned}$$

We apply the proof of Theorem 3.1 to δ_1 and δ_3, keeping in mind that we are now considering the difference of two solutions generated from two distinct initial densities. The resulting estimates on δ_1 and δ_3, while similar to (3.26), will thus have an extra term on the right hand side involving ρ_0 and $\rho_{0,n}$. Specifically, one can conclude that, for $j = 1, 3$,

$$\delta_j(t) \leq \frac{\|\rho_0 - \rho_{0,n}\|_{\dot{B}^{-1}_{q,\infty}}}{M} + C(M,T,\varepsilon_0)\int_0^t \delta_j(s)(2-\ln\,\delta_j(s))\,ds, \tag{4.2}$$

We must estimate $\|\rho_0 - \rho_{0,n}\|_{\dot{B}^{-1}_{q,\infty}}$. Write

$$\|\rho_0 - \rho_{0,n}\|_{\dot{B}^{-1}_{q,\infty}} \leq \|\rho_0 - \chi_n\rho_0\|_{\dot{B}^{-1}_{q,\infty}} + \|\chi_n\rho_0 - \chi_n S_n\rho_0\|_{\dot{B}^{-1}_{q,\infty}}.$$

Since ρ_0 is compactly supported, for sufficiently large n, $\|\rho_0 - \chi_n\rho_0\|_{\dot{B}^{-1}_{q,\infty}} = 0$. It remains to estimate $\|\chi_n\rho_0 - \chi_n S_n\rho_0\|_{\dot{B}^{-1}_{q,\infty}}$. By Bernstein's Lemma, for q sufficiently large,

$$\begin{aligned}
\|\chi_n\rho_0 - \chi_n S_n\rho_0\|_{\dot{B}^{-1}_{q,\infty}} &= \sup_{j\in\mathbb{Z}} 2^{-j}\|\dot{\Delta}_j(\chi_n(\rho_0 - S_n\rho_0))\|_{L^q} \\
&\leq \sup_{j<0} 2^{-j}\|\dot{\Delta}_j(\chi_n(\rho_0 - S_n\rho_0))\|_{L^q} + \sup_{j\geq 0} 2^{-j}\|\dot{\Delta}_j(\chi_n(\rho_0 - S_n\rho_0))\|_{L^q} \\
&\leq \sup_{j<0} 2^{-j}2^{jd(1-1/q)}\|\dot{\Delta}_j(\chi_n(\rho_0 - S_n\rho_0))\|_{L^1} + \|\chi_n\|_{L^q}\|\rho_0 - S_n\rho_0\|_{L^\infty} \\
&\leq \|\chi_n\|_{L^1}\|\rho_0 - S_n\rho_0\|_{L^\infty} + \|\chi_n\|_{L^q}\|\rho_0 - S_n\rho_0\|_{L^\infty} \\
&\leq (\|\chi_n\|_{L^1} + \|\chi_n\|_{L^q})\sum_{k\geq n} 2^{-k\alpha}2^{k\alpha}\|\Delta_k\rho_0\|_{L^\infty} \\
&\leq Cn^d 2^{-n\alpha} \leq C2^{-n\beta}
\end{aligned}$$

for all $\beta < \alpha$, where we performed a change of variables on χ_n to get the second to last inequality, and where C depends on the initial data.

Combining these estimates gives, for sufficiently large n and q,

$$\|\rho_0 - \rho_{0,n}\|_{\dot{B}^{-1}_{q,\infty}} \leq C2^{-n\beta}$$

for any $\beta < \alpha$. Substituting this estimate into (4.2) gives, for any $\beta < \alpha$, and for $j = 1, 3$,

$$\delta_j(t) \leq \frac{C2^{-n\beta}}{M} + C(M, T, \varepsilon_0)\int_0^t \delta_j(s)(2 - \ln\,\delta_j(s))\,ds.$$

By Osgood's Lemma, for $j = 1, 3$,

$$-\log(2 - \log\delta_j(t)) + \log\left(2 - \log\left(\frac{C2^{-n\beta}}{M}\right)\right) \leq C(M, T, \varepsilon_0)t.$$

Taking the exponential twice gives, for $j = 1, 3$,

$$\delta_j(t) \leq e^{2-2e^{-C(M,T,\varepsilon_0)t}}\left(\frac{C2^{-n\beta}}{M}\right)^{e^{-C(M,T,\varepsilon_0)t}}. \tag{4.3}$$

Now consider the term $\delta_2(t) = \|(\rho_n^\nu - \rho_n^0)(t)\|_{\dot{B}^{-1}_{q,\infty}}$. In this case, the two solutions in the difference are generated from the same initial data. However, as ρ_n^0 satisfies the inviscid equation, when taking the difference of ρ_n^ν and ρ_n^0, we see that an equation analogous to (3.1) holds, but with the extra term $\nu\Delta\rho_n^0$ on the right hand side. Applying the proof of Theorem 3.1 to this slightly modified equation results in the estimate

$$\delta_2(t) \leq \frac{C\nu T\|\Delta\rho_n^0\|_{L^\infty(0,T;\dot{B}^{-1}_{q,\infty})}}{M} + C(M, T, \varepsilon_0)\int_0^t \delta_2(s)(2 - \ln\,\delta_2(s))\,ds. \tag{4.4}$$

To estimate $\|\Delta\rho_n^0\|_{\dot{B}^{-1}_{q,\infty}}$, first observe that, by Bernstein's Lemma,

$$\|\Delta\rho_n^0\|_{\dot{B}^{-1}_{q,\infty}} \leq \|\rho_n^0\|_{\dot{B}^{1}_{q,\infty}}.$$

Using the compact support of ρ_n^0 and Bernstein's Lemma, we can write

$$\begin{aligned}
\|\rho_n^0\|_{\dot{B}^1_{q,\infty}} &\leq \sup_{j\leq 0}\|\dot{\Delta}_j\rho_n^0\|_{L^q} + \sup_{j>0} 2^{-j}2^j\|\dot{\Delta}_j\nabla\rho_n^0\|_{L^q} \\
&\leq \|\rho_n^0\|_{L^q} + \|\nabla\rho_n^0\|_{L^q} \leq C(m(B_n(t)))^{1/q}\|\rho_n^0\|_{C^1},
\end{aligned}$$

where the support of $\rho_n^0(t)$ is contained in $B_n(t)$, a ball with radius $R_n(t)$. By (2.6), it follows that $R_n(t)$ satisfies

$$R_n(t) = R_n(0) + \frac{Ct}{1 - T\|\rho_{0,n}\|_{L^\infty}} \|\rho_{0,n}\|_{L^1 \cap L^\infty}.$$

Since $R_n(0) = 2n$, and $\|\rho_{0,n}\|_{L^1\cap L^\infty} \leq \|\rho_0\|_{L^1\cap L^\infty}$, for sufficiently large n, $m(B_n(t))$ can be bounded above by $C(T, \rho_0)n^d$. Therefore,

$$\|\Delta \rho_n^0\|_{\dot{B}^{-1}_{q,\infty}} \leq \|\rho_n^0\|_{\dot{B}^1_{q,\infty}} \leq C(T, \rho_0) n^{d/q} \|\rho_n^0\|_{C^1}. \tag{4.5}$$

Moreover, by (2.4), (2.7), (2.8), and the estimates

$$\begin{aligned}
&\|\rho_{n,0}\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}, \\
&\|\rho_{n,0}\|_{C^\alpha} \leq \|\chi_n\|_{C^\alpha} \|S_n\rho_0\|_{C^\alpha} \leq C\|\rho_0\|_{C^\alpha},
\end{aligned}$$

it follows that

$$\begin{aligned}
\|\rho_n^0\|_{C^1} &\leq \|\rho_n^0\|_{L^\infty} + \|\nabla \rho_n^0\|_{L^\infty} \\
&\leq \frac{\|\rho_{n,0}\|_{L^\infty}}{1 - \|\rho_{n,0}\|_{L^\infty} t} + (1 + Ct\|\rho_{n,0}\|_{L^\infty}) \|(1 - t\rho_{n,0})^{-1}\|^2_{L^\infty} \|\nabla \rho_{n,0}\|_{L^\infty} \|\nabla X_n^{-1}(t)\|_{L^\infty} \\
&\leq C(T, \|\rho_0\|_{L^\infty}) \left(1 + \|\nabla \rho_{n,0}\|_{L^\infty} \|\nabla X_n^{-1}(t)\|_{L^\infty}\right) \\
&\leq C(T, \|\rho_0\|_{L^1}, \|\rho_0\|_{C^\alpha}) \left(1 + \|\rho_{n,0}\|_{C^1}\right) \\
&\leq C(T, \|\rho_0\|_{L^1}, \|\rho_0\|_{C^\alpha}) \left(1 + \|\chi_n\|_{C^1} \|S_n \rho_0\|_{C^1}\right).
\end{aligned}$$

Note that $\|\chi_n\|_{C^1} \leq \frac{C}{n} \leq C$, and

$$\|S_n\rho_0\|_{C^1} \leq \sum_{j=-1}^{n} (\|\Delta_j \rho_0\|_{L^\infty} + \|\Delta_j \nabla \rho_0\|_{L^\infty}) \leq C\|\rho_0\|_{C^\alpha} \sum_{j=-1}^{n} 2^{j(1-\alpha)} \leq C 2^{n(1-\alpha)}.$$

Thus,

$$\|\rho_n^0\|_{C^1} \leq C(T, \|\rho_0\|_{L^1}, \|\rho_0\|_{C^\alpha}) 2^{n(1-\alpha)}.$$

Substituting this estimate into (4.5) gives, for n sufficiently large,

$$\|\Delta \rho_n^0\|_{\dot{B}^{-1}_{q,\infty}} \leq C(T, \rho_0) n^{d/q} 2^{n(1-\alpha)} \leq C(T, \rho_0) 2^{n(1-\beta)}$$

for any $\beta < \alpha$. Setting $\nu = 2^{-n}$ in (4.4) and applying the above estimate gives

$$\delta_2(t) \leq \frac{C2^{-n\beta}}{M} + C(M, T, \varepsilon_0) \int_0^t \delta_2(s)(2 - \ln\, \delta_2(s))\, ds.$$

Again applying Osgood's Lemma and taking the exponential twice gives

$$\delta_2(t) \leq e^{2 - 2e^{-C(M,T,\varepsilon_0)t}} \left(\frac{C2^{-n\beta}}{M} \right)^{e^{-C(M,T,\varepsilon_0)t}}. \tag{4.6}$$

Combining the estimates for δ_1, δ_2, and δ_3 and using that $\nu = 2^{-n}$ gives, for all $\beta < \alpha$ and for ν sufficiently small,

$$\begin{aligned}
\|(\rho^\nu - \rho^0)(t)\|_{\dot{B}^{-1}_{q,\infty}} &\leq \delta_1(t) + \delta_2(t) + \delta_3(t) \\
&\leq Ce^{2 - 2e^{-C(M,T,\varepsilon_0)t}} \left(\frac{C\nu^\beta}{M} \right)^{e^{-C(M,T,\varepsilon_0)t}} \leq C(M, T, \varepsilon_0) \nu^{\beta e^{-C(M,T,\varepsilon_0)t}},
\end{aligned}$$

establishing (4.1).

4.1. **Convergence in L^∞.** We now apply an interpolation argument to show that the vanishing viscosity limit actually holds in the L^∞-norm. By Bernstein's Lemma, for any fixed $N \geq 1$,

$$\begin{aligned}
\|\rho^\nu - \rho^0\|_{L^\infty} &\leq \|\check{\psi}_{-N} * (\rho^\nu - \rho^0)\|_{L^\infty} + \sum_{j=-N}^{N} \|\dot{\Delta}_j(\rho^\nu - \rho^0)\|_{L^\infty} + \sum_{j=N+1}^{\infty} \|\dot{\Delta}_j(\rho^\nu - \rho^0)\|_{L^\infty} \\
&\leq C2^{-N\frac{d}{p_0}} \|\check{\psi}_{-N} * (\rho^\nu - \rho^0)\|_{L^{p_0}} + \sum_{j=-N}^{N} 2^{j(d/q+1)} \|\rho^\nu - \rho^0\|_{\dot{B}^{-1}_{q,\infty}} + \sum_{j=N+1}^{\infty} \|\dot{\Delta}_j(\rho^\nu - \rho^0)\|_{L^\infty} \\
&\leq C2^{-N\frac{d}{p_0}} \|\rho^\nu - \rho^0\|_{L^{p_0}} + (C + 2^{N(d/q+1)}) \|\rho^\nu - \rho^0\|_{\dot{B}^{-1}_{q,\infty}} + 2^{-N\alpha}(\|\rho^\nu\|_{C^\alpha} + \|\rho^0\|_{C^\alpha}) \\
&\leq C2^{-N\alpha}(\|\rho^\nu\|_{L^{p_0}} + \|\rho^0\|_{L^{p_0}} + \|\rho^\nu\|_{C^\alpha} + \|\rho^0\|_{C^\alpha}) + C2^{N(d/q+1)} \|\rho^\nu - \rho^0\|_{\dot{B}^{-1}_{q,\infty}} \\
&\leq C(M, T, \varepsilon_0) \left(2^{-N\alpha} + 2^{N(d/q+1)} \nu^{\beta e^{-C(M,T,\varepsilon_0)t}}\right),
\end{aligned}$$

where we used that $d > p_0$ to get the fourth inequality, and where we applied (4.1) to get the last inequality. To optimize the rate of convergence, we choose N such that $2^{-N\alpha} = 2^{N(d/q+1)} \nu^{\beta e^{-C(M,T,\varepsilon_0)t}}$. This gives

$$N = \frac{-1}{1 + \alpha + d/q} \log_2 \left(\nu^{\beta e^{-C(M,T,\varepsilon_0)t}}\right).$$

After substituting this value of N into the above calculation, we conclude that

$$\|(\rho_\nu - \rho)(t)\|_{L^\infty} \leq C(M, T, \varepsilon_0) \nu^{\beta C(\alpha,d,q) e^{-C(M,T,\varepsilon_0)t}}.$$

This completes the proof of Theorem 4.1. □

Acknowledgements

The author was supported by Simons Foundation Grant No. 429578.

References

[1] J. Azzam and J. Bedrossian. Bounded mean oscillation and the uniqueness of active scalar equations. *Trans. Amer. Math. Soc.*, 367: 3095-3118, 2015. 1

[2] H. Bahouri and J.-Y. Chemin. Equations de transport relatives à des champs de vecteurs non-lipschitziens et mécanique des fluides. *Arch. Rational Mech. Anal.*, 127:159-181, 1994. 8

[3] J. Bedrossian, A.L. Bertozzi, and N. Rodr ́iguez. Local and global well-posedness for aggregation equations and Patlak-Keller-Segel models with degenerate diffusion. *Nonlinearity*, 24(6): 1683-1714, 2011. 1

[4] A. Blanchet, J.A. Carrillo, and P. Laurencot. Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions. *Calc. Var.* 35:133-168, 2009. 1

[5] A. Bertozzi, T. Laurent, and F. Léger. Aggregation via the Newtonian potential and aggregation patches. *M3AS*, 22, supp. 1, 2012. 1, 5

[6] J.-M. Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Annales de l'Ecole Normale Supérieure*, 14:209-246, 1981. 2

[7] J.-Y. Chemin. *Perfect Incompressible Fluids*, Vol 14 of *Oxford Lecture Series in Mathematics and its Applications*. Clarendon Press, Oxford 1998. 2, 3

[8] E. Cozzi, G. Gie, and J. Kelliher. The aggregation equation with Newtonian potential: The vanishing viscosity limit. *J. Math. Anal. Appl.*, 453(2):841?893, 2017. 1, 5

[9] S. Resnick. Dynamical problems in nonlinear advective partial differential equations. Ph.D. thesis, University of Chicago, 1995. 3

[10] Y. Taniuchi, T. Tashiro, and T. Yoneda. On the two-dimensional Euler equations with spatially almost periodic initial data. *J. Math. Fluid Mech.*, 12(4): 594-612, 2010. 7, 11

[11] M. Vishik. Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type. *Ann. Sci. École Norm. Sup.* (4), 32(6): 769-812, 1999. 1, 8

Department of Mathematics, Oregon State University
E-mail address: cozzie@math.oregonstate.edu