THE AXISYMMETRIC EULER EQUATIONS WITH VORTICITY IN BORDERLINE SPACES OF BESOV TYPE

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Abstract. Borderline spaces of Besov type consist of tempered distributions satisfying the property that the partial sums of their $B^0_{\infty,1}$-norm diverge in a controlled way. Misha Vishik established uniqueness of solutions to the two and three-dimensional incompressible Euler equations with vorticity whose $B^0_{\infty,1}$ partial sums diverge roughly at a rate of $N \log N$. In two dimensions, he also established conditions on the initial data for which solutions in his uniqueness class exist. In this paper, we extend existence results of Vishik to the three-dimensional Euler equations with axisymmetric velocity. We also study the inviscid limit of solutions of the Navier-Stokes equations with initial vorticity in these Besov type spaces.

1. Introduction

We consider the Navier-Stokes equations modeling incompressible viscous fluid flow in $\mathbb{R}^3$, given by

$$(NS) \quad \begin{cases} \partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu = -\nabla p_\nu \\ \text{div } u_\nu = 0 \\ u_\nu|_{t=0} = u^0_\nu, \end{cases}$$

and the Euler equations modeling incompressible non-viscous fluid flow in $\mathbb{R}^3$, given by

$$(E) \quad \begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p \\ \text{div } u = 0 \\ u|_{t=0} = u^0. \end{cases}$$

In three dimensions, breakdown of smooth solutions to the Navier-Stokes and Euler equations remains open and has proved to be one of the most difficult problems in fluid mechanics (see [10] for details). For the case of axisymmetric solutions without swirl, however, global existence and uniqueness for (NS) and (E) has been

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established under various assumptions on the initial data. In what follows, we say a vector field v is axisymmetric if it can be written as

$$v(t,x) = v^r(t,r,z)e_r + v^z(t,r,z)e_z,$$

where $z = x_3$, $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$, (e_r, e_θ, e_z) is the cylindrical basis of $\mathbb{R}^3$, and v^r and v^z do not depend on θ. For fluids with axisymmetric divergence-free velocity u_ν solving (NS), the vorticity ω_ν can be written as

$$\omega_\nu = (\partial_z u_\nu^r - \partial_r u_\nu^z)e_\theta := \omega_\nu^\theta e_\theta. \tag{1.1}$$

Moreover, ω_ν satisfies the equality

$$\partial_t \omega_\nu + u_\nu \cdot \nabla \omega_\nu - \nu \Delta \omega_\nu = r^{-1}\omega_\nu u_\nu^r. \tag{1.2}$$

Similarly, for axisymmetric velocity u solving (E), the vorticity ω satisfies

$$\omega = (\partial_z u^r - \partial_r u^z)e_\theta := \omega^\theta e_\theta, \tag{1.3}$$

and

$$\partial_t \omega + u \cdot \nabla \omega = r^{-1}\omega u^r. \tag{1.4}$$

Letting $\alpha = \frac{\omega_\nu^\theta}{r}$, one can use (1.2) to conclude that $||\alpha(t)||_{L^p} \leq ||\alpha^0||_{L^p}$ for all $p \in [1,\infty]$. When $\nu = 0$, (1.4) implies that the L^p-norms of α are conserved.

In [14], Ukhovskii and Yudovich proved that when initial velocity is axisymmetric and satisfies $u^0 \in L^2(\mathbb{R}^3)$, $\omega(u^0) = \omega^0 \in L^2 \cap L^\infty(\mathbb{R}^3)$, and $r^{-1}\omega^0 \in L^2 \cap L^\infty(\mathbb{R}^3)$, there exists a unique global in time solution to (NS) and (E) with $u \in L^\infty([0,T]; L^2 \cap L^p(\mathbb{R}^2))$ for some $p \in (3,6]$ and with $\omega \in L^1([0,T]; L^2(\mathbb{R}^3))$ (in fact, for (NS) the assumption that ω^0 is bounded is unnecessary to prove existence). These conditions on the initial data follow when, for example, u^0 belongs to $H^s(\mathbb{R}^3)$ for $s > \frac{7}{2}$. Shirota and Yanagisawa proceeded to show in [11] that there exists a unique solution u to (NS) in H^s for $s > \frac{5}{2}$ with u^0 axisymmetric and belonging to $H^s(\mathbb{R}^3)$. Finally, in a recent paper of Abidi (see [1]), the author demonstrated existence and uniqueness of a solution to (NS) in $H^{\frac{1}{2}}(\mathbb{R}^3)$ with axisymmetric initial velocity in $H^{\frac{1}{2}}(\mathbb{R}^3)$. For our purposes, the most relevant result appears in a recent paper of Abidi, Hmidi, and Keraani regarding the Euler equations. In [2], the authors prove that if u^0 is an axisymmetric vector field in the critical Besov space $B_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3)$ (see Definition 2) with $p \in [1,\infty]$, and if $r^{-1}\omega^0$ belongs to the Lorentz space $L^{3,1}(\mathbb{R}^3)$ (see Definition 4), then there exists a unique solution u to (E) in $C(\mathbb{R}^+; B_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3))$.

In this paper, we study the three-dimensional Euler equations with axisymmetric initial velocity under the assumption that ω^0 does not necessarily belong to $B_{p,1}^{\frac{3}{p}}(\mathbb{R}^3)$ for any $p \in [1,\infty]$. We do require, however, that the partial sums of the $B_{\infty,1}^0$-norm of ω^0 diverge in a controlled way. Specifically, we assume ω^0 belongs to the space $B_\Gamma(\mathbb{R}^3)$. This space was introduced by Vishik in [15]. Roughly speaking, the space $B_\Gamma(\mathbb{R}^3)$ consists of tempered distributions which satisfy the property that the partial sums of the $B^0_{\infty,1}$-norm diverge no faster than $\Gamma(N)$ (see Definition 3 in Section 2 for more information about B_Γ spaces). In [15], Vishik proved that the solution to (E) is unique as long as vorticity remains in the space $L^\infty([0,T]; B_{\Gamma_1}(\mathbb{R}^d))$ for $d \geq 2$ with $\Gamma_1(N)$ behaving roughly like $N \log N$. He also proved that for $d = 2$, a solution in his uniqueness class exists locally in time for ω^0 in $B_\Gamma(\mathbb{R}^2)$ with $\Gamma(N) = \log N$, and the solution exists globally in time for $\omega^0 \in B_\Gamma$ with $\Gamma(N) = \log^{\frac{1}{2}}(N)$.

The first objective of this paper is to extend the short time existence results established in [15] to three dimensions when initial velocity is axisymmetric. We show that a solution in Vishik's uniqueness class exists for short time under essentially the same assumptions on the initial vorticity as those required in the two-dimensional setting, although we must assume that $r^{-1}\omega^0$ belongs to the Lorentz space $L^{3,1}(\mathbb{R}^3)$ in order to control the growth of L^p-norms of vorticity as time evolves.

To explain our approach for extending Vishik's existence results, we take a moment to clarify his general strategy in the two-dimensional case. Broadly speaking, Vishik's strategy in [15] for showing existence of solutions in his uniqueness class consists of three steps (not necessarily achieved in the following order): (i) Construct a sequence $\{u^n\}$ of smooth solutions to (E) (Cauchy in a certain Besov norm) which lie in his uniqueness class. (ii) Establish an upper bound on the B_{Γ_1}-norms of the corresponding smooth vorticities $\{\omega^n\}$ which is independent of n. (iii) Pass to the limit and show, using the uniform bound established in step (ii), that the limit u is indeed a solution to (E) with vorticity in B_{Γ_1}.

We follow this general strategy for the three-dimensional case. The main difficulty is step (ii), the establishment of a uniform upper bound on the B_{Γ_1}-norms of the smooth vorticities. In [15], Vishik utilizes the property that the vorticity is transported along particle trajectories for two-dimensional flows. Specifically, he proves an estimate of the form

$$\|f \circ g(t)^{-1}\|_{B_{\Gamma_1}} \leq C\|f\|_{B_\Gamma} 2^{C\int_0^t \lambda(\tau)\,d\tau}, \tag{1.5}$$

where g is the particle trajectory map corresponding to the velocity solution to (E), $\lambda(t)$ is (essentially) the B_{Γ_1}-norm of the vorticity at time t, and $f \in B_\Gamma$ is arbitrary. In the two-dimensional case, if we set $f = \omega^0$, then the left hand side represents the vorticity at positive times. One can then use (1.5) and a standard argument in ordinary differential equations to establish an upper bound on the B_{Γ_1}-norm of $\omega(t)$ for $t > 0$. In three dimensions, when $f = \omega^0$, the left hand side of (1.5) no longer represents the time evolution of vorticity. Therefore, we must prove an estimate similar to (1.5) in three dimensions, with the left hand side replaced by $\|\omega(t)\|_{B_{\Gamma_1}}$ (see Proposition 6). As one would expect, the challenge is to bound the vorticity stretching term in the B_{Γ_1}-norm. To achieve this, we use a decomposition of the vorticity from [2], as well as properties of axisymmetric flows, specifically the equality $\omega \cdot \nabla u = \omega r^{-1} u^r$ and the conservation of the quantity $\alpha = r^{-1}\omega^\theta$ along particle trajectories. We also rely on (1.5), keeping in mind that f is arbitrary.

The second objective of this paper is to extend the results of [7] to the axisymmetric setting. In [7], with Kelliher we studied the vanishing viscosity limit in the plane for Vishik's existence and uniqueness class. Our goal is to extend the results of [7] to the three-dimensional case with axisymmetric initial velocity. Specifically, we apply the techniques of [7] to prove two results regarding the three-dimensional fluid equations with axisymmetric initial velocity. In the first result we prove that when $\Gamma(N) = \log^{\kappa}(N)$ for $\kappa \in [0, 1)$, there exists a unique solution u to (E) in $L^{\infty}(\mathbb{R}^+; H^1(\mathbb{R}^3))$ with $u^0 \in H^1(\mathbb{R}^3)$, $\omega^0 \in B_{\Gamma}(\mathbb{R}^3)$ and $r^{-1}\omega^0 \in L^{3,1}(\mathbb{R}^3)$. With $\Gamma(N) = \log N$, we prove an analogous existence and uniqueness statement for short time. In the second result, we show that the vanishing viscosity limit holds in $L^{\infty}([0, T]; L^2(\mathbb{R}^3))$ for these types of solutions to (E). The proof of the vanishing viscosity result closely follows the argument used in [7] and relies on new estimates from [2] for solutions with axisymmetric initial data.

The paper is organized as follows. In Section 2, we define the Littlewood-Paley operators and various function spaces. We also state a few useful lemmas and discuss properties of axisymmetric solutions to (E) with vorticity in B_{Γ} spaces. In Sections 3 and 4, we establish conditions on initial vorticity under which there exists a solution to the axisymmetric Euler equations in Vishik's uniqueness class. In Section 5 we show that if initial vorticity belongs to a certain B_{Γ} space, then there exists a unique velocity in $H^1(\mathbb{R}^3)$ solving the axisymmetric Euler equations, extending the results from [7] to the axisymmetric setting. In Section 6, we prove an inviscid limit result.

2. Background and Preliminary Lemmas

We first define the Littlewood-Paley operators (for a more thorough discussion of these operators, we refer the reader to [13]). We let $\phi \in S(\mathbb{R}^d)$ satisfy supp $\phi \subset \{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, and for every $j \in \mathbb{Z}$ we let $\phi_j(\xi) = \phi(2^{-j}\xi)$ (so $\check{\phi}_j(x) = 2^{jn}\check{\phi}(2^j x)$). We define $\psi_n \in S(\mathbb{R}^d)$ by the equality

$$\psi_n(\xi) = 1 - \sum_{j \geq n} \phi_j(\xi)$$

for all $\xi \in \mathbb{R}^d$, and for $f \in S'(\mathbb{R}^d)$ we define the operator S_n by

$$S_n f = \check{\psi}_n * f.$$

In the following sections we will make frequent use of the Littlewood-Paley operators. For $f \in S'(\mathbb{R}^d)$ and $j \in \mathbb{Z}$, we define the Littlewood-Paley operators Δ_j by

$$\Delta_j f = \begin{cases} 0, & j < -1, \\ \check{\psi}_0 * f, & j = -1, \\ \check{\phi}_j * f, & j > -1. \end{cases}$$

We will also need the paraproduct decomposition introduced by J.-M. Bony in [4]. We recall the definition of the paraproduct and remainder used in this decomposition.

Definition 1. Define the paraproduct of two functions f and g by

$$T_f g = \sum_{\substack{i,j \\ i \leq j-2}} \Delta_i f \Delta_j g = \sum_{j=1}^{\infty} S_{j-1} f \Delta_j g.$$

We use $R(f,g)$ to denote the remainder. $R(f,g)$ is given by the following bilinear operator:

$$R(f,g) = \sum_{\substack{i,j \\ |i-j| \leq 1}} \Delta_i f \Delta_j g.$$

Bony's decomposition then gives

$$fg = T_f g + T_g f + R(f,g).$$

We now define the Besov spaces.

Definition 2. Let $s \in \mathbb{R}$, $(p,q) \in [1,\infty] \times [1,\infty)$. We define the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^d)$ to be the space of tempered distributions f on $\mathbb{R}^d$ such that

$$||f||_{B^s_{p,q}} := ||S_0 f||_{L^p} + \left(\sum_{j=0}^{\infty} 2^{jqs} ||\Delta_j f||^q_{L^p} \right)^{\frac{1}{q}} < \infty.$$

When $q = \infty$, we write

$$||f||_{B^s_{p,\infty}} := ||S_0 f||_{L^p} + \sup_{j \geq 0} 2^{js} ||\Delta_j f||_{L^p}.$$

We define the B_Γ spaces as in [15].

Definition 3. We define B_Γ to be the set of all f in $S'(\mathbb{R}^d)$ satisfying

$$\sum_{j=-1}^{N} ||\Delta_j f||_{L^\infty} = O(\Gamma(N))$$

with the norm

$$||f||_{B_\Gamma} = \sup_{N \geq -1} \frac{1}{\Gamma(N)} \sum_{j=-1}^{N} ||\Delta_j f||_{L^\infty}.$$

Finally, we must define the Lorentz spaces.

Definition 4. The nonincreasing rearrangement of a measurable function f is given by

$$f^*(t) := \inf \left\{ s, m(\{x : |f(x)| > s\}) \leq t \right\}.$$

For $p, q \in [1,\infty]$, we define the Lorentz space $L^{p,q}(\mathbb{R}^d)$ to be the set of tempered distributions f satisfying $||f||_{L^{p,q}} < \infty$, where

$$||f||_{L^{p,q}} = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}, q \in [1,\infty) \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), \ q = \infty. \end{cases}$$

We remark that Lorentz spaces relate to L^p spaces by the equality $L^{p,p} = L^p$. We refer the reader to [12] and [5] for other properties of Lorentz spaces.

Before we state the main theorem, we give a few lemmas which we use throughout the paper. We begin with Bernstein's Lemma. We refer the reader to [6] for a proof of the lemma.

Lemma 1. *(Bernstein's Lemma) Let r_1 and r_2 satisfy $0 < r_1 < r_2 < \infty$, and let p and q satisfy $1 \leq p \leq q \leq \infty$. There exists a positive constant C such that for every integer k , if u belongs to $L^p(\mathbb{R}^d)$, and supp $\hat{u} \subset B(0, r_1\lambda)$, then*

$$\sup_{|\alpha|=k} ||\partial^\alpha u||_{L^q} \leq C^k \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} ||u||_{L^p}. \tag{2.1}$$

Furthermore, if supp $\hat{u} \subset C(0, r_1\lambda, r_2\lambda)$, then

$$C^{-k}\lambda^k ||u||_{L^p} \leq \sup_{|\alpha|=k} ||\partial^\alpha u||_{L^p} \leq C^k \lambda^k ||u||_{L^p}. \tag{2.2}$$

We also make frequent use of the following lemma, which is proved in [7].

Lemma 2. *Let u be a divergence-free vector field in $L^2_{loc}(\mathbb{R}^d)$ with vorticity ω. Then there exists an absolute constant C such that for all $q \geq 0$,*

$$||\Delta_q \nabla v||_{L^\infty} \leq C||\Delta_q \omega||_{L^\infty}.$$

We now state a key lemma regarding the vorticity equation in three dimensions. This lemma allows us to control the growth of the L^p-norm of $\omega(t)$ as time evolves using the $L^{3,1}$-norm of $r^{-1}\omega^0$. We refer the reader to Proposition 2 and Corollary 1 of [8] for a proof of the lemma. (In fact, the author proves a more general lemma for Lorentz spaces $L^{p,q}$, but the lemma for L^p spaces follows by the equality $L^{p,p} = L^p$.)

Lemma 3. *Let u be a divergence-free vector field with coefficients in the space $L^1([0,T]; B^1_{\infty,\infty}(\mathbb{R}^3))$, and let ω satisfy the vorticity equation*

$$\partial_t \omega + u \cdot \nabla \omega = r^{-1}\omega u^r.$$

There exists a constant C such that for all $p \in [1,\infty]$ and $t \in [0,T]$,

$$||\omega(t)||_{L^p} \leq ||\omega^0||_{L^p} \exp(C_1 t ||r^{-1}\omega^0||_{L^{3,1}}). \tag{2.3}$$

Remark 2.4. An inequality similar to that in (2.3) holds for the Navier-Stokes vorticity. Indeed, by (1.2) we have that for all $p \in [1,\infty]$,

$$||\omega_\nu(t)||_{L^p} \leq ||\omega^0_\nu||_{L^p} + \int_0^t ||r^{-1}u_\nu(s)||_{L^\infty} ||\omega_\nu(s)||_{L^p}\, ds. \tag{2.5}$$

Using the Biot-Savart law, we can bound $||r^{-1}u_\nu(s)||_{L^\infty}$ above by $||r^{-1}\omega_\nu^\theta(s)||_{L^{3,1}}$ (see for example [8] and [2]). Moreover, we have that $||r^{-1}\omega_\nu^\theta(s)||_{L^{3,1}} \leq ||r^{-1}\omega_\nu^\theta(0)||_{L^{3,1}}$. This follows from the inequality $||r^{-1}\omega_\nu^\theta(s)||_{L^p} \leq ||r^{-1}\omega_\nu^\theta(0)||_{L^p}$ for all $p \in [1,\infty]$ and a standard interpolation argument. Finally, an application of Gronwall's Lemma yields an estimate analogous to (2.3).

3. A Priori Estimates

Throughout this section, we assume that $\Gamma : \mathbb{R} \to [1,\infty)$ is a locally Lipschitz continuous monotonically nondecreasing function satisfying the following conditions:

(i) $\Gamma(\alpha) = 1$ for $\alpha \in (-\infty, -1]$, $\lim_{\alpha\to\infty}\Gamma(\alpha) = \infty$.
(ii) There exists a constant $C > 0$ such that $C^{-1}\Gamma(\beta) \leq \Gamma(\alpha) \leq C\Gamma(\beta)$ for α, $\beta \in [-1,\infty)$, $|\alpha - \beta| \leq 1$.
(iii) There is a constant $C > 0$ such that

$$C\Gamma(\alpha) \geq \int_\alpha^\infty 2^{-(\xi-\alpha)}\Gamma(\xi)\, d\xi,\ \ \alpha \in [-1,\infty).$$

Let $\Gamma_1(\alpha) = (\alpha+2)\Gamma(\alpha)$ for $\alpha \in [-1,\infty)$, $\Gamma_1(\alpha) = 1$ for $\alpha \in (-\infty,-1)$, and assume that

(iv) Γ_1 satisfies (iii) above;
(v) Γ_1 is convex;
(vi) $\int_1^\infty \Gamma_1(\alpha)^{-1}\, d\alpha = \infty$.

Example 3.1. The function $\Gamma(\alpha) = \log^\kappa(\alpha+3)$ for $\alpha > -1$ and $\kappa \in (0,1]$ satisfies (i)-(vi).

We consider the vorticity formulation to the axisymmetric Euler equations, given by

$$\begin{aligned} &\partial_t\omega + u\cdot\nabla\omega = u^r\frac{\omega}{r},\\ &u = \mathcal{K} * \omega,\\ &\omega|_{t=0} = \omega^0 \in B_\Gamma \cap L^{p_0} \cap L^{p_1},\ \text{and}\ r^{-1}\omega^0 \in L^{3,1}(\mathbb{R}^3), \end{aligned} \tag{3.2}$$

where $p_0 < 3 < p_1$, and where $\mathcal{K}$ represents the Biot-Savart kernel in $\mathbb{R}^3$. Here we are using (1.4).

Remark 3.3. The assumption that ω^0 belongs to $L^{p_0} \cap L^{p_1}(\mathbb{R}^3)$ is utilized in [15] to prove estimates using the Biot-Savart law. These estimates give information about the modulus of continuity of the velocity and, in turn, stretching properties of the particle trajectory map (these properties are described below). The assumption that $r^{-1}\omega^0$ belongs to $L^{3,1}(\mathbb{R}^3)$ will be used to control to L^p-norms of vorticity as time evolves via Lemma 3.

For fixed $T_1 > 0$, we assume that we are given a solution to (3.2) satisfying

$$u \in \mathcal{K} * C([0,T_1]; B_{\Gamma_1} \cap L^{p_0} \cap L^{p_1}(\mathbb{R}^3)). \tag{3.4}$$

We first summarize properties of u satisfying (3.4) and of the particle trajectory map g corresponding to u. These properties can be found in [15] and will be important later in this section. As in [15], let

$$\lambda(t) = \max\left(\sup_{0\le\tau\le t} \|\omega(\tau)\|_{B_{\Gamma_1}}, 1\right),\ t \in [0,T_1], \tag{3.5}$$

and let

$$\tilde{\Gamma}_1(t,m) = \begin{cases} \lambda(t)^{-1}, & -\infty < m \le m_1, \\ 1 + (m+1)\Gamma_1'(-1+), & m_1 \le m \le -1, \\ \Gamma_1(m), & -1 < m < \infty, \end{cases}$$

with $m_1 = -1 - (1-\lambda(t)^{-1})\Gamma_1'(-1+)^{-1}$. Note that for $\alpha > -1$,

$$\Gamma_1'(\alpha) = \Gamma(\alpha) + (\alpha+2)\Gamma'(\alpha) \ge \Gamma(\alpha) \ge 1,$$

and therefore $\Gamma_1'(-1+) \ge 1$. By definition of m_1 and $\lambda(t)$, we infer that $m_1 \ge -2$.

As follows from arguments in [15], there exists an absolute constant C_0 such that the velocity vector field u satisfies

$$|u(t,x) - u(t,y)| \le C_0\lambda(t)\tilde{\Gamma}_1(t, -\log_2|x-y|)|x-y| \tag{3.6}$$

whenever $x \ne y$. Denote by $\mu(t,m)$ the solution to

$$\begin{aligned} &\dot{\mu}(t,m) = -C_0(\log_2 e)\lambda(t)\tilde{\Gamma}_1(t,\mu(t,m)), \\ &\mu(0,m) = m \in \mathbb{R}. \end{aligned} \tag{3.7}$$

In [15], it is established using (3.6) that for each x, y in $\mathbb{R}^3$, the particle trajectory map corresponding to u satisfies

$$|g(t,x) - g(t,y)| \le 2^{-\mu(t,m)},\ t \in [0,T_1], \tag{3.8}$$

where $m = -\log_2|x-y|$.

In the next section, we will prove existence of a solution to (3.2) which satisfies (3.4) by considering a sequence of global in time smooth velocity solutions (i.e. $u \in L^\infty_{loc}(\mathbb{R}^+; C^r(\mathbb{R}^3))$ for all $r > 1$) and passing to the limit in $B^0_{\infty,1}(\mathbb{R}^3)$. The following theorem gives a uniform bound on the B_{Γ_1}-norms of the corresponding smooth sequence of vorticities on a sufficiently short time interval under certain assumptions on Γ. This bound will allow us to conclude that the limit of the smooth sequence lies in Vishik's uniqueness class for short time.

Theorem 1. *Assume* $\Gamma : \mathbb{R} \to [1,\infty)$ *satisfies*

$$(\alpha+2)\Gamma'(\alpha) \le C \text{ for a.e. } \alpha \in [-1,\infty). \tag{3.9}$$

Also assume u^0 is an axisymmetric vector field in $L^2(\mathbb{R}^3)$, and ω^0 satisfies

$$\|\omega^0\|_{B_\Gamma \cap L^{p_0} \cap L^{p_1}} \le C. \tag{3.10}$$

Let u be a regular solution to (3.2) satisfying (3.4), with initial data u^0. Then there exists $T > 0$ and a continuous function $\tilde{\lambda}$ defined on $[0,T]$ such that

$$\|\omega(t)\|_{B_{\Gamma_1}} \le \tilde{\lambda}(t), \ t \in [0,T].$$

Both T and the function $\tilde{\lambda}$ depend only on C in (3.10), $\|r^{-1}\omega^0\|_{L^{3,1}}$, and Γ.

Theorem 1 above is identical to a theorem proved in [15] for the two-dimensional Euler equations. In order to establish the above theorem in the two-dimensional case, Vishik estimates the time evolution of the B_{Γ_1} norm of initial vorticity under composition with the particle trajectory map. His estimate in fact holds in dimension three with ω^0 replaced with arbitrary f in $B_\Gamma(\mathbb{R}^3)$. This estimate will be useful to us, so we state it here.

Proposition 4. *(Vishik) Assume $\Gamma : \mathbb{R} \to [1,\infty)$ satisfies (3.9), and let f in $B_\Gamma(\mathbb{R}^3)$ be arbitrary. Let u be a regular solution to (3.2) which satisfies (3.4) for given $T_1 > 0$, and let g be the particle trajectory map corresponding to u. Define λ as in (3.5). Then*

$$\|f \circ g(t)^{-1}\|_{B_{\Gamma_1}} \le C\|f\|_{B_\Gamma} 2^{C\int_0^t \lambda(\tau)\,d\tau} \tag{3.11}$$

for all $t \in [0,T_1]$.

The following proposition follows from the proof of Proposition 4 in [15] and will be useful to us in what follows. We outline the proof of the proposition here and refer the reader to [15] for further details.

Proposition 5. *(Vishik) Assume Γ satisfies (3.9). Let f belong to $B_\Gamma(\mathbb{R}^3)$ and let μ and λ be as above. Also assume m and t are such that $\mu(t,m-1) > 0$. Then the following inequality holds for any $\beta \in (0,1]$:*

$$\sum_{l \ge m} 2^{-\beta\mu(t,l-1)}\|\Delta_l f\|_{L^\infty} \le C\|f\|_{B_\Gamma} 2^{C\int_0^t \lambda(\tau)\,d\tau}\Gamma(\mu(t,m-1))2^{-\beta\mu(t,m-1)}. \tag{3.12}$$

Proof. Set $\rho(t,l) = 2^{-\beta\mu(t,l-1)}$. Then, following an argument identical to that in (5.15) of [15], we see that

$$\sum_{l \ge m} \rho(t,l)\|\Delta_l f\|_{L^\infty} \le \|f\|_{B_\Gamma} \int_m^\infty -\partial_\xi \rho(t,\xi)\Gamma(\xi)\,d\xi. \tag{3.13}$$

We estimate the integral in (3.13) by applying (ii) and the inequality

$$\Gamma(\xi) \le \Gamma(\mu(t,\xi))2^{C\int_0^t \lambda(\tau)\,d\tau}, \tag{3.14}$$

which is proved in [15] using (3.9) and (3.7). Specifically, we write

(3.15)

$$\int_m^\infty -\partial_\xi \rho(t,\xi)\Gamma(\xi)\,d\xi = \int_m^\infty \partial_\xi(\beta\mu(t,\xi-1))2^{-\beta\mu(t,\xi-1)}\Gamma(\xi)\,d\xi$$

$$= \int_{m-1}^\infty \partial_\xi(\beta\mu(t,\xi))2^{-\beta\mu(t,\xi)}\Gamma(\xi+1)\,d\xi \leq C\int_{m-1}^\infty \partial_\xi(\beta\mu(t,\xi))2^{-\beta\mu(t,\xi)}\Gamma(\xi)\,d\xi$$

$$\leq C2^{C\int_0^t\lambda(\tau)\,d\tau}\int_{m-1}^\infty \partial_\xi(\beta\mu(t,\xi))2^{-\beta\mu(t,\xi)}\Gamma(\mu(t,\xi))\,d\xi$$

$$\leq C\beta 2^{C\int_0^t\lambda(\tau)\,d\tau}\int_{\mu(m-1,t)}^\infty 2^{-\beta\mu}\Gamma(\mu)\,d\mu.$$

We conclude that

$$\int_m^\infty -\partial_\xi \rho(t,\xi)\Gamma(\xi)\,d\xi \leq C2^{C\int_0^t\lambda(\tau)\,d\tau}\int_{\mu(t,m-1)}^\infty 2^{-\beta\mu}\Gamma(\mu)\,d\mu$$

$$\leq C2^{C\int_0^t\lambda(\tau)\,d\tau}\int_{\beta\mu(t,m-1)}^\infty 2^{-\beta\mu}\Gamma(\beta\mu)\,d\mu \leq C2^{C\int_0^t\lambda(\tau)\,d\tau}2^{-\beta\mu(t,m-1)}\Gamma(\mu(t,m-1)).$$

Here we used (3.9) and the property $0 < \beta\mu(t,m-1) \leq \mu(t,m-1)$ to get the second inequality. To get the last inequality, we applied a change of variables and (iii). Substituting this estimate into (3.13) yields (3.12). □

For the axisymmetric Euler equations, vorticity is not transported along particle trajectories; hence the left hand side of (3.11) above does not represent vorticity at positive times. While Proposition 4 will be useful, we must prove the following analogous proposition, which estimates the B_{Γ_1}-norm of vorticity at positive times for solutions to (3.2) satisfying (3.4). The following proposition is the key ingredient in the proof of Theorem 1.

Proposition 6. *Assume $u^0 \in L^2(\mathbb{R}^3)$ is an axisymmetric vector field, and let u be a regular solution to (3.2) which satisfies (3.4). Let λ and T_1 be as above. Then*

$$\|\omega(t)\|_{B_{\Gamma_1}} \leq C\|\omega_0\|_{B_\Gamma} 2^{C\int_0^t\lambda(\tau)\,d\tau} \tag{3.16}$$

for all $t \in [0, T_1]$.

Proof. We first apply the B_{Γ_1} norm to the equality

$$\omega(t,x) = \omega^0(g(t)^{-1}(x)) + \int_0^t \left(u^r\frac{\omega}{r}\right)(s, g(s)\circ g(t)^{-1}(x))\,ds$$

and utilize Proposition 4 to write

$$\|\omega(t)\|_{B_{\Gamma_1}} \leq C\|\omega_0\|_{B_\Gamma} 2^{C\int_0^t\lambda(\tau)\,d\tau} + \int_0^t \left\|\left(u^r\frac{\omega}{r}\right)(s, g(s)\circ g(t)^{-1}(\cdot))\right\|_{B_{\Gamma_1}}. \tag{3.17}$$

To estimate the time integral above, we will decompose the vorticity as in [2]. Let ω_l be the unique solution to the problem

$$\partial_t \omega_l + (u\cdot\nabla)\omega_l = \omega_l\cdot\nabla u,$$
$$\omega_l\big|_{t=0} = \Delta_l \omega^0.$$

By linearity and uniqueness (which follows from [15]),

$$\omega = \sum_{l\geq -1} \omega^l. \tag{3.18}$$

Moreover, in [2] it is shown (see Proposition 4.2) that for each $l \geq -1$,

$$\|\omega^l(t)\|_\infty \leq \|\Delta_l\omega^0\|_{L^\infty} e^{Ct\|r^{-1}\omega^0\|_{L^{3,1}}}. \tag{3.19}$$

For fixed but arbitrary $\beta\in(0,\frac{1}{2})$, we choose m satisfying

$$\beta\mu(t,m-1) > 2N > \beta\mu(t,m-2), \tag{3.20}$$

and consider

(3.21)
$$\int_0^t \left\| \left(u^r\frac{\omega}{r}\right)(s, g(s)\circ g(t)^{-1}(\cdot))\right\|_{B_{\Gamma_1}} ds \leq \int_0^t \sum_{l\leq m}\left\| \left(u^r\frac{\omega^l}{r}\right)(s, g(s)\circ g(t)^{-1}(\cdot))\right\|_{B_{\Gamma_1}} ds$$
$$+\int_0^t \sum_{l\geq m+1}\left\| \left(u^r\frac{\omega^l}{r}\right)(s, g(s)\circ g(t)^{-1}(\cdot))\right\|_{B_{\Gamma_1}} ds.$$

(The reasons for our choice of m will become clear in what follows. Note that (3.20) ensures that $m > 2N$, as $\mu(t,x)$ is decreasing in t and increasing in x.) We begin by estimating the first term on the right hand side of (3.21). Using (3.19), the inequality $\|r^{-1}u^r\|_{L^\infty} \leq \|r^{-1}\omega^0\|_{L^{3,1}}$, and the definition of the B_Γ-norm, we can write the series of estimates

$$\sum_{j\leq N}\sum_{l\leq m}\left\|\Delta_j\left\{\left(u^r\frac{\omega^l}{r}\right)(s, g(s)\circ g(t)^{-1}(\cdot))\right\}\right\|_{L^\infty} \leq CN\sum_{l\leq m}\|\omega^l(s)\|_{L^\infty}\left\|r^{-1}u^r(s)\right\|_{L^\infty}$$
$$\leq CN\sum_{l\leq m}\|\Delta_l\omega^0\|_{L^\infty}e^{Cs\|r^{-1}\omega^0\|_{L^{3,1}}}\left\|r^{-1}\omega^0\right\|_{L^{3,1}}$$
$$\leq CN\Gamma(m)\|\omega^0\|_{B_\Gamma}e^{Cs\|r^{-1}\omega^0\|_{L^{3,1}}}\left\|r^{-1}\omega^0\right\|_{L^{3,1}}.$$

Integrating in time gives

$$\begin{aligned}&\int_0^t\sum_{j\leq N}\sum_{l\leq m}\left\|\Delta_j\left\{\left(u^r\frac{\omega^l}{r}\right)(s, g(s)\circ g(t)^{-1}(\cdot))\right\}\right\|_{L^\infty} ds\\ &\qquad \leq CN\Gamma(m)\|\omega^0\|_{B_\Gamma}e^{Ct\|r^{-1}\omega^0\|_{L^{3,1}}}.\end{aligned} \tag{3.22}$$

We now estimate the quantity

$$\int_0^t \sum_{l\geq m+1} \left\| \left(u^r \frac{\omega^l}{r} \right) (s, g(s)\circ g(t)^{-1}(\cdot)) \right\|_{B_{\Gamma_1}} ds. \tag{3.23}$$

We first note that $\frac{\omega^l}{r}$ is advected by the flow. Specifically,

$$\frac{\omega^l}{r}(s, g(s)\circ g(t)^{-1}(x)) = \frac{\Delta_l \omega^0}{r}(g(t)^{-1}(x)), \tag{3.24}$$

so we can rewrite the integrand in (3.23) as

$$\sum_{l\geq m+1} \left\| \frac{\Delta_l \omega^0}{r}(g(t)^{-1}(\cdot)) u^r(s, g(s)\circ g(t)^{-1}(\cdot)) \right\|_{B_{\Gamma_1}}. \tag{3.25}$$

We again utilize Proposition 4 and write

$$\begin{aligned} &\sum_{l\geq m+1} \left\| \frac{\Delta_l \omega^0}{r}(g(t)^{-1}(\cdot)) u^r(s, g(s)\circ g(t)^{-1}(\cdot)) \right\|_{B_{\Gamma_1}} \\ &\qquad \leq C 2^{C\int_0^t \lambda(\tau)\, d\tau} \sum_{l\geq m+1} \left\| \frac{\Delta_l \omega^0}{r}(\cdot) u^r(s, g(s,\cdot)) \right\|_{B_\Gamma}. \end{aligned} \tag{3.26}$$

Thus it remains to estimate

$$\sum_{l\geq m+1} \left\| \frac{\Delta_l \omega^0}{r}(\cdot) u^r(s, g(s,\cdot)) \right\|_{B_\Gamma}. \tag{3.27}$$

Observe that in cylindrical coordinates, the operator $(\Delta_l \omega^0)\cdot\nabla$ takes the form

$$(\Delta_l\omega^0)\cdot\nabla = (\Delta_l\omega^0)^r \partial_r + \frac{1}{r}(\Delta_l\omega^0)^\theta \partial_\theta + (\Delta_l\omega^0)^z \partial_z = \frac{1}{r}(\Delta_l\omega^0)^\theta\partial_\theta, \tag{3.28}$$

since $\Delta_l\omega^0$ is the curl of the axisymmetric vector field $\Delta_l u^0$ and therefore only has a θ component. It follows that

$$\begin{aligned} \frac{\Delta_l\omega^0}{r} u^r(s, g(s,\cdot)) &= \frac{1}{r}(\Delta_l\omega^0)^\theta e_\theta(u^r(s,g(s,\cdot))) = \frac{1}{r}(\Delta_l\omega^0)^\theta \partial_\theta(u^r(s,g(s,\cdot))e_r), \\ &= \frac{1}{r}(\Delta_l\omega^0)^\theta\partial_\theta(u^r(s,g(s,\cdot))e_r + u^z(s,g(s,\cdot))e_z) = (\Delta_l\omega^0)\cdot\nabla(u(s,g(s,\cdot))). \end{aligned}$$

Above we used the property that $u^r(s, g(s,x))$ does not depend on θ to get the second equality and the property that $u^z(s,g(s,x))$ does not depend on θ to get the third equality. The last equality follows from (3.28). The above analysis implies that

$$\sum_{l\geq m+1} \left\| \frac{\Delta_l\omega^0}{r}(\cdot) u^r(s,g(s,\cdot)) \right\|_{B_\Gamma} = \sum_{l\geq m+1} \left\| (\Delta_l\omega^0)\cdot\nabla(u(s,g(s,\cdot))) \right\|_{B_\Gamma}.$$

The divergence-free property of $\Delta_l\omega^0$ and Bernstein's Lemma allow us to write

$$\begin{aligned}\sum_{l\geq m+1}\sum_{j\leq N}&\left\|\Delta_j\left\{(\Delta_l\omega^0)\cdot\nabla(u(s,g(s,\cdot)))\right\}\right\|_{L^\infty}\\ &\leq C\sum_{i=1}^{3}\sum_{l\geq m+1}\sum_{j\leq N}2^j\left\|\Delta_j\left\{(\Delta_l\omega_i^0)(u(s,g(s,\cdot)))\right\}\right\|_{L^\infty}\\ &\leq C\sum_{i=1}^{3}\sum_{l\geq m+1}\sum_{j\leq N}2^{2j}\left\|\Delta_j\left\{(\Delta_l\omega_i^0)(u(s,g(s,\cdot)))\right\}\right\|_{L^3}.\end{aligned}$$

The product inside the L^3-norm can be written as the sum of the three terms determined by Bony's paraproduct decomposition:

$$(\Delta_l\omega_i^0)u(s,g(s,\cdot))=T_{\Delta_l\omega_i^0}u(s,g(s,\cdot))+T_{u(s,g(s,\cdot))}\Delta_l\omega_i^0+R(\Delta_l\omega_i^0,u(s,g(s,\cdot))).$$

Since $m>2N$, it is clear that for $j\leq N$ and $l\geq m+1$, with N sufficiently large ($N>2$),

$$\Delta_j(T_{\Delta_l\omega_i^0}u(s,g(s,\cdot)))=\Delta_j(T_{u(s,g(s,\cdot))}\Delta_l\omega_i^0)=0,$$

leaving only $\Delta_jR(\Delta_l\omega_i^0,u(s,g(s,\cdot)))$. To estimate this term, we use properties of Littlewood-Paley operators and write

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j\leq N}\sum_{l\geq m+1}&2^{2j}\|\Delta_jR(\Delta_l\omega_i^0,u(s,g(s,\cdot)))\|_{L^3}\\ &\leq C\sum_{j\leq N}\sum_{l\geq m+1}2^{2j}\sum_{k\geq j-3}\|\Delta_k(\Delta_l\omega^0)\|_{L^\infty}\|\Delta_k(u(s,g(s,\cdot)))\|_{L^3}\\ &\leq C2^{2N}\sum_{k\geq m}2^{-\beta\mu(s,k-1)}\|\Delta_k\omega^0\|_{L^\infty}2^{\beta\mu(s,k-1)}\|\Delta_k(u(s,g(s,\cdot)))\|_{L^3},\end{aligned}\tag{3.29}$$

where β corresponds to the β used in our initial choice of m. It follows from (3.29), Proposition 5, and Lemma 7 below that

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j\leq N}\sum_{l\geq m+1}&2^{2j}\|\Delta_jR(\Delta_l\omega_i^0,u(s,g(s,\cdot)))\|_{L^3}\\ &\leq C2^{2N}\|\omega^0\|_{B_\Gamma}2^{C\int_0^s\lambda(\tau)\,d\tau}\Gamma(\mu(s,m-1))2^{-\beta\mu(s,m-1)}.\end{aligned}$$

Combining the above estimates, we can finally conclude that

$$\begin{aligned}\int_0^t\sum_{j\leq N}\sum_{l\geq m+1}&\left\|\Delta_j\left\{\frac{\Delta_l\omega^0}{r}(\cdot)u^r(s,g(s,\cdot))\right\}\right\|_{L^\infty}ds\\ &\leq C2^{2N}\|\omega^0\|_{B_\Gamma}2^{C\int_0^t\lambda(\tau)\,d\tau}\int_0^t\Gamma(\mu(s,m-1))2^{-\beta\mu(s,m-1)}\,ds\\ &\leq Ct2^{2N}\|\omega^0\|_{B_\Gamma}2^{C\int_0^t\lambda(\tau)\,d\tau}\Gamma(m)2^{-\beta\mu(t,m-1)},\end{aligned}\tag{3.30}$$

where we used the property that Γ is monotonically nondecreasing and that $\mu(t,m)$ is decreasing in t to get the last inequality. It follows from our choice of m, (3.14), (3.9), and the inequality $\partial_m \mu(t,m) \leq 1$ that

$$\begin{aligned}\Gamma(m) &\leq C\Gamma(\mu(t,m))2^{C\int_0^t \lambda(\tau)\,d\tau}\\ &\leq C\Gamma(2N)2^{C\int_0^t \lambda(\tau)\,d\tau} \leq C\Gamma(N)2^{C\int_0^t \lambda(\tau)\,d\tau}.\end{aligned} \tag{3.31}$$

Combining (3.30) and (3.31), we infer that

$$\int_0^t \sum_{l\geq m+1} \left\| \frac{\Delta_l \omega^0}{r}(\cdot) u^r(s, g(s,\cdot)) \right\|_{B_\Gamma} ds \leq C\|\omega^0\|_{B_\Gamma} 2^{C\int_0^t \lambda(\tau)\,d\tau}.$$

We insert this estimate into (3.26) and combine (3.23), (3.24), (3.25), and (3.26). We conclude that

$$\int_0^t \sum_{l\geq m+1} \left\| \left(u^r \frac{\omega^l}{r}\right)(s, g(s)\circ g(t)^{-1}(\cdot)) \right\|_{B_{\Gamma_1}} ds \leq C 2^{C\int_0^t \lambda(\tau)\,d\tau} \|\omega^0\|_{B_\Gamma}. \tag{3.32}$$

We return to the first term on the right hand side of (3.21). We combine (3.22) with (3.31) and write

$$\int_0^t \sum_{l\leq m} \left\| \left(u^r \frac{\omega^l}{r}\right)(s, g(s)\circ g(t)^{-1}(\cdot)) \right\|_{B_{\Gamma_1}} ds \leq C 2^{C\int_0^t \lambda(\tau)\,d\tau} \|\omega^0\|_{B_\Gamma} e^{Ct\|r^{-1}\omega^0\|_{L^{3,1}}}. \tag{3.33}$$

We insert the estimates from (3.32) and (3.33) into (3.21), and we apply the resulting estimate to (3.17). This yields (3.16).

To complete the proof of Proposition 6, we must prove the following lemma.

Lemma 7. *If μ, u, and g are as above, then for all $\beta \in (0, \frac{1}{2})$ there exists C depending only on the initial data and s such that*

$$\sup_{k\geq m} 2^{\beta\mu(s,k-1)} \|\Delta_k(u(s, g(s,\cdot)))\|_{L^3} \leq C.$$

Moreover, $C = C(s)$ belongs to $L^\infty_{loc}(\mathbb{R}^+)$.

Proof. The proof of Lemma 7 relies on membership of u to $L^\infty(\mathbb{R}^+; W_3^1(\mathbb{R}^3))$ and on stretching properties of the particle trajectory map g, as given in (3.8). To simplify notation in what follows, we temporarily suppress the time dependance.

We first apply a change of variables and utilize the property that $\int \check{\phi} = 0$ to write

$$\begin{aligned}\Delta_k(u(g(x))) &= 2^{kd} \int_{\mathbb{R}^3} \check{\phi}(2^k y) u(g(x-y))\,dy\\ &= \int_{\mathbb{R}^3} \check{\phi}(z)[u(g(x-2^{-k}z)) - u(g(x))]\,dz\\ &= \int_{\mathbb{R}^3} \check{\phi}(z) \frac{u(g(x-2^{-k}z)) - u(g(x))}{|g(x-2^{-k}z) - g(x)|^\alpha} |g(x-2^{-k}z) - g(x)|^\alpha\,dz\end{aligned}$$

for any fixed $\alpha \in (0,1)$. It follows from Minkowski's inequality and (3.8) that

(3.34)
$$\begin{aligned}\|\Delta_k(u(g(\cdot)))\|_{L^3} &\leq \int_{\mathbb{R}^3} |\check{\phi}(z)| \left\| \frac{u(g(\cdot - 2^{-k}z)) - u(g(\cdot))}{|g(\cdot - 2^{-k}z) - g(\cdot)|^\alpha} |g(\cdot - 2^{-k}z) - g(\cdot)|^\alpha \right\|_{L^3} dz \\ &\leq \int_{\mathbb{R}^3} |\check{\phi}(z)| 2^{-\alpha\mu(s,M)} \left\| \frac{u(g(\cdot - 2^{-k}z)) - u(g(\cdot))}{|g(\cdot - 2^{-k}z) - g(\cdot)|^\alpha} \right\|_{L^3} dz,\end{aligned}$$

where $M = -\log_2 |2^{-k}z|$. The quantity
$$\left\| \frac{u(g(\cdot - 2^{-k}z)) - u(g(\cdot))}{|g(\cdot - 2^{-k}z) - g(\cdot)|^\alpha} \right\|_{L^3}$$
can be estimated using the membership of u to the space $L^\infty_{loc}(\mathbb{R}^+; F^\alpha_3(\mathbb{R}^3))$ for $\alpha < 1$. We define $F^\alpha_p(\mathbb{R}^3)$ as in [3]. For $\alpha \in (0,1)$ and for $p \in [1,\infty]$, let $F^\alpha_p(\mathbb{R}^3)$ to be the set of functions v in $L^p(\mathbb{R}^3)$ such that there exists a function U in $L^p(\mathbb{R}^3)$ satisfying

$$\frac{|v(x) - v(y)|}{|x-y|^\alpha} \leq U(x) + U(y) \tag{3.35}$$

for all x, y in $\mathbb{R}^3$. Define

$$\|v\|_{F^\alpha_p} = \|v\|_{L^p} + \inf\{\|U\|_{L^p} : U \text{ satisfies } (3.35)\}. \tag{3.36}$$

In [3], the authors show equivalence of $F^\alpha_p(\mathbb{R}^3)$ and the Triebel-Lizorkin space $F^\alpha_{p,\infty}(\mathbb{R}^3)$. Moreover, $W^\alpha_p(\mathbb{R}^3) \hookrightarrow F^\alpha_{p,\infty}(\mathbb{R}^3)$, so for all $t > 0$,

(3.37)
$$\begin{aligned}\|u(t)\|_{F^\alpha_3} &\leq C\|u(t)\|_{W^\alpha_3} \leq C(\|u(t)\|_{L^3} + \|\omega(t)\|_{L^3}) \leq C(\|u(t)\|_{L^2\cap L^\infty} + \|\omega(t)\|_{L^3}) \\ &\leq C(\|u^0\|_{L^2} + \|\omega^0\|^{\gamma_1}_{L^{p_0}} \|\omega^0\|^{\gamma_2}_{L^{p_1}} + \|\omega^0\|_{L^3}) e^{C\gamma_3 t \|r^{-1}\omega^0\|_{L^{3,1}}},\end{aligned}$$

where we used Proposition 3.2 of [15], which states that

$$\|u\|_{L^\infty} \leq C\|\omega\|^{\frac{p_0(p_1-3)}{3(p_1-p_0)}}_{L^{p_0}} \|\omega\|^{\frac{p_1(3-p_0)}{3(p_1-p_0)}}_{L^{p_1}}, \tag{3.38}$$

combined with Lemma 3, to get the last inequality above. Here γ_1, γ_2, and γ_3 depend on the dimension, p_0, and p_1.

We now reintroduce the time variable s. It follows from measure-preserving properties of the particle trajectory map and translation invariance of the L^3-norm that for any $U \in L^3(\mathbb{R}^3)$ satisfying (3.35) with $v = u(s,\cdot)$,

$$\begin{aligned}\left\| \frac{u(s, g(s, \cdot - 2^{-k}z)) - u(s, g(s,\cdot))}{|g(s, \cdot - 2^{-k}z) - g(s,\cdot)|^\alpha} \right\|_{L^3} &\leq 2\|U(g(s,\cdot))\|_{L^3} \\ &\leq 2\|u(s)\|_{L^3} + 2\|U\|_{L^3}.\end{aligned} \tag{3.39}$$

Taking the infimum of (3.39) over all such U, we conclude from (3.36) and (3.37) that

$$\left\| \frac{u(s,g(s,\cdot-2^{-k}z))-u(s,g(s,\cdot))}{|g(s,\cdot-2^{-k}z)-g(s,\cdot)|^{\alpha}} \right\|_{L^3} \leq 2\|u(s)\|_{F_3^{\alpha}} \leq C(s),$$

where $C(s)$ is given by the last line of (3.37). Substituting this estimate into (3.34) gives

$$\|\Delta_k(u(s,g(s,\cdot)))\|_{L^3} \leq C(s)\int_{\mathbb{R}^3} |\check{\phi}(z)| 2^{-\alpha\mu(s,M)}\,dz. \tag{3.40}$$

It remains to estimate

$$\int_{\mathbb{R}^3} |\check{\phi}(z)| 2^{-\alpha\mu(s,M)}\,dz.$$

First note that

$$M = -\log_2|2^{-k}z| = -\log_2 2^{-k} - \log_2|z| = k - \log_2|z|.$$

We consider two cases: $|z|<2$ and $|z|\geq 2$. When $|z|<2$, $M>k-1$, so $\mu(s,M)>\mu(s,k-1)$, which means $2^{-\alpha\mu(s,M)} \leq 2^{-\alpha\mu(s,k-1)}$. We can then write

$$\begin{aligned} \int_{\mathbb{R}^3} |\check{\phi}(z)| 2^{-\alpha\mu(s,M)}\,dz &= \int_{|z|<2} |\check{\phi}(z)| 2^{-\alpha\mu(s,M)}\,dz + \int_{|z|\geq 2} |\check{\phi}(z)| 2^{-\alpha\mu(s,M)}\,dz \\ &\leq C2^{-\alpha\mu(s,k-1)} + \int_{|z|\geq 2} |\check{\phi}(z)| 2^{-\alpha\mu(s,M)}\,dz. \end{aligned} \tag{3.41}$$

We now turn to the case $|z|\geq 2$. Since $\mu(s,m)$ is concave (see Proposition 5.3 of [15]), it follows that

$$\begin{aligned} 2^{-\mu(s,M)} &= 2^{-\mu(s,k-\log_2|z|)} = 2^{-\mu\left(s,\frac{2k-2\log_2|z|}{2}\right)} \\ &\leq 2^{-\frac{1}{2}\mu(s,2k)}2^{-\frac{1}{2}\mu(s,-2\log_2|z|)} \leq 2^{-\frac{1}{2}\mu(s,k)}2^{-\frac{1}{2}\mu(s,-2\log_2|z|)}, \end{aligned} \tag{3.42}$$

where we used the fact that μ is increasing in m. Since $|z|\geq 2$, it follows that $-2\log_2|z| \leq -2$. Moreover, for $m\leq -2$, $\tilde{\Gamma}_1(t,m)=\lambda(t)^{-1}$. Substituting this information into (3.7) and solving for μ, we conclude that for $m\leq -2$,

$$\mu(s,m) = m - C_0(\log_2 e)s.$$

Therefore, when $|z|\geq 2$,

$$\mu(s,-2\log_2|z|) = -2\log_2|z| - C_0(\log_2 e)s,$$

and

$$2^{-\frac{\alpha}{2}\mu(s,-2\log_2|z|)} = 2^{\alpha\log_2|z|}2^{\frac{\alpha}{2}C_0(\log_2 e)s} = |z|^{\alpha}2^{\frac{\alpha}{2}C_0(\log_2 e)s}.$$

Substituting this information into (3.41) and observing that $|z|^\alpha|\check{\phi}(z)|$ is integrable, we can write

(3.43)

$$\int_{\mathbb{R}^3} |\check{\phi}(z)| 2^{-\alpha\mu(s,M)}\, dz \le 2^{\frac{\alpha}{2} C_0 (\log_2 e) s}(2^{-\alpha\mu(s,k-1)} + 2^{-\frac{\alpha}{2}\mu(s,k)}) \le C(s) 2^{-\frac{\alpha}{2}\mu(s,k-1)}.$$

Combining (3.43) and (3.40) gives

$$\|\Delta_k(u(s, g(s,\cdot)))\|_{L^\infty} \le C(s) 2^{-\frac{\alpha}{2}\mu(s,k-1)}.$$

We set $\beta = \frac{\alpha}{2}$. Lemma 7 follows. □

This competes the proof of Proposition 6. □

We can now complete the proof of Theorem 1. We follow [15]. First, note that Proposition 6 implies that

$$\lambda(t) \le \hat{C} 2^{C \int_0^t \lambda(\tau)\, d\tau}, \tag{3.44}$$

where $\hat{C} = \max\{1, C\|\omega^0\|_{B_\Gamma}\}$. Set $\gamma(t) = \hat{C} 2^{C\int_0^t \lambda(\tau)\, d\tau}$. Then $\lambda(t) \le \gamma(t)$ and $\dot{\gamma} \le C\gamma^2$. Now let $\tilde{\lambda}$ satisfy the differential equation

$$\dot{\tilde{\lambda}} = C\tilde{\lambda}^2, \ \tilde{\lambda}(0) = \gamma(0), \tag{3.45}$$

where C is as in (3.44). Choose $T > 0$ to be less than the blow-up time for (3.45). Using Gronwall's lemma, one can show that $\gamma(t) - \tilde{\lambda}(t) \le 0$ for all $t \in [0, T]$. Therefore $\lambda(t) \le \gamma(t) \le \tilde{\lambda}(t)$ for all $t \in [0, T]$, and Theorem 1 follows.

4. Construction of the Flow

Throughout this section, we assume that Γ satisfies (i)-(vi). We prove the following theorem.

Theorem 2. *Assume ω^0 belongs to $L^{p_0} \cap L^{p_1} \cap B_\Gamma(\mathbb{R}^3)$ with $p_0 < 3 < p_1$ and with Γ satisfying (3.9). Also assume $r^{-1}\omega^0$ belongs to $L^{3,1}(\mathbb{R}^3)$ and u^0 is an axisymmetric vector field in $L^2(\mathbb{R}^3)$. Then there exists $T > 0$ and a solution ω to (3.2) with initial data ω^0 which satisfies*

$$\omega \in L^\infty([0, T]; L^{p_0} \cap L^{p_1} \cap B_{\Gamma_1}(\mathbb{R}^3)).$$

Proof. Our strategy for proving Theorem 2 is virtually identical to that employed in Section 8 of [15] to prove existence for the two-dimensional Euler equations. The idea is to construct a sequence of smooth solutions to (3.2) and to show that the sequence of smooth velocities are Cauchy in the space $B^0_{\infty,1}(\mathbb{R}^3)$. One can then show that the limit of the sequence satisfies the conditions of the theorem.

As the proof of Theorem 2 is similar to that in Section 8 of [15], we will omit many of the details and instead refer the reader to [15]. We must, however, take some care in constructing a sequence of smooth solutions to the axisymmetric equations. Given ω^0 and u^0 as in Theorem 2, we construct the sequence $u^{n,0} = S_n u^0$. Note that membership of ω^0 to $B_\Gamma(\mathbb{R}^3)$ ensures that, for each fixed n, $u^{n,0}$ belongs to $C^r(\mathbb{R}^3)$ for all $r > 1$. This follows by the series of estimates

$$\begin{aligned}\|u^{n,0}\|_{C^r} &= \sup_{-1\le j\le n} 2^{jr}\|\Delta_j S_n u^0\|_{L^\infty} \le C\|u^0\|_{L^2} + \sup_{0\le j\le n} 2^{j(r-1)}\|\Delta_j\omega^0\|_{L^\infty} \\ &\le \|u^0\|_{L^2} + 2^{n(r-1)}\Gamma(n)\|\omega^0\|_{B_\Gamma}.\end{aligned}$$

By Theorem 4.2.1 in [6], there exists $T^* \in [0,\infty]$ and a unique solution u^n to (E) in $\cap_{r>1} L^\infty_{loc}([0,T^*); C^r(\mathbb{R}^3))$ with initial data $u^{n,0}$. In addition, if $T^* < \infty$, then

$$\int_0^{T^*} \|u^n(t)\|_{C^1_*}\,dt = \infty.$$

By (3.38), Lemma 3, and the inequality $\|r^{-1}\omega^{n,0}\|_{L^{3,1}} \le \|r^{-1}\omega^0\|_{L^{3,1}}$ (see, for example, Lemma 1 of [8]),

$$\begin{aligned}\|u^n(t)\|_{C^1_*} &\le C\|u^n\|_{L^\infty} + \sup_{j\ge 0}\|\Delta_j\omega^n(t)\|_{L^\infty} \le C\|u^n\|_{L^\infty} + \|\omega^{n,0}\|_{L^\infty} e^{Ct\left\|r^{-1}\omega^{n,0}\right\|_{L^{3,1}}} \\ &\le C\|\omega^0\|^{\gamma_1}_{L^{p_0}}\|\omega^0\|^{\gamma_2}_{L^{p_1}} e^{\gamma_3 t\left\|r^{-1}\omega^0\right\|_{L^{3,1}}} + \Gamma(n)\|\omega^0\|_{B_\Gamma} e^{Ct\left\|r^{-1}\omega^0\right\|_{L^{3,1}}}.\end{aligned}$$

Therefore, $T^* = \infty$. Note that by Young's convolution inequality,

$$\|\omega^{n,0}\|_{L^{p_0}\cap L^{p_1}} \le \|\omega^0\|_{L^{p_0}\cap L^{p_1}} \text{ and } \|u^{n,0}\|_{L^2} \le \|u^0\|_{L^2}. \tag{4.1}$$

Moreover,

$$\begin{aligned}&\|\omega^{n,0}\|_{B_\Gamma} = \sup_{N\ge -1}\frac{1}{\Gamma(N)}\sum_{j\le N}\|\Delta_j S_n\omega^0\|_{L^\infty} \le \|\omega^0\|_{B_\Gamma}, \text{ and} \\ &\left\|r^{-1}\omega^{n,0}\right\|_{L^{3,1}} \le \left\|r^{-1}\omega^0\right\|_{L^{3,1}}.\end{aligned} \tag{4.2}$$

We have thus constructed a sequence $\{u^n\}$ of global in time smooth solutions to (3.2) with initial data satisfying the conditions of Theorem 2.

We now proceed as in [15]. Since ω^n belongs to $L^\infty_{loc}([0,\infty); C^r(\mathbb{R}^3))$ for all $r > 0$, it follows that ω^n also belongs to $L^\infty_{loc}([0,\infty); B_{\Gamma_1}(\mathbb{R}^3))$ (see Proposition 8.7 of [15]). Therefore, by (4.1), (4.2), Lemma 3, and Theorem 1, there exists $T > 0$ and a constant $C > 0$ independent of n such that

$$\|\omega^n\|_{L^\infty([0,T];L^{p_0}\cap L^{p_1}\cap B_{\Gamma_1})} \le C. \tag{4.3}$$

From this uniform bound we will be able to conclude that the limit of the sequence of smooth solutions satisfies Theorem 2.

Indeed, it can be shown using an argument identical to that in Section 8 of [15] that the sequence $\{u^n\}$ is a Cauchy sequence in $L^\infty([0,T]; B^0_{\infty,1}(\mathbb{R}^3))$. Let u satisfy

$$u_n \to u \in L^\infty([0,T]; B^0_{\infty,1}(\mathbb{R}^3)). \tag{4.4}$$

Then, following arguments identical to those in [15], one can show using (4.3) that

$$\omega \in L^\infty([0,T]; B_{\Gamma_1}(\mathbb{R}^3)).$$

Moreover, it can be shown using (4.4), (4.3), and a standard weak-* compactness argument that

$$||\omega||_{L^\infty([0,T];L^{p_0}\cap L^{p_1})} \leq C.$$

We refer the reader to arguments in Section 8 of [15] which show that the limit u is indeed a solution to (E). This completes the proof of Theorem 2. ☐

5. Uniqueness of Velocity in $H^1(\mathbb{R}^3)$

In the next two sections we extend the results of [7] to the three-dimensional Euler equations with axisymmetric initial velocity. In this section we prove the following theorem.

Theorem 3. *Let u^0 be an axisymmetric vector field belonging to $L^2(\mathbb{R}^3)$ and let ω^0 belong to $B_\Gamma \cap L^2(\mathbb{R}^3)$ with $\Gamma(N) = \log^\kappa(N)$ for $0 \leq \kappa \leq 1$. Also assume $r^{-1}\omega^0$ is in the Lorentz space $L^{3,1}(\mathbb{R}^3)$. For $\kappa = 1$, there exists $T_0 > 0$ and a unique solution $(u, \nabla p)$ to (E) with u in the space $L^\infty([0,T_0]; H^1(\mathbb{R}^3))$ and with $u|_{t=0} = u^0$. If $\kappa \in [0,1)$, then for every $T > 0$ there exists a unique solution $(u, \nabla p)$ to (E) with u in $L^\infty([0,T]; H^1(\mathbb{R}^3))$ and with $u|_{t=0} = u^0$.*

Proof. Let u^0 satisfy the conditions of Theorem 3, and let u_p be the unique solution to (E) in $C(\mathbb{R}^+; B^1_{\infty,1}(\mathbb{R}^3))$ with initial data $S_p u^0$ (for a proof that $S_p u^0$ is axisymmetric, see [2], Proposition 3.1). We will show the sequence $\{u_p\}$ is Cauchy in $L^\infty([0,T]; L^2(\mathbb{R}^3))$ for $\Gamma(N)$ growing sufficiently slowly with N. We fix positive integers m and n, with $m > n$, and we let u_m and u_n be solutions to (E) with initial data $S_m u^0$ and $S_n u^0$, respectively. An energy argument identical to that in [9] yields the estimate

$$||u_n - u_m||_{L^\infty([0,T];L^2)} \leq C||u^0_m - u^0_n||_{L^2} e^{\int_0^T ||\nabla u_n||_{L^\infty}}. \tag{5.1}$$

Since ω^0 belongs to $L^2(\mathbb{R}^3)$, Bernstein's inequality and boundedness of Calderon-Zygmund operators on L^2 imply that $||u^0_m - u^0_n||_{L^2} \leq C2^{-n}||\omega^0||_{L^2}$. We can conclude from (5.1) that

$$||u_n - u_m||_{L^\infty([0,T];L^2)} \leq C2^{-n} e^{\int_0^T ||\nabla u_n||_{L^\infty}}.$$

It remains to estimate the growth of the quantity $||\nabla u_n||_{L^\infty}$ with n. To do this, we use estimates proved in [2]. Specifically, in the proof of Proposition 4.4 of [2], the authors show that if the axisymmetric initial velocity belongs to $B^1_{\infty,1}(\mathbb{R}^3)$ and if $r^{-1}\omega^0$ is in $L^{3,1}(\mathbb{R}^3)$, then for fixed $\hat{N} \in \mathbb{N}$,

$$||\omega(t)||_{B^0_{\infty,1}} \leq C||\omega^0||_{B^0_{\infty,1}}(2^{-\hat{N}} e^{CU(t)} + \hat{N} e^{Ct||r^{-1}\omega^0||_{L^{3,1}}}),$$

where $U(t) = \int_0^t ||u||_{B^1_{\infty,1}}$ and C is an absolute constant. Applying this estimate to ω_n and using the inequality $||r^{-1}\omega_n^0||_{L^{3,1}} \leq ||r^{-1}\omega^0||_{L^{3,1}}$, it follows that

$$(5.2)\qquad \begin{aligned} ||\omega_n(t)||_{B^0_{\infty,1}} &\leq C||\omega_n^0||_{B^0_{\infty,1}}(2^{-\hat{N}}e^{CU_n(t)} + \hat{N}e^{Ct||r^{-1}\omega_n^0||_{L^{3,1}}}) \\ &\leq C||\omega_n^0||_{B^0_{\infty,1}}(2^{-\hat{N}}e^{CU_n(t)} + \hat{N}e^{Ct||r^{-1}\omega^0||_{L^{3,1}}}), \end{aligned}$$

with $U_n(t)$ now equal to $\int_0^t ||u_n||_{B^1_{\infty,1}}$. As in [2], we choose $\hat{N} = CU_n(t) + 1$. Using Bernstein's Lemma, energy conservation, Lemma 2, and the definition of $B^0_{\infty,1}$, we can conclude that $||u_n||_{B^1_{\infty,1}} \leq C||u_n||_{L^2} + \sum_{j\geq 0} ||\Delta_j \omega_n||_{L^\infty} \leq C||u^0||_{L^2} + ||\omega_n||_{B^0_{\infty,1}}$. This estimate, combined with our choice of $\hat{N}$ and (5.2), gives

$$(5.3)\qquad \begin{aligned} ||\omega_n(t)||_{B^0_{\infty,1}} &\leq C||\omega_n^0||_{B^0_{\infty,1}} e^{Ct||r^{-1}\omega^0||_{L^{3,1}}}(U_n(t)+1) \\ &\leq Ce^{C_1 t}||\omega_n^0||_{B^0_{\infty,1}}\left(\int_0^t ||\omega_n(s)||_{B^0_{\infty,1}} ds + t||u^0||_{L^2} + 1\right), \end{aligned}$$

where the constant C_1 depends on $||r^{-1}\omega^0||_{L^{3,1}}$. An application of Gronwall's inequality yields

$$(5.4)\qquad ||\omega_n(t)||_{B^0_{\infty,1}} \leq Ce^{C_1 t}||\omega_n^0||_{B^0_{\infty,1}}(t+1)e^{Ct||\omega_n^0||_{B^0_{\infty,1}} e^{C_1 t}}.$$

Again by Bernstein's Lemma, energy conservation, Lemma 2, and the definition of $B^0_{\infty,1}$, it follows that $||\nabla u_n||_{L^\infty} \leq ||u^0||_{L^2} + ||\omega_n||_{B^0_{\infty,1}}$. Using this estimate, (5.4), and the estimate $||\omega_n^0||_{B^0_{\infty,1}} \leq ||\omega^0||_{B_\Gamma}\Gamma(n)$, we write

$$(5.5)\qquad \begin{aligned} &||u_n - u_m||_{L^\infty([0,T];L^2)} \leq C2^{-n}e^{\int_0^T ||\nabla u_n||_{L^\infty}} \\ &\leq C2^{-n}\exp\left(\int_0^T Ce^{C_1 t}\Gamma(n)(t+1)e^{Ct\Gamma(n)e^{C_1 t}}\right) \\ &\leq C2^{-n}\exp\left(e^{CT\Gamma(n)e^{C_1 T}} - 1\right) \end{aligned}$$

for sufficiently large n, where we integrated in time to get the last inequality. We conclude that the sequence $\{u_p\}$ is Cauchy in $L^\infty([0,T];L^2(\mathbb{R}^3))$ for sufficiently small T when $\Gamma(n) = \log(n)$ and for every T when $\Gamma(n) = \log^\kappa(n)$ with $\kappa \in [0,1)$. From this we conclude that $\{u_p\}$ converges to a vector field u in $L^\infty([0,T];L^2(\mathbb{R}^3))$.

A straightforward argument shows that since $\{u_p\}$ converges to u in the space $L^\infty([0,T];L^2(\mathbb{R}^3))$, we can pass to the limit in (E). To see that u belongs to $L^\infty([0,T];H^1(\mathbb{R}^3))$, we observe that by Lemma 3, we have the estimate

$$(5.6)\qquad ||\omega_n(t)||_{L^2} \leq ||\omega^0||_{L^2}e^{Ct||r^{-1}\omega^0||_{L^{3,1}}}$$

for all n, where we also used the inequality $||r^{-1}\omega_n^0||_{L^{3,1}} \leq ||r^{-1}\omega^0||_{L^{3,1}}$. A weak compactness argument shows that $||\omega(t)||_{L^2} \leq ||\omega^0||_{L^2}e^{Ct||r^{-1}\omega^0||_{L^{3,1}}}$, giving $u \in L^\infty([0,T];H^1(\mathbb{R}^3))$. Finally, the membership of u to $L^\infty([0,T];L^2(\mathbb{R}^3))$ allows us to uniquely determine ∇p from u. We have thus constructed a weak solution $(u, \nabla p)$ to (E) satisfying the assumptions of Theorem 3.

To show uniqueness of u in $L^\infty([0,T]; H^1(\mathbb{R}^3))$, we first observe that by the existence proof, $||u_p - u||_{L^\infty([0,T];L^2(\mathbb{R}^2))}$ approaches 0 as p approaches infinity. Since the sequence u_p is uniquely determined by the initial data u_0, two solutions to (E) with the same initial data and initial vorticity in B_Γ will have the same approximating sequence and will therefore be equal on $[0,T]$. □

6. The Vanishing Viscosity Limit

In this section we prove that the vanishing viscosity limit holds for the entire class of solutions given in Theorem 3. We have the following theorem.

Theorem 4. *Assume u^0 and $\omega^0 = \nabla \times u^0$ satisfy the conditions of Theorem 3, and let u be the solution to (E) with initial data u^0. If u_ν is the unique solution to (NS) from* [1] *in the space $H^{\frac{1}{2}}(\mathbb{R}^3)$ with the same initial data u^0, then the following estimate holds for fixed $T > 0$:*

$$||u_\nu - u||_{L^\infty([0,T];L^2(\mathbb{R}^3))} \leq \sqrt{\nu e^{C_1 T}} \exp\left(e^{C_1 T \Gamma\left(-\frac{1}{2}\log(\nu e^{C_1 T})\right) e^{C_1 T}}\right). \tag{6.1}$$

We remark that the right hand side of (6.1) converges to 0 as ν approaches 0 for short time when $\Gamma(N) = \log(N)$ and for any finite time when $\Gamma(N) = \log^\kappa(N)$ with $\kappa \in [0,1)$.

Proof. The proof that the vanishing viscosity limit holds in $L^\infty([0,T], L^2(\mathbb{R}^3))$ is similar to that in [7]. Given our solution $(u, \nabla p)$ to (E) from Theorem 3 with initial data u^0, we again construct the sequence of solutions $\{u_n\}$ to (E) with initial data $S_n u^0$ (this notation is admittedly somewhat confusing as we used $\{u_p\}$ to denote a sequence of velocities in Section 5), and we write

$$||u_\nu - u||_{L^\infty([0,T];L^2)} \leq ||u_\nu - u_n||_{L^\infty([0,T];L^2)} + ||u_n - u||_{L^\infty([0,T];L^2)}. \tag{6.2}$$

To estimate $||u_n - u||_{L^\infty([0,T];L^2)}$, we apply the same argument as that used to show existence in Section 5, but with u in place of u_m, to conclude that

$$||u_n - u||_{L^\infty([0,T];L^2)} \leq C 2^{-n} e^{\int_0^T ||\nabla u_n||_{L^\infty}}. \tag{6.3}$$

To estimate $||u_\nu - u_n||_{L^\infty([0,T];L^2)}$, we again apply the energy argument from [9], this time to the solutions u_ν and u_n to (NS) and (E), respectively. Before applying Gronwall's Lemma, the energy argument yields the inequality

(6.4)

$$||(u_\nu - u_n)(t)||_{L^2}^2 \leq ||u_n^0 - u^0||_{L^2}^2 + C\nu \int_0^t ||\nabla u_\nu(s)||_{L^2} ||(\nabla u_\nu - \nabla u_n)(s)||_{L^2}\, ds$$
$$+ \int_0^t \int_{\mathbb{R}^3} |(u_\nu - u_n)(s)|^2 |\nabla u_n(s)|^2 \, dx\, ds.$$

Note that we now have an extra term on the right hand side, resulting from the fact that $\nu > 0$. To handle this term, we bound $||\nabla u_\nu||_{L^2}$ and $||\nabla u_n||_{L^2}$ with $||\omega_\nu||_{L^2}$ and $||\omega_n||_{L^2}$, respectively, and we apply Remark 2.4 and (5.6). We also use the membership of ω^0 to $L^2(\mathbb{R}^3)$ and Bernstein's Lemma to again write $||u_n^0 - u^0||_{L^2}^2 \leq C2^{-2n}$. These bounds combined with an application of Gronwall's Lemma to (6.4) yield

$$||u_\nu - u_n||_{L^\infty([0,T];L^2)} \leq C\left(2^{-n} + \sqrt{\nu e^{C_1 T}}\right) e^{\int_0^T ||\nabla u_n||_{L^\infty}}, \tag{6.5}$$

where C_1 depends on $||r^{-1}\omega^0||_{L^{3,1}}$. Combining (6.2), (6.3), and (6.5) gives

$$||u_\nu - u||_{L^\infty([0,T];L^2)} \leq C\left(2^{-n} + \sqrt{\nu e^{C_1 T}}\right) e^{\int_0^T ||\nabla u_n||_{L^\infty}}.$$

To complete the proof, we apply the bound $e^{\int_0^T ||\nabla u_n||_{L^\infty}} \leq \exp\left(e^{CT\Gamma(n)e^{C_1 T}} - 1\right)$ from (5.5) in Section 5, and we let $n = -\frac{1}{2}\log(\nu T)$. This completes the proof of Theorem 4. □

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