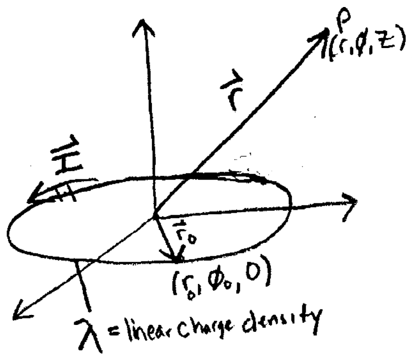
Magnetic Field Due to a Ring of Charge

<u>Question</u>

Calculate the magnetic vector potential $\vec{A}$ due to a spinning ring of charge at some point P in space. Given that the ring has a constant linear charge density λ, a constant angular velocity ω, and a constant tangential velocity $\vec{v}$ for any point along the ring. The ring has a total charge Q. See Figure I for a schematic diagram of the problem. Simplify the solution into an equation that a mathematical program like Maple can compute.

Figure I: Ring of Charge in Space



<u>Solution</u>

*There is a right hand rule that states if you point your thumb in the direction of current around a ring and curl your fingers around the ring, your fingers will point in the direction of the magnetic field **B** induced by the current flowing through the wire. In this case we're given no direction of current, so I'm going to say that it is flowing in the counterclockwise (CCW) direction. Thus, the right hand rule for the ring of charge in our given coordinate system can be seen in figure II.*

Figure II: Magnetic Field Lines due to Ring of Charge in Space

In general the equation for the magnetic vector potential $\vec{A}$ is as follows,

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}(\vec{r}_0)|d\vec{r}_0|}{|\vec{r}-\vec{r}_0|}$$

*The Current **I** for the ring is equal to the tangential velocity **v** multiplied by the charge per unit length of the ring λ .*

$$\vec{I} = \lambda\vec{v} \qquad (2)$$

The linear charge density of the ring λ can be found very easily in knowing

$$\lambda = \frac{Q}{2\pi r_0} \qquad (3)$$

In other words, if it is known for some line of charge that the linear charge density is constant throughout then the linear charge density is simply the total charge Q on the ring divided over the length of the ring.

The velocity can be found with respect to some period T at constant angular momentum. Since velocity is simply the distance travelled per unit time and velocity is in the $\hat{\phi}_0$ direction. Thus tangential velocity is simply the circumference of the ring divided by the period. Where the period T is the time it takes any point on the ring to complete a single revolution.

$$\vec{v} = \frac{2\pi r_0}{T}\hat{\phi}_0 \qquad (4)$$

*Since we already know how to find the current **I** in terms of λ and **v** we can now simply plug in our values for λ and **v** to derive an equation for the current in the ring.*

$$\vec{I} = \lambda\vec{v} = \left(\frac{Q}{2\pi r_0}\right)\left(\frac{2\pi r_0}{T}\right)\hat{\phi}_0 = \frac{Q}{T}\hat{\phi}_0$$

The $|d\vec{r}_0|$ component of our integrand can easily be obtained by symmetry arguments. In *cylindrical coordinates,*

$$d\vec{r} = dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z}$$

In the case of our current problem, as seen in figure I. The dr_o is zero because there is no change in r_0 as the vector traces out every point on the ring. In addition the way we have our problem set up, the ring is strictly in the xy-plane which means dz_0 is zero as well. Thus

$$d\vec{r}_0 = dr_0\hat{r} + r_0 d\phi_0\hat{\phi} + dz_0\hat{z} = 0 + r_0 d\phi_0\hat{\phi} + 0$$

And so our $|d\vec{r}_0|$ *component of the integrand is simply* $r_0 d\phi_0$.

*Looking at figure 1 again, we need to find vector equations for the **r** and **r**$_o$ vectors. We know that at all points along the **r**$_o$ vector the z_o component is equal to zero. Thus the general equations for the two vectors in space are below*

$$\vec{r} = r\hat{r} + (\phi\hat{\phi} + z\hat{z}$$

$$\vec{r}_0 = r_0\hat{r} + (\phi_0\hat{\phi} + 0\hat{z}$$

Now we can simply plug all of the parts we calculated into equation 1 as follows

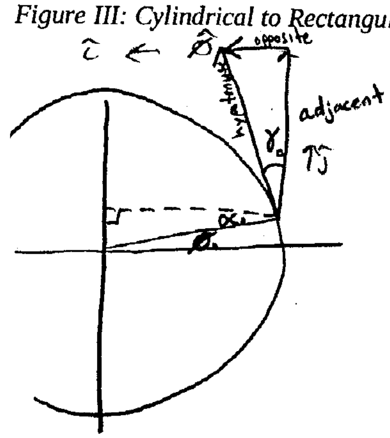
$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{(\frac{Q}{T})\hat{\phi}_0 r_0 d\phi_0}{(r^2 + r_0^2 - 2rr_0\cos(\phi - \phi_0) + (z - z_0)^2)^{1/2}}$$

Note: The denominator is simply the solution of $|\vec{r} - \vec{r}_0|$ *in cylindrical coordinates which is as follows*

$$|\vec{r} - \vec{r}_0| = \sqrt{r^2 + r_0^2 - 2rr_0\cos(\phi - \phi_0) + (z - z_0}$$

We can now simplify our problem by inserting things that we know about the problem. We know that r_o, Q, and T are all constants and can be treated like such throughout the problem. We also know that z_o is equal to zero. We incur a problem though. Even a mathematically calculating program like maple cannot compute this integral in the form we currently have it. A simple fix is to change our coordinate systems. We will convert from the cylindrical coordinate system to the rectangular coordinate system. The only thing we need to convert is the $\hat{\phi}_0$ into rectangular coordinates. First we will draw a picture of what the $\hat{\phi}_0$ direction looks like and convert it to rectangular coordinates. In general a table that gives you this conversion can be found. We will rely on symmetry arguments however. If we draw some tangent line along the unit circle, arbitrarily choosing a small angle for φ we can draw a line parallel to the x axis and perpendicular to the y axis that intersects our circle at the point M. We know the angle α by the rule of opposite interior angles is equal to φ(thus φ= α). We then use geometry arguments to determine that the angle γ is also equal to φ as well. So we have an end result of φ= α= γ. See Figure III below.

Figure III: Cylindrical to Rectangular



Solve for x and y components using the SOH CAH TOA rules

$\cos(\phi) = \frac{a}{h}$ $\sin(\phi) = \frac{o}{h}$

$\tan(\phi) = \frac{o}{a}$

$\therefore -\sin(\phi)\ \hat{i} + \cos(\phi)\ \hat{j}$

is the direction of $\hat{\phi}$

We now take the x and y components of our $\hat{\phi}_0$ direction vector. These point in the $\hat{i}$ and $\hat{j}$ rectangular vector directions respectively. Using simple trigonometry in the form of SOH CAH TOA we find these components to be as follows.

$$\hat{\phi}_0 = -\sin(\phi_0)\hat{i} + \cos(\phi_o)\hat{j}$$

Thus we obtain two much simpler integrals to compute. The final integral components plugging in all of the components we've solved for. The end result after plugging in our unknowns is as follows

$$\bar{A}=\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{(\frac{Q}{T})((-\sin(\phi_0)\hat{i}+\cos(\phi_0)\hat{j})r_0d\phi_0}{(r^2+r_0^2-2rr_0\cos(\phi-\phi_0)+(z)^2)^{1/2}}$$

This is an equation that a program designed to compute mathematical equations can obtain a solution to.

The magnetic vector potential decreases in size as the radius away from the ring in any given direction is increased. This makes physical sense because it is an analog of the magnetic field which decreases as the distance away from the ring of constant current is increased. If we analyze the integrand and say that it is for a point along the positive x axis a radius r away from the ring our expression simplifies down since our r components of our vectors are constants, our z is equal to zero, and we are left with two simple integrals, one in the x direction and one in the y direction as follows

$$\bar{A}=\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{(\frac{Q}{T})((-\sin(\phi_0)r_0d\phi_0}{(r^2+r_0^2+2rr_0\cos(\phi_0))^{1/2}}\hat{i}$$

And the other vector component of the magnetic potential along the x axis would be

$$\bar{A}=\frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{(\frac{Q}{T})\cos(\phi_0)r_0d\phi_0}{(r^2+r_0^2+2rr_0\cos(\phi_0))^{1/2}}\hat{j}$$

Let Maple compute these integrals and it is simple to analyze the behavior of the magnetic potential due to a ring of current in the xy-plane along the x axis. If you perform similar tricks for a few different points in space you can see that the general behavior of the magnetic potential simply points in the $\hat{\phi}_0$ *direction at all points and is therefore tangent to the ring at all given points in space. To be clearer, the magnetic potential is tangent to the current flowing through the ring in a given direction. This makes physical sense because the magnetic field is perpendicular to the ring and the magnetic potential compared to the magnetic field is an analog of the electric potential compared to the electric field which is perpendicular.*