

Student responses to chain rule problems in thermodynamics

Ian W. Founds¹, Paul J. Emigh¹, and Corinne A. Manogue¹

¹*Department of Physics, Oregon State University, Corvallis OR 97331*

Students often struggle with the many partial derivatives used in the study of thermodynamics. This project explores how students respond to chain rule problems in an upper-level undergraduate thermodynamics course. This project’s dataset is composed of anonymized student responses to two such problems. We used an emergent coding method to sort responses by solution method. Observed solution methods include variable and differential substitution, implicit differentiation, differential division, and chain rule diagrams. The change of students’ solution methods between assignments was also observed. Responses were later analyzed to identify conceptual errors. Students make specific errors that provide insight into their lack of conceptual understanding of the solution methods.

I. INTRODUCTION

Thermodynamic variables are often related in complex ways. Partial derivatives express these relationships, which typically correspond to physically measurable attributes of the system of interest. The evaluation of such partial derivatives often involves algebraic manipulation of multiple equations and the use of complicated chain rules. Such mathematical techniques can be difficult for students and experts alike [1-6].

In recent years, physics education research has expanded into upper-division courses, including thermodynamics. Some of this research focuses on student understanding of partial derivatives in both math and physics contexts [1-7].

In this paper, we investigate the solution methods and errors in student responses to two chain rule problems, one with thermodynamic context and one without, analogous to the problems examined by Kustusch *et al.* [1,2]. Our results can be used to help curriculum developers better prepare students to solve problems of this type, which are common in thermodynamics.

II. METHODS AND ANALYSIS

The participants in this study are students who were enrolled in a junior-level thermodynamics course at Oregon State University (OSU), known as *Energy and Entropy*. This course is part of OSU’s *Paradigms in Physics*, a reformed upper-level undergraduate physics program [8,9]. Students in this program work interactively in class to learn and apply the course content, and are encouraged to continue collaborating in this manner on the program’s intensive homework assignments. Most of the students previously completed two quarters of vector calculus as well as an introductory differential equations course. Most also had experience applying relevant mathematical concepts in prior Paradigms courses.

Data was gathered from student responses to two prompts given as part of the course (see Table I). In each

prompt, students were given two “equations of state,” with overlapping variables, and asked to evaluate a particular partial derivative.

The Quiz prompt (N = 29) was assigned as a graded quiz on the last day of the course, the Friday of the third course week. Students had previously responded to the same prompt on a graded quiz at the end of the first week, and the Quiz prompt was posted online several days in advance of the assignment. This prompt has no explicit or implicit physical context.

The Final prompt (N = 27) was assigned on the final exam, which took place on the Monday following the quiz. The Final prompt has an explicit thermodynamic context (the equations of state for a Van der Waals gas).

We used an emergent, or open, coding scheme to identify and categorize students’ solution methods and errors [10]. Student responses that had little or no coherent work or contained invalid methods were labeled as “Other.” We refer to the five solution methods that emerged from student work (described in detail in the next section) as *variable substitution*, *differential substitution*, *implicit differentiation*, *differential division*, and *chain rule diagrams*. We believe these methods to be exhaustive. After sorting the responses by method, we separated student errors into two categories: *conceptual* and *mathematical*. We define conceptual errors to be errors pertaining to the

TABLE I. Two written prompts given at different points at the end of a junior-level thermodynamics class.

<i>Given the definitions below, evaluate the requested partial derivative.</i>	
Quiz prompt	Final prompt
$U = x^2 + y^2 + z^2$ (1)	$S = Nk_B \left(\ln \left(\frac{N-Vb}{NC} T^{\frac{3}{2}} \right) + \frac{5}{2} \right)$ (4)
$z = \ln(y-x)$ (2)	$U = \frac{3}{2} Nk_B T - \frac{aN^2}{V}$ (5)
Find $\left(\frac{\partial U}{\partial z} \right)_y$ (3)	Find $\left(\frac{\partial U}{\partial V} \right)_S$ (6)

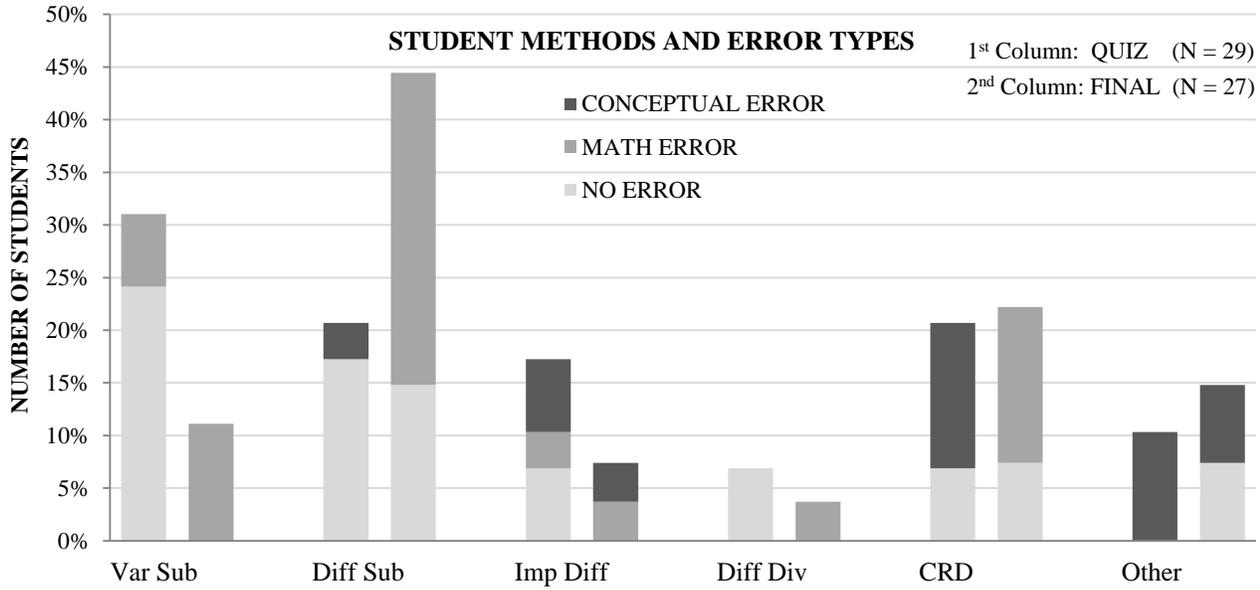


FIG. 1. The distribution of solution methods per assignment and error types per solution method in responses to the Quiz prompt and the Final prompt.

solution methods themselves (e.g., algebraic manipulation of partial derivatives and differentials or the construction and reading of chain rule diagrams). Other algebraic errors, as well as sign errors, inadvertently dropped terms, and computational errors, are referred to as mathematical errors and were not examined further.

III. STUDENT SOLUTION METHODS AND CONCEPTUAL ERRORS

In this section, we describe the observed solution methods and discuss their prevalence in the dataset. We also identify conceptual errors that occurred within each method. The results are summarized in Fig. 1. In this figure, any student who made both a math and a conceptual error is counted only in the conceptual error category. There were two cases of this on the Quiz, but none on the Final exam. Responses that reflected multiple solution methods were counted as each of the methods. This occurred twice on the Quiz and once on the Final exam.

Variable Substitution (Var Sub): Students who used this method solved for and eliminated the variable that is not present in the partial derivative. Throughout this paper,

$$\begin{aligned}
 e^z &= y - x & x &= y - e^z \\
 u &= (y - e^z)^2 + y^2 + z^2 \\
 \left(\frac{\partial u}{\partial z}\right)_y &= -2(y - e^z)e^z + 2z
 \end{aligned}$$

FIG. 2. A student applying the Var Sub method to the Quiz prompt correctly.

we refer to variables that must be eliminated as *excess variables*. The excess variable is x in the Quiz prompt, and T in the Final prompt. Once the excess variable is replaced, the requested partial derivative can be evaluated. An example of a student using this method is shown in Fig. 2.

Both prompts can be solved using Var Sub. However, the Final prompt was intentionally designed to make isolating the excess variable more difficult than on the Quiz prompt. The algebra to solve Equation (4) for T is thus more challenging than to solve Equation (2) for x .

Var Sub was the most common solution method used on the Quiz prompt (31%). It was substantially less common on the Final prompt (11%). Only one student used Var Sub in response to both prompts. No students who used this method made a conceptual error.

Differential Substitution (Diff Sub): Students who used Diff Sub found the total differential of each given equation and eliminated the differential corresponding to

$$\begin{aligned}
 du &= 2x dx + 2y dy + 2z dz & x &= y - e^z \\
 z &= \ln(y - x) & \Rightarrow e^z &= y - x & \Rightarrow e^z dz &= dy - dx \\
 & & & & \Rightarrow dx &= dy - e^z dz \\
 \Rightarrow du &= 2(y - e^z)(dy - e^z dz) + 2y dy + 2z dz \\
 du &= (4y - 2e^z)dy + (2z - 2e^z(y - e^z))dz \\
 \Rightarrow \left(\frac{\partial u}{\partial z}\right)_y &= 2z - 2e^z(y - e^z)
 \end{aligned}$$

FIG. 3. A student applying the Diff Sub method to the Quiz prompt correctly.

$$dU = \left(\frac{\partial U}{\partial y}\right)_T (dy) + \left(\frac{\partial U}{\partial T}\right)_y dT$$

$$\left(\frac{\partial U}{\partial v}\right)_S = \left(\frac{\partial U}{\partial v}\right)_T + \left(\frac{\partial U}{\partial T}\right)_v \left(\frac{\partial T}{\partial v}\right)_S$$

FIG. 4. A student applying the Diff Div method to the Final prompt correctly.

the excess variable. Then they factored out the remaining differentials and “identified” the requested partial derivative as the coefficient of the appropriate differential (dz for the Quiz prompt and dV for the Final prompt). An example response is shown in Fig. 3.

Diff Sub was often used on the Quiz prompt (21%). One of these responses contained a conceptual error. In this response, a student took the total differential of Equation (1), applied $dy = 0$, and then equated the resulting equation with the requested partial derivative, as below.

$$\left(\frac{\partial U}{\partial z}\right)_y = 2x dx + 2z dz \quad (\text{incorrect relation})$$

Diff Sub was substantially more common on the Final prompt (44%). Five of the six students who used Diff Sub on the quiz also did so on the final exam. Four students changed from Var Sub to Diff Sub. No student made conceptual errors during Diff Sub on the Final exam.

Implicit Differentiation (Imp Diff): Students who used Imp Diff wrote a multivariable chain rule, but gave no explicit justification for how they determined the chain rule. For example, the correct chain rule for the Quiz prompt is:

$$\left(\frac{\partial U}{\partial z}\right)_y = \left(\frac{\partial U}{\partial x}\right)_{y,z} \left(\frac{\partial x}{\partial z}\right)_y + \left(\frac{\partial U}{\partial z}\right)_{x,y}$$

The authors are able to produce this chain rule, without written work, either via implicit differentiation or in some cases from memory. We saw no evidence that would allow us to distinguish between these two possibilities of how students categorized as using this method mentally approached the prompts.

Imp Diff was somewhat common on the Quiz prompt (17%). Two of these responses contained conceptual errors. In one response, a student wrote a chain rule that included the wrong partial derivatives. In the other response, a student left out one of the necessary partial derivatives and also included a differential in one of their partial derivatives. Fewer students used Imp Diff on the Final prompt (7%). One of these responses contained conceptual errors. This student used the wrong partial derivatives in their chain rule and was missing one partial derivative.

Differential Division (Diff Div): Students who used Diff Div wrote a multivariable chain rule by dividing the total differential of the requested partial derivative’s dependent variable (U for both prompts) by the differential

$$\frac{dU}{dT} = \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_S + \left(\frac{\partial U}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_S$$

FIG. 5. A student applying the CRD method to the Final prompt correctly. Branches and their corresponding partial derivatives in the chain rule have been labeled A-D.

of the requested partial derivative’s independent variable (dz for the Quiz prompt and dV for the Final prompt). The requested partial derivative’s constant variable (y for the Quiz prompt and S for the Final prompt) was assigned to partial derivatives formed by the division. Figure 4 shows an example of this solution method.

Diff Div was the least used solution method on both the Quiz prompt (7%) and the Final exam (4%). No student made a conceptual error during this method.

Chain Rule Diagram (CRD): Students who used CRD wrote a multivariable chain rule by drawing and reading a chain rule diagram (see Fig. 5 for an example). Such a diagram helps keep track of how the differentials of the system’s state variables are related. In a chain rule diagram, each branch of each path to the differential of the independent variable represents a different partial derivative. The chain rule for a given partial derivative is found by multiplying all partial derivatives along each unique path from the requested partial derivative’s dependent variable, at the top of the diagram, to the requested partial derivative’s independent variable, at the bottom. The results from all such paths are then summed to give the chain rule. The chain rule diagrams taught in the Paradigms program label the diagram with differentials rather than the variables used in math texts [4].

CRD was often used on the Quiz prompt (21%). Four of these responses contained at least one conceptual error. These four students built diagrams representing partial derivatives that did not correspond to the requested partial derivative, effectively finding the chain rule for a *different* derivative. Three of these four students also misread their diagrams. Misreading a chain rule diagram usually produces a physically meaningless chain rule composed of physically meaningful, but incorrectly related, partial derivatives. CRD was also common on the Final prompt (22%). Four students used it on both the quiz and the final exam; two students changed from Var Sub to CRD. Three of the four students who made conceptual errors on the quiz using CRD used it on the final exam without making a conceptual error. In fact, no students made a conceptual error when using CRD on the Final exam.

Other: Some students did not produce work resembling any method that can lead to a correct solution. On the quiz, three students simplified the total differential of Equation (1) by applying $dy = 0$ and then set the requested partial derivative equal to all or part of the resulting expression. For example, one student wrote:

$$\left(\frac{\partial U}{\partial z}\right)_y = 2zdz \quad (\text{incorrect quiz response})$$

On the Final exam, two students found the total differential of Equation (5) and then equated the requested partial derivative to the coefficient of the dV differential as below.

$$\left(\frac{\partial U}{\partial z}\right)_y = a \frac{N^2}{V^2} \quad (\text{incorrect final exam response})$$

Two students applied the thermodynamic identity without making conceptual errors; however, this is not productive for the Final prompt.

IV. CONCLUSIONS

One of the goals of OSU's Paradigms in Physics program is to expand students' mathematical skills and problem-solving abilities, allowing them to respond to physics problems in multiple ways. When asked to solve chain rule problems, we found that upper-level physics students tended to use a variety of solution methods, rather than all students choosing the same method. However, we also observed that students tended to favor certain methods, and that different methods were favored when responding to a mathematically complex problem with explicit thermodynamic context than to a simpler problem with no explicit or implicit physical context.

In particular, the most prevalent solution method on the Quiz prompt was variable substitution (Var Sub), despite the fact that it was not explicitly taught during the course. Though substitution of variables is a common strategy for solving math and physics problems, in this context the step of identifying which variable must be replaced is not necessarily obvious. We suspect that the course material (*i.e.*, quizzes and homework assignments) or experiences in prior courses may have prompted students to eliminate the (correct) excess variable, but we currently have no evidence for or against either possibility. Many students abandoned Var Sub on the Final prompt, which would involve more challenging algebra. It is possible that these students recognized the difficulty of the method when the excess variable is deeply nested within an equation, and switched to an alternate method that does not have this feature.

All other solution methods were taught as part of the course. Diff Sub and CRD were introduced first and had the most class time devoted to them. Imp Diff and Diff Div were shown in class only once, when the instructor reviewed the solution to a quiz early in the term that

contained the same Quiz prompt discussed in this article. Imp Diff, also known as the chain rule, is also typically taught in multivariable calculus.

Most students who abandoned Var Sub changed to Diff Sub or CRD. This is not surprising given that Diff Sub and CRD were the primary methods discussed in class. Since we observed that these methods were more effective, it may be valuable to encourage their use in the future.

Students had difficulty "identifying" partial derivatives as the coefficients of differentials in total differentials. These students instead equated partial derivatives with terms and included the attached differentials. Students also had difficulties involving partial derivatives' constant variables. Some students equated partial derivatives that differed only by their constant variables. Another difficulty is that some students did not correctly handle excess variables. When evaluating partial derivatives, these students either treated the excess variable as a constant or left it unchanged by the derivative operator.

We intend to follow up on this limited study by conducting further probes of how students respond to the kinds of chain rule problems that are common in thermodynamics. One method for gaining new insight into student ideas would be to collect data at additional strategic points throughout the course: for example, on pretests or quizzes early in the course. Additionally, we hope to provide prompts with parallel contexts; either all explicitly thermodynamic, or not. Doing so will eliminate the possibility of students responding differently based on prompts' contextual differences. Such an improved study would permit us to conduct a better assessment of the effectiveness of course material in preparing students for chain rule problems. This in turn might lead to an overall improvement in student understanding of partial derivatives and their implications to thermodynamics.

ACKNOWLEDGEMENTS

The authors thank David Roundy for designing the prompts, Guenter Schneider for instructing the spring-quarter 2016 thermodynamics class, and Mackenzie Lenz for being the teaching assistant during the class. Additional thanks go to Mike Vignal for his work as the interrater reliability tester during initial coding scheme development, the rest of the OSU PER group for their continued support, and the students who participated in this study. This work was supported in part by NSF grant DUE-1323800.

[1] M. B. Kustusich *et al.*, Phys. Rev. ST Phys. Ed. Res., **10** (2014).
 [2] M. B. Kustusich *et al.*, PERC 2012 (2012).
 [3] D. Roundy *et al.*, PERC 2013 (2013).
 [4] T. Dray *et al.*, PRIMUS (2017).
 [5] D. Roundy *et al.*, Am. J. Phys., **82** (2014).
 [6] J. R. Thompson *et al.*, PERC 2011 (2011).

[7] J. R. Thompson *et al.*, PERC 2005 (2005).
 [8] C. A. Manogue *et al.*, Am. J. Phys., **69** (2001)
 [9] C. A. Manogue *et al.*, Phys. Today, **56** (2003)
 [10] M. B. Miles & A. M. Huberman. *Qualitative Data Analysis: A Methods Sourcebook* (2014)