

Experts' understanding of partial derivatives using the Partial Derivative Machine

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Partial derivatives are used in a variety of different ways within physics. Thermodynamics, in particular, uses partial derivatives in ways that students often find especially confusing. We are at the beginning of a study of the teaching of partial derivatives, with a goal of better aligning the teaching of multivariable calculus with the needs of students in STEM disciplines. In this paper, we report on an initial study of expert understanding of partial derivatives across three disciplines: physics, engineering and mathematics. We report on the central research question of how disciplinary experts understand partial derivatives, and how their concept images of partial derivatives differ, with a focus on experimentally measured quantities. In particular, for these experts, we investigate which representations are cued by their interactions with the partial derivative machine. Our interviews made use of the Partial Derivative Machine (PDM), a mechanical system featuring four observable and controllable properties, of which any two are independent. Using the PDM, we probed expert understanding of partial derivatives in an experimental context without a known functional form. Whereas the physicists and engineers were quick to use measurements to find a numeric approximation for a derivative, the mathematicians repeatedly returned to speculation as to the functional form; although they were comfortable drawing qualitative conclusions about the system from measurements, they were reluctant to approximate the derivative through measurement. On a theoretical front, we found ways in which existing frameworks for the concept of derivative could be expanded to include numerical approximation.

I. INTRODUCTION

Thermo is hard. In a recent national workshop on the upper-division physics curriculum, approximately one-third of the faculty indicated that they are uncomfortable enough with the content of thermodynamics that they would be reluctant to teach it. There are a number of reasons *why* thermodynamics is hard. One reason is various kinds of partial derivative manipulations that need to be performed to solve many theoretical problems. For instance, our group has analyzed expert problem solving in this context using a framework of epistemic games.^{1,2} Other research groups have identified a variety of other difficulties, both mathematical and conceptual.³⁻⁶

In this paper, we consider two further issues that make thermodynamics hard. First, the independent variables in thermodynamics are measurable (and changeable) physical variables such as pressure and volume, rather than immutable background markers such as space and time.⁷ Furthermore, which of these variables are independent and which are dependent varies with the context. In particular, the conjugate pair associated with heating, namely temperature and entropy, is known to be troublesome for students.⁸

Second, many important physical quantities in thermodynamics are actually partial derivatives of other physical quantities. Thermodynamics involves an apparent surfeit of variables in the sense that extensive variables such as volume have intensive conjugate pairings such as pressure that have independent operational definitions and are independently measurable, and may seem to be independently controllable. Because of this apparent surfeit of variables, thermodynamics is typically the first time

that physics students encounter scenarios in which the quantities held fixed when taking a partial derivative are ambiguous. In mathematics courses, students are taught that when taking partial derivatives, all the *independent* variables are held fixed. Nevertheless, we have found that most students come into our course with a firm belief that when taking a partial derivative *everything else is held fixed*.

Two years ago, one of us (DR) developed the partial derivative machine (PDM), a simple mechanical device of springs and pulleys as a classroom manipulative. (See Section II for a more complete description of the PDM.) Classroom activities involving the PDM exhibit many of the same features as experiments and calculations that students encounter in thermodynamics. All the same issues about independent and dependent, extensive and intensive variables arise that we described above. And the question of which variables to hold fixed also arises, but in a somewhat simpler context in that the variables involved are concrete and tangible (lengths and forces). Our hope was that students would benefit from classroom experience with the PDM, and our classroom experience with the device has been positive.

However, we have now become aware of a more fundamental underlying problem. In the first experiment with the PDM, students were asked to find a partial derivative from experimental data. Anecdotally, it became clear that many students did not immediately understand that a derivative can be effectively approximated by the ratio of small numerical differences. Furthermore, our deliberately open-ended prompt revealed important differences of interpretation and notation between different disciplines.

We do not believe that the issues we have observed with partial derivatives are limited to students. Indeed, we hypothesize that many of the issues we have observed are due to the ways in which different disciplines use and think about derivatives and partial derivatives. In this study, we conducted small group interviews with experts in several STEM disciplines. By studying experts' thinking about derivatives and partial derivatives, we hoped to obtain a better benchmark for comparison in the study of students' thinking about those same ideas. The group setting provided a means for participants to listen and respond to each other's ideas, rather than just the interviewers'. Our overall research question is *in what ways do disciplinary experts in physics, engineering and mathematics think about partial derivatives?*. This paper focuses on the aspect of this general question relating to *how disciplinary experts understand partial derivatives in relation to experimentally measurable quantities*. This aspect leads to additional subquestions such as how experts use and understand notation, and how they connect their understanding of the derivative with the experimental process.

In the course of gaining insight into this question, we describe how the responses were similar and differed across disciplines and consider the role and affordances of the partial derivative machine in the experts' responses. In the remainder of this paper, we give a description of the Partial Derivative Machine (PDM), describe the method we used to study our research question and give the results of our analyses from the expert interviews. We close by considering both the pedagogical and the research implications of our results.

II. THE PARTIAL DERIVATIVE MACHINE

We have developed and used two versions of the Partial Derivative Machine (PDM). The first version of this device is documented in Ref. 9, and features a central system that is attached to four strings. The simplified version of this device—which will be discussed in this paper—is shown in Fig. 1, and consists of an anchored elastic system, which is constructed of springs and strings. In both versions, the elastic system may be manipulated using two strings independently. Each of these two strings has a scalar position that can be measured with a measuring tape and a tension that can be adjusted by adding to or removing weights from a hanger. Detailed instructions for constructing a Partial Derivative Machine, including a parts list and photographs of additional central systems, are available on our Paradigms website.¹⁰

The usefulness of the PDM emerges because it is an exact mechanical analogue for a thermodynamic system. The system contains a potential energy U (analogous to the internal energy) that cannot be directly measured. The system has four directly measurable—and controllable—state properties: two positions x and y and

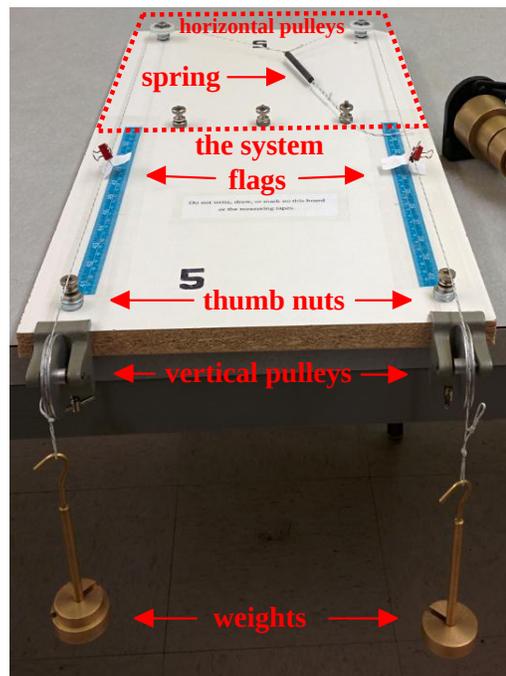


FIG. 1. The Partial Derivative Machine. The machine consists of a system (inside the dotted quadrilateral) which may be manipulated by pulling on two strings. Each string features a flag, which may be used to measure its position, and a hanger for weights ranging from 10 g to 250 g which may be used to fix the tension in the string. In addition, thumb nuts may be used to fix the position of each string independently.

two tensions F_x and F_y . These four state properties play roles analogous to volume, entropy, pressure and temperature in a thermodynamic system.

Although the PDM has *four* measurable and manipulable properties, like its analogous thermal system, it only has *two* degrees of freedom. One cannot independently control the tension and position of a single string, unless one uses the other string to do so. In the PDM, the two degrees of freedom are physically manifest: each corresponds to one string. We can choose to manipulate that string either by changing its position, or by changing its tension.

As in thermodynamics, the choice of which properties to treat as independent variables is context-dependent. While it is experimentally easiest to control the two weights as independent variables while measuring positions, it is sometimes theoretically more convenient to view the positions as the independent variables. Most notably, when using work to determine the potential energy, the positions are the “natural” variables, as seen in the total differential that is analogous to the thermodynamic identity, namely

$$dU = F_x dx + F_y dy. \quad (1)$$

We can relate this total differential to the mathematical

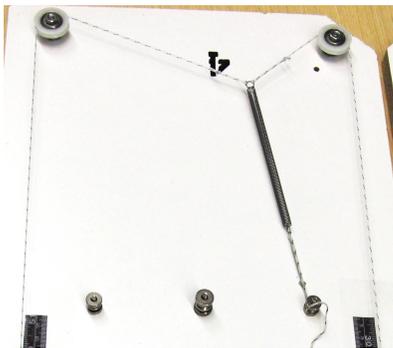


FIG. 2. The system we used in the PDM for for the expert interviews. The system contains a single spring, attached by string to an off-center post. The “observable” strings are routed around two horizontal pulleys towards the front of the machine where they are measured, and are attached to weights.

expression

$$dU = \left(\frac{\partial U}{\partial x} \right)_y dx + \left(\frac{\partial U}{\partial y} \right)_x dy. \quad (2)$$

By equating coefficients of dx and dy , we can find expressions for the two tensions as partial derivatives of the potential energy,

$$F_x = \left(\frac{\partial U}{\partial x} \right)_y, \quad F_y = \left(\frac{\partial U}{\partial y} \right)_x, \quad (3)$$

which enables us to clarify the interdependence of the four directly observable quantities.

The PDM was developed as a way to provide a mathematical introduction to thermodynamics prior to our junior-level course in thermal physics, *Energy and Entropy*. This introduction uses seven contact hours, and covers the range of mathematical topics generally taught in undergraduate thermodynamics, including total differentials, integration along a path, partial derivatives, chain rules, mixed partial derivatives, Maxwell relations, and Legendre transformations. Throughout both this mathematical introduction and *Energy and Entropy*, a focus is placed on connecting the mathematical expressions with tangible reality,¹¹ in many cases using the PDM.

III. THEORETICAL GROUNDING

A. Concept Images and Concept Definitions

Thompson¹² argued that the development of coherent meanings is at the heart of the mathematics that we want teachers to teach and what we want students to learn. Focusing on meanings emphasizes one’s thinking, rather than any normative evaluation of the correctness of particular approaches or thinking. Thompson argued that meanings reside in the minds of the person producing

it and the person interpreting it. Given this focus, we hypothesized that we could study experts’ meanings by studying their images, definitions, and representations for a concept and using these to model their meaning for an idea. We rely on Vinner’s¹³ language of concept images and concept definitions as an orienting framework. Vinner described the concept image as “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p. 152). A concept definition is a verbal definition that accurately explains the concept in a non-circular way. While we were primarily interested in experts’ concept images, as illustrated by our tasks and method, we also considered their concept definitions that in some cases underpin those images. We see both the concept image and concept definition as a means to operationalize and explore the meanings that experts had for derivatives.

Vinner’s definition of concept image explicitly allows a particular concept image to involve many properties and many mental pictures. We believe that mathematicians, engineers, and physicists have multifaceted and detailed concept images for derivative. However, Browne¹⁴ showed that middle-division physics students did not necessarily move spontaneously between various facets even when changing to a different facet might make solving a particular problem easier for these relative novices. Our own classroom experience bears out this observation. Therefore, one of our intentions was to explore which facets of the concept image of derivative are cued for different content experts by an open-ended prompt involving numerical data from the PDM.

B. A framework for student understanding of derivatives

The framework developed by Zandieh for student understanding of derivative is a valuable tool in this work.¹⁵ While it is focused on students’ thinking, many of the conceptions described in the framework naturally transfer to experts’ thinking about partial derivatives. This framework is aimed at mapping student concept images for derivative at the level of first-year calculus. It begins by breaking the formal symbolic definition of the derivative, namely

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (4)$$

into three *process-object layers*. These three layers are the *ratio layer*, in which one finds a ratio of changes, the *limit layer* in which one takes the limit as the changes become small, and finally the *function layer*, in which one recognizes that this could be done for any value of x , and thus describes a function. These three layers are each required in a complete understanding of derivative. Moreover, each of these layers can be seen both as a *process* and as a reified *object*. As a *process*, each layer

is a procedure that you could use to find a value. But alternatively, one can understand each of these layers as a static *object*, which exists independently, and can be and is acted upon by other processes.

Zandieh identifies an orthogonal dimension of *representation* (or alternatively *context*) with four possible representations: graphical, verbal, symbolic, and “paradigmatic physical.” Each of these representations exists for each process-object layer. We introduced Zandieh’s symbolic representation in the previous paragraph and will here briefly outline the graphical representation of derivative, which is slope. At the *ratio layer*, the graphical representation is the slope of the secant line to a curve (which itself is the graphical representation of a function). At the *limit layer*, one has the slope of a tangent line. And finally, at the *function layer*, one recognizes that the slope of the tangent line is itself a function that could be visualized as a curve.

This framework is particularly valuable because it makes explicit the three process-object layers that exist in the concept of the derivative, and which can be used separately. In Section IV B we will introduce our perspective on the different representations of the derivative, which is expanded beyond that considered by Zandieh in order to explicitly include physical representations at a level beyond that treated by Zandieh.

IV. BACKGROUND AND LITERATURE

The purpose of this section is threefold. First, we describe in what ways students and experts have been shown to think about the concepts of derivative and partial derivative, and hone in on particular difficulties for students. Second, we articulate various meanings for derivative and partial derivative that come from both research literature and our own experience working with students and colleagues. In this subsection, we also briefly elucidate several language issues that have arisen as we ourselves, from our different disciplinary perspectives, have discussed these various concepts of derivative and we detail the specific language choices that we have made in this paper. Third, we consider the importance of studying experts’ thinking about derivatives and partial derivatives as a means to identify important learning goals for students in physics, mathematics and engineering. The overarching purpose of the section is to demonstrate that while mathematics and physics education research have gained insight into students’ thinking about derivatives, they have not fully explored thinking about partial derivatives. Understanding how experts think about these ideas is a natural first step to exploring how we might want students to reason about them.

A. Students’ Ways of Thinking about Derivative

A number of researchers have identified difficulties students have in thinking about rate of change of one variable functions. These difficulties range from students thinking about a graph as representing its derivative, confounding average and instantaneous rate of change,¹⁶ conceptualizing rate as the slope or steepness of a graph,¹⁷ using statistical meanings for average to interpret average rate of change,¹⁸ and inattention to how fast quantities are changing with respect to one another. Some students conflate the average rate of change of a function with the average value of a function and therefore compute average rate of change by computing an arithmetic mean. Such students do not distinguish between the graph of the function and the graph of the function’s rate of change.¹⁹

Researchers have suggested that some of these difficulties might be attributable to students not conceiving of rate of change as a quotient of two quantities. For instance, students often discuss the rate of change as a slope but do not speak of slope as a quotient (the change in a function’s value being so many times as large as the corresponding change in its argument). Instead, they talk about slope as the function’s steepness.¹⁵ As another example, students often use a tangent line and rely on visual judgments to sketch the derivative function.²⁰ In yet another study, students who were able to correctly rank the slope at points on a graph were less able to find the sign of the derivative at those points.²¹ While this approach is not necessarily problematic, thinking about sliding tangent lines does not necessitate images of variation. Zandieh also argued that students can answer many standard calculus questions without needing to think about functions as relationships between variables nor needing to think about the rate of change of one quantity with respect to another. For example, students often respond to the directive “find the derivative of $g(x)$ ” by acting on a symbolic expression using standard rules for differentiation, without thinking about functions or rates of change. Indeed, researchers have documented that these issues keep students from thinking about the derivative as a ratio of changes in quantities.^{15,22,23}

There is only limited literature that addresses how students think about rates of change in the context of functions of two variables. In a mathematics setting, Yerushalmy²⁴ provides an indication of natural questions that might arise as students conceptualize rate of change in multivariable settings. She illustrated students’ struggles with how to think about dependence in a system with three quantities and how to represent multiple quantities and their changes in a single graph. Her students struggled to describe the change in a particular direction of a linear function of two variables. Part of the reason may have been that there are infinitely many directions in which to move from a given point at a constant rate, yet, in general, each direction yields a different slope.

In a physics setting, researchers have investigated

	graphical	numerical	symbolic	verbal	experimental
slope	•				
ratio of small changes		•			
difference quotient			•		
rate of change				•	
name the experiment					•

TABLE I. Representations of derivatives. Each column describes one form of representation, extending the Rule of Four²⁹ to include an experimental representation. Each row corresponds to a separate way of understanding the concept of derivative.

mixed partial derivatives and differentials in thermodynamics.^{3,25} They address students’ ability to translate back and forth between “physical processes” and partial derivatives, and found that students were more able to go from a partial derivative to a physical process than the other way around. Similarly, researchers investigating physical chemistry have found that students need help in interpreting mathematics in a thermodynamics context.²⁶ The physics literature also includes discussions of graphical representations of slope.^{27,28}

It is clear that students do not necessarily reason about derivatives and rates of change as we might hope. Furthermore, very little literature has considered how these issues extend to functions of more than one variable, particularly how students might think about partial derivatives. This is surprising, given that many scenarios in mathematics, physics and engineering require thinking about systems or scenarios in which many variables may be varying simultaneously.

B. Summary of Important Meanings for Derivative and Partial Derivatives

1. What is a derivative?

Researchers have identified the need for students to be fluent in thinking about the derivative using multiple perspectives. In this section, we articulate these multiple perspectives and describe why each is important. Derivatives are commonly described as *slopes*, as *ratios of small changes*, as *difference quotients*, and as *rates of change*, among others. However, these phrases are used in different ways in different disciplines, and by different individuals within those disciplines. For instance, the description of *slope* as “rise over run” could be either numerical or symbolic, as well as graphical. This complexity reflects the multifaceted concept images of the derivative held by experts.

A common framework for multiple representations is

the *Rule of Four* introduced by the Calculus Consortium (see Hughes Hallett *et al.*²⁹), in which key concepts are presented *graphically, numerically, symbolically, and verbally*. In recent work,³⁰ we proposed adding a fifth representation, based on *experiment*. In order to be clear in our discussion of the different facets of expert concept images of the derivative, we will associate each of the verbal descriptions of derivatives given in the previous paragraph with just one of these representations, as shown in Table I. We have based this association on our interpretation of the technical usage of these common descriptions within the mathematics community. The usage may be unfamiliar to our physics readers, but we hope that this unfamiliarity will help the reader notice the nuanced differences in language. In particular, both the use of *rate of change* to refer to rates that do *not* involve time and the technical term *difference quotient* are largely limited to mathematics. We will use the conventions indicated in Table I throughout this paper, except where explicitly stated otherwise.

Therefore, we consider five different ways to understand and think about the concept image of derivative, each of which is useful in different scenarios.

1. The *slope* of the tangent line to a curve. This representation describes the slope as a geometric measure of the steepness or slant of a graph of a function.
2. A numerical *ratio of small changes*, by which we mean an explicit numerical quotient, involving actual values of the “rise” and the “run”. By “small changes” we mean small enough that the quotient represents a reasonable estimate of the derivative (within the physical context of the problem).
3. The result of algebraic manipulation of a symbolic expression. Formally, these manipulations involve the *difference quotient* $\frac{f(x+\Delta x)-f(x)}{\Delta x}$, but in practice a memorized set of derivative rules is used instead. (For the purposes of this paper, we will conflate these two algebraic manipulations.) This meaning for derivative is a process applied to an algebraic object, but does not include an image of that process as measuring how one quantity changes with respect to another.
4. The *rate of change* of one quantity with respect to another, which is here used to mean a description in *words* of that change. (Outside of mathematics, the word “rate” implies that the second quantity is assumed to be time, but we will use this term more generically.) In this representation, the derivative measures covariation, i.e. how one physical quantity changes with respect to another.
5. Finally, as introduced in our previous work,³⁰ particular derivatives can be associated with particular experiments, such as measuring the change of volume in a piston of gas as weights are added to

the top of the piston. Determining which experiment might correspond to which derivative provides a representation of the derivative, which we call *name the experiment*.

In the second case, it is most natural to think of the derivative as a *number*. One picks a point at which to take the derivative and computes a numerical ratio. While that number will be different at other points—making the derivative actually a function—this aspect of the derivative may often be ignored. When considering the slope of the tangent line to a curve, it is clearer that the derivative is a function, but it is also natural to think of the derivative as a number, the slope at a single point on the curve. When using the symbolic approach, the derivative is *inherently* a function, and while that function could be evaluated at a point, its value cannot be determined until after its functional form is known.

Most of these aspects of the concept image of derivative (namely, 1, 3, 4, 5) find approximate analogues in Zandieh’s framework.¹⁵ Interestingly, the second aspect, which is not present in Zandieh’s framework, turned out to be the most important one in the analysis of our interviews and is a major theme throughout this paper.

2. What is a partial derivative?

Partial derivatives differ from *ordinary* derivatives in important ways. How we understand this difference can vary with how we understand derivatives.

1. A tangent line turns into a tangent plane in three dimensions, and a partial derivative becomes the slope of the plane *in a given direction* in the domain, at a given point.
2. When considering a ratio of small changes, a *partial* derivative requires that we specify not only which quantities are changing, but also which quantities to hold fixed.
3. The algebraic procedure to find a partial derivative of a symbolic expression is identical to that for an ordinary derivative, provided there are no interdependencies among the variables in the expression.
4. The verbal description of derivatives as rates of change must explicitly mention the independent variable(s) in order to describe a partial derivative. e.g. the derivative of volume with respect to pressure would be “the rate of change of volume as pressure is changed, with either temperature or entropy held constant.”
5. The representation of derivatives in terms of experiments is designed precisely to take into account which physical quantities are controlled, and which are not. As such, it is particularly well suited to descriptions of partial derivatives.

In thermal physics, and other areas of mathematics, the quantities that are being held fixed are context-dependent. In general, one has a set of interrelated variables, of which a few may be fixed. The number of independent variables is itself context-dependent, and in physical situations we are seldom provided with symbolic equations connecting the set of interdependent variables. More often we rely on physical intuition and argumentation to establish how many variables may be controlled independently. “If I fix the pressure, temperature and number of molecules, I could measure the volume and the mass, therefore I believe I have three independent degrees of freedom.”

How we respond to the ambiguity provided by abundant physical variables depends on our concept of a derivative. However, if students’ concept of derivative is not rooted in an image of measuring how fast one quantity changes with respect to one or more other quantities, then it is unlikely they will understand derivative in the ways we intend.

C. The Need to Study Experts from Mathematics, Engineering and Physics

In the previous two sections, we have made the case that there are a variety of ways to think about derivatives and partial derivatives, but students have difficulty thinking in the ways we might intend because of their inability to think about a derivative as measuring the ratio of small numerical changes between quantities. This is the case even for functions of a single variable. Earlier, we also argued that most real-world scenarios involve reasoning about multiple quantities and the relationships between them, a goal that seems especially problematic for students who have difficulty reasoning even about simple systems. Our reasons for studying the thinking of a variety of experts across mathematics, engineering and physics were a) we believed they would be accustomed to working with situations involving multiple quantities and relationships, b) we believed their experience would allow us to observe sophisticated reasoning patterns that we could only hope to observe in extremely advanced students, c) we anticipated their ways of thinking about partial derivatives could help us identify “end goals” for how we want students to think and d) we expected their thinking would vary across disciplinary areas, allowing us to better understand how students in these fields might need to reason in different ways about derivatives and partial derivatives. This last expectation was motivated in part by the obvious impact our own diverse disciplinary backgrounds had on our internal discussions.

As mentioned earlier, the overarching purpose of the section was to demonstrate that while mathematics and physics education research have gained insight into students’ thinking about derivatives, they have not fully explored thinking about partial derivatives. Furthermore, understanding how experts think about these ideas is a

natural first step to exploring how we might want students to reason about them. In the subsequent sections, we describe how we studied experts’ thinking about these ideas.

V. METHOD

A. Design and Conduct of the Expert interviews

To gain insight into our research question, we performed three expert interviews, each of which lasted approximately one hour. We interviewed seven experts in physics, engineering or mathematics, divided into three disciplinary groups of two or three. For the physicists we interviewed one associate professor and one full professor. Both use computational methods in their research, which is in astrophysics and optics. We interviewed three engineers: one chemical engineer who is a full professor with considerable research and teaching expertise in thermodynamics, and two assistant professors who study student thinking and epistemology in engineering. Finally, we interviewed two mathematicians who are both assistant professors and whose research is in mathematics education at the collegiate level. All of those we interviewed have doctoral degrees in their disciplinary areas and work in physics, engineering, or mathematics departments.

These interviews were most closely aligned with the notion of a clinical interview in which one focuses on gaining insight into another’s thinking through systematic questioning and open ended prompts. At the same time, it is atypical that clinical interviews occur in group format because of the difficulty of ascertaining an individual’s thinking. Our use of groups was purposeful. We believed that the conversations between the experts and their questioning of each others’ responses would be just as important as the questioning and prompts from the interview team. Indeed, we saw that some of the most interesting data we collected came from the experts debating each others’ responses.

B. Prompts and Purpose of Task

Our overall research question was to explore *in what ways do disciplinary experts in physics, engineering, and mathematics think about partial derivatives*. Important subquestions that arise in answering that question include how experts use notation, how they think about derivatives and partial derivatives, how they think about the various variables and which, if any, should be held fixed, and how they relate those ways of thinking to the PDM. A pedagogical design feature of the PDM is the lack of any symbolic expressions relating the given variables. We exploited this property of the PDM to explore our experts’ understandings of partial derivatives outside the world of symbolic manipulation.

We began by introducing the experts to the PDM, and showing them how to manipulate the machine. We then gave the experts the following prompt:

$$\textit{Find } \frac{\partial x}{\partial F_x}.$$

which was written on a whiteboard. Thus, we asked experts to “find” a partial derivative for which they are given no functional form. The PDM allows for measurement of changes in the positions and tensions at discrete data points and the prompt was designed in such a way that a ratio of changes in quantities would be the only easy response. However, we deliberately asked the prompt in an open ended way so that it would not cue a particular aspect of the concept of derivatives. All parts of the system were chosen to be visible in order to encourage discussion of the possibility of finding an analytic expression by which one could determine the derivative, although the actual determination of such an analytical expression would have been prohibitively difficult.

When we provided this prompt, we did not define either x or F_x , but rather let the interviewees discuss what these quantities might mean. After they had discussed the meanings of these terms, and we agreed that their meaning was sufficiently clear to us, we clarified if necessary that x was the position of one flag (i.e. one string) and that F_x was the tension in that string, which was determined by the weight (and that y and F_y were defined similarly for the other string). We note that this aspect of the task came at the cost of not explicitly exploring graphical representations of tabular experimental data with all groups of participants.

One thermodynamic system that the PDM is designed to mimic is a gas, described by the four variables pressure (p), volume (V), temperature (T), and entropy (S). The internal energy U involves these variables through the thermodynamic identity

$$dU = p dV + T dS. \quad (5)$$

A prompt analogous to ours in this context would be to find $\partial V/\partial p$, which is related to *compressibility*, but which does not have any inherent information about whether temperature (for *isothermal compressibility*) or entropy (for *isoentropic compressibility*) should be held fixed, corresponding to two different physical properties of the system.

What did we expect? In the absence of a functional expression for x , we expected the experts to respond to our prompt by computing ratios of measured small changes in x and F_x . With two other quantities in the game (y and F_y), we expected the experts to recognize that the question was ambiguous, since they were not told which of these two quantities to hold constant, and ask for clarification. We also anticipated some notational confusion, due to our use of x for the *independent* variable, and perhaps due to our use of subscripts to label components, which is not standard in all disciplines.

C. Analytical Method

The analysis of the data collected for this study relied on systematically creating descriptions of experts' thinking about rate of change (i.e. their concept images and definitions). These hypotheses resulted from in-the-moment observations and short reflections by the researchers between interviews. We also created descriptions of experts' thinking about derivatives and partial derivatives from the interviews using retrospective analysis. The retrospective analyses involved making interpretations and hypotheses about participants' thinking by analyzing videos of the interviews after they were completed. Thus, we worked from two sets of observations: those formed during the interviews and those formed from analysis of the videos as a whole. These analyses helped us to think about the categories of concept images and definitions we present in the analyses. This data corpus provided a means to describe patterns in experts' thinking, which in turn helped us focus on various concept images and definitions they appeared to have for derivative. Differing interpretations due to our own diverse disciplinary backgrounds were discussed and resolved as a group.

VI. RESULTS

During the course of the three interviews, a number of themes emerged, each of which provided insight into the experts' concept images and definitions for derivatives and partial derivatives. These themes were a combination of issues we noticed as we did the interviews and issues that emerged as we conducted the data analysis described in the analytical method section. In the sections below we describe these themes and articulate how we saw each group of interviewees in the context of that theme. Where it is possible, we provide transcript excerpts from the interviews to support the claims we make.

A. Identifying x and F_x

Since the invention of the first partial derivative machine, we have conducted informal "interviews" with colleagues. During these interviews, we have noticed that different individuals interpreted the symbols in the algebraic expression

$$\frac{\partial x}{\partial F_x}$$

differently and we began to expect that there might be disciplinary reasons for these differences. Therefore, we chose, in these interviews, *not* to tell the interviewees our own meaning for the symbols (at first) but rather to let them explore in their groups what these symbols might

mean. All three groups spent significant time thinking about and debating over how to identify the quantities x and F_x . We expected that the physicists and probably the engineers would share our understanding that x was the position of the pointer on the x measuring tape and that F_x was the tension in the x labeled string. We expected that the mathematicians would try to invoke the mathematics convention that a subscript indicates a partial derivative. We were surprised by several other unexpected types of confusion that our notation caused.

The physicists initially identified x with the elongation of the spring and then wanted F_x to represent a force in the same direction as the spring. Their mental focus was clearly on the internal mechanics of the system and they (at first) ignored the measuring tapes on the PDM as a potential method for measuring a position x . When they recognized that their interpretation of the symbols would lead to a "total" derivative rather than a partial derivative, they then switched to an interpretation in which they wanted the two strings to be perpendicular to each other, even going so far as to manipulate the PDM to make this true. Presumably, they were invoking a physics convention that x (and y) are independent rectangular coordinates and F_x (and F_y) are perpendicular rectangular components of a single, net force vector. We might have anticipated some of this interpretation since one of the interviewees has such a strong connection of the symbol F_x to the x component of a force that she even referred to F_x in this way in the interview with the mathematicians.

The engineers immediately showed a preference for having x represent the horizontal coordinate (the strings were vertical) but acknowledged, with laughter, that this must not be the case because it would have trivialized the problem. Unlike the physicists, they rapidly made use of the left and right measuring tapes and were happy to invent their own notation, calling the position of one pointer x_L and the position of the other pointer x_R . They did not have any difficulty identifying F_x as the weight on one of the strings.

The mathematicians were not only puzzled by the meaning of the subscript, as we expected, but also expected the capital symbol F to represent a function rather than a force.

1. Physicists

The physicists began the task thinking that the position x might be the elongation of the spring and F_x a force in the direction of the spring. They rapidly corrected this one-dimensional interpretation when they realized that they were asked to find a partial derivative.

EXCERPT 1

5:00

P2: So we have, probably it's the elongation of the of the spring, in this case. So, right, so this is tied here [points to spring's anchored string], so if we put two

forces like this [pulls on both strings] we need to, so, x , if this is x [points to spring], right, we need to make the forces go on this direction [points in direction along spring]. So...

P1: But then it's a partial derivative, right?

P2: Right.

P1: So my feeling is that it cannot be that simple, or it would be a total derivative.

They continued discussing the possible meanings of x and \vec{F} and moved rapidly to language about *balance* (of forces) and *perpendicular*. Notice the agreement in Excerpt 2 between the two interviewees.

EXCERPT 2 5:57

P2 : Or if we, if we... Okay so this is one way [pulls on right-hand string] or the other way is if we make it like this [pointing to spring], right, so if there's a force, if we can somehow balance it like this [pointing to spring and strings] and there's a force [pointing in the direction of the spring] going this way because now this is perpendicular [pointing to right-hand string where it is attached to the spring], right? So this force will not count in x [still gesturing around the spring]. Then if we're able to balance it with the right weights, then, you know, if we do like this [pulling on left-hand string], right, maybe, you know the force that comes from this side. So here we need to put a force that makes it [pulling on right-hand string], right.

P1 : At a right angle.

P2 : Huh.

P1 : That sounds like a good idea.

P2 : Mhmm, and then, you know, then in this case, right, then [pulling on weights to make a right angle] all the force on this [points in direction of the spring] way is parallel to x , so that would be x in this case.

P1 : Yeah, so then F_x would be just this weight [points at F_x].

P2 : In this case, right.

P2 : In the other case we have to do, have to analyze in two axes as you said, and then figure out, you know, one of the two, so you have to find the force on one of the axes and the... So, so so, the idea is to, we need either to analyze x [points at spring], right, in two directions and then we'll have a net force, right, and analyze it into components in the two axes or we just make it into one axis and then we have one force parallel to it, right?

P1 : I like your idea. I say if we put some weight at a right angle [points toward string and spring angle at top of PDM], then we change both weights [points at both weights] in order to keep the right angle, and we measure displacement [points at measuring tape on left side] and the force and we have...

Immediately after Excerpt 2, the interviewer clarified the meanings of x , y , F_x , and F_y . From this stage on, the physicists had no difficulty using this notation.

At the end of this portion of the interview, after the interviewers interpretation of the symbols has been explained, P1 volunteers the interpretation that, at the beginning of the interview he was trying to measure the spring constant, which explains why he was looking at the internal mechanics of the system.

EXCERPT 3 8:37

P1 Yeah, I was more thinking we wanted to measure the spring constant.

2. Engineers

The engineers began with the professional but humorous recognition that there is an interpretation of the symbols that would make the problem trivial:

EXCERPT 4 3:09

E3 : I think it's easy if x is this way [gestures perpendicular to the strings].

E1 : Yeah, exactly!

[Laughter]

E3 : We don't know anything about that one, do we? [Laughter] So we're done.

They then gave a quick interpretation in words which seems, like the physicists' interview, to focus on the internal mechanics of the system,

EXCERPT 5 3:25

E3 : Okay, x is going to be the spring.

E1 : Sounds good.

E2 : Okay so... $\partial x / \partial F_x$ [pronounced $d x d F x$] is how much it moves per unit force, sort of, could we do it that way?

But within 1/2 minute, their attention focused on the strings and weights:

EXCERPT 6 4:02

E3 : Does this [points to the written interview prompt $\frac{\partial x}{\partial F_x}$] mean \vec{F} in the direction, the x direction, only?

E1 : So F_x [pronounced F of x], so I think it's a force in the x direction... which the question is, does it matter [points back and forth to each string] which side you're on?

They then spent 2-3 minutes discussing how they expected the system to behave as they added weights, based on the geometry of the system, especially the angle of the spring. During this analysis, they discussed whether these symbols should refer to one side of the PDM or both:

EXCERPT 7 5:48

E2 : Do we want to go net, or do we want to pick one to privilege? Obviously the right one is the real one and the left is fake.

- E1 : So we could do each one separately
 E3 : Mhmm.
 E1 : And then verify that... that looks correct.
 E3 : Mhmm.
 E1 : In principle you could do one and figure out how the other one...
 E3 : So it would be $\partial x/\partial F_x$ for this one [points to x string] and then figure that out and then do it for this one [points to y string] and then maybe some verification by comparing those two?

After they decided to take data and began to construct the outline of a table, they discussed how to label the columns of the table. This discussion returned them to the meaning of the symbols. At this stage, they agreed to add *left* and *right* subscripts to their symbols.

EXCERPT 8 8:07

- E1 : So I would say... I would propose we do an x left and an x right for each... but wherever we put the, the weights.
 E2 : Okay so we're going to do two trials.
 E1 : Because they're going to move opposite [motions in opposite direction for the x and y flags] directions, right?
 E3 : Okay. So this [referencing table of w_{left} x_{left} w_{right} and x_{right}] would capture the thing that you want to do?

3. Mathematicians

From the initial moments of the interview, the two mathematicians puzzled over the meaning of the subscript on F_x , noting that “we have not seen this type of notation before” and “it looks like a derivative but we are unsure what the symbols are.” While they attended to the position of the strings, they did not interpret F_x as the force related to x until the interviewer explicitly suggested this to them part way through the interview. In Excerpt 9, the mathematicians explain their confusion with the notation and in doing so, reveal that they often associate a derivative with the process of differentiating an explicitly defined function. This was a theme we observed throughout the interview with the mathematicians, yet only observed briefly in the other groups.

EXCERPT 9 2:24

- M1: I'm not familiar with the notation F_x [pronounced F sub x].
 M2: Me neither.
 M1: So, should we talk about what we think that might mean?
 M2: Well, so, it's a par... So, usually what we say, well, we... We, I'm saying, my experience has been like $\frac{\partial x}{\partial y}$ [pronounced $d x d y$], right? The partial derivative of x with respect to the y , right? So, like y is some function of x or... what's that... Or you might

have a function like little f of x , y ,

$$f(x, y)$$

um, is equal to some function of x and y , and so you take, you know potentially that would be the partial derivative of that function...

- M1: With re... like of either x or y with respect to that variable, yeah
 M2: Yeah
 M1: Um, okay yeah, so I would think similarly, yeah, like that... That symbol is the partial of x with respect to some function. So I, do you think that big F sub x means that it's a function that has x ? Has at least as one of its variables, at least one of its variables, or do you think it means something else?
 M2: Uh, no, I think that's what it means, that, yeah...

A few minutes later, the mathematicians began to focus on the meaning of the subscripts and revealed how problematic the notation was for them. In particular, we think their uncertainty reflects possible disciplinary differences in notation, an issue we anticipated prior to the interviews.

EXCERPT 10 4:53

- M2: So what if, what if we also had like an F_y ? So how would that... be different, you know. So they're both capital F , so yeah, boy I wish I could remember the meaning of the subscript [M1 points to F_x on whiteboard] like if that already, like the original functions both F , capital F is a function of x and y . And then when we see this notation [pointing to F_x and F_y on whiteboard], F sub x , F sub y that means you've done something to that original function of x and y . You know, like if there's, that means that's the derivative with respect to x [points at F_x] and that's the derivative with respect to y [points to F_y].

At 5:34 the interviewer interrupted to clarify that F means “force” and that the x subscript indicates the component of the force in the x direction. The mathematicians immediately grasped that there could then be a y and an F_y , although—like the engineers—they did not guess that y would be the position of the other string and F_y the tension on that string.

B. What is a (partial) derivative?

The prompt to find a partial derivative yielded an interesting picture of how our experts understood the concept of derivative. In the end, all of our experts found an approach to measure the derivative experimentally—which practically requires application of the *ratio* layer of Zandieh¹⁵ (see Section III B). Each group immediately proceeded to explore whether their changes were sufficiently small. This reflects a strong recognition of the

limit layer of Zandieh,¹⁵ and a recognition that a measurement of the derivative must account for this. We saw a large difference between disciplines in the prominence of the *function* layer of Zandieh,¹⁵ which we recognize as an exploration of the dependence of the derivative on the forces F_x and F_y . And finally, we saw a large difference in the degree of comfort with a numerical approximation for the derivative.

The physicists and engineers were very comfortable with the derivative as a number, and quickly computed this number as a ratio of small changes, with a ΔF corresponding to 50 g or 100 g. In both cases, they took an additional measurement in order to verify that their changes were sufficiently small that they were within the linear regime. Their comfort with a single numerical answer for the derivative suggests to us that the physicists and engineers were satisfied with a derivative that omits the function layer of Zandieh.¹⁵ Both groups did acknowledge and discuss that there is a functional dependence of the derivative on the force.

Our mathematicians, in contrast, saw the derivative as a function, and expressed concern about numerical approximation. Interestingly, although the mathematicians were persistent in seeking a symbolic functional form for the derivative, in the process they were quite comfortable and creative with drawing conclusions about the derivative through experimentation, and specifically investigated the functional behavior of the derivative in how it changes when different parameters are modified.

Unlike the physicists, the engineers went on to mention other representations of the derivative in their discussions, such as the slope of a graph, and a symbolic expression derived from statics. The mathematicians made use of essentially every representation for derivative *excerpt* for the ratio of small changes.

1. Physicists

After some technical difficulties, at 13:56 the physicists had collected their first data, which was sufficient to find a simple ratio of changes. In Excerpt 11, they discussed whether two values each for x and F_x are sufficient. In particular, they expressed the idea that “because it is a derivative” it may need smaller increments.

EXCERPT 11 14:29

P2: The other question is, because it is a derivative, does it need to have smaller increments?

P1: Umm...

P2: The difference is...

P1: Yeah... unless the system is linear or not.

P2: So maybe we should try...

P1: Try it with fifty.

They went ahead and took an additional data point to verify that their ΔF_x value was small enough that they were working in the linear regime, which they did in Excerpt 12. They computed the derivative as a ratio of

their Δx and ΔF_x , confirmed that the two sizes of ΔF_x gave similar answers, and concluded that they had made a measurement of the derivative.

EXCERPT 12 15:48

P1: And, um, and now we have everything, we can do Δx over ΔT or F [P1 had previously referred to the force as “tension”]. And because we held constant F_y , it uses the partial derivative.

P2: Okay.

P1: And that is 1.5 over one hundred [pointing to table of x and F_x values] which is

$$\frac{\Delta x}{\Delta F} = \frac{1.5}{100}$$

They proceeded to discuss the possibility that this derivative might not be a constant, i.e. might depend on the value of F_x , acknowledging that that the derivative is a function, although they did not feel that this it was necessary to explore this dependence in order to answer the prompt.

2. Engineers

We begin with the engineers at the same stage as we began with the physicists, after they had taken two measurements each of x and y (x_{left} and x_{right} in their notation) with different values of F_x and the same F_y , and thus had enough data to compute an estimate of the derivative as a ratio of changes. In Excerpt 13 the engineers discussed whether to go on taking more data.

EXCERPT 13 14:17

E1: So shall we put another hundred grams on to see if it's linear?

E2: I think so. Yeah.

E3: I'm feeling like we should be writing some huge equation to describe this and not have to mess with this, but I'm unwilling to start that procedure.

Also, in Excerpt 13, E3 suggested that perhaps they ought to be performing a symbolic calculation, but was reluctant to do so, and the subject was dropped. On another occasion, the engineers spent some time discussing the possibility of using statics to find a symbolic solution, and also dropped the idea. They clearly recognized the possibility of a symbolic expression, but were unwilling to pursue it.

At this point the engineers computed values for the derivative for two values of F_x using a ratio of changes, and found somewhat different numerical values. In Excerpt 14 they discussed how to interpret these differing ratios.

EXCERPT 14 18:49

E3: Are those the same?

E1: Well so, I would recommend, let's let's crank it a bunch and see if we come up with the same number.

E3: Yeah.

E1: And then if it's not, we could either, we could either plot it and take slopes, or we could say, hey is that good enough.

E3: Right.

After some further discussion they concluded that the two slopes were the same within their experimental error, and that they had a good measurement of the derivative. In checking that the response is linear they addressed the *limit layer* in a way that could be surprising: they *increased* the size of their change in order to show that it was sufficiently small. This expert behavior reflected a recognition that experimental uncertainty would make *smaller* changes harder to measure.

The engineers returned to the idea of graphing and finding the slope much later in the interview. After being prompted with how they would perform an analysis if they *had* sufficient experimental data, the engineers returned in Excerpt 15 to the idea of the slope of a graph to explain how they could find the F_x dependence of the partial derivative.

EXCERPT 15 38:43

E2: I would like to see it, to see them, like what you said about slope, so like that. So what would the x ...

E1: So we could, I mean we could do a plot, right, of mass so...

E3: Versus uhh...

E1: Yeah, we would want on the x axis. Well we would want, um, force, right? [E3 begins making plot of x and F_x] That's our independent variable and length or mass.

E3: And wouldn't it be like delta, well it would be force no no, of course not. [Labels plot with x on vertical axis and F_x on the horizontal axis] F_x and then this would be...

E1: And that would be x .

E3: x .

E1: And then we could plot that and we could plot that... we could do a series of graphs set at different L_2 's. [points to the subscript L_2 on $\left(\frac{\partial x}{\partial F_x}\right)_{L_2}$]

E3: Uh huh.

E1: And then just at any value, the slope of that would be this derivative.

E3: Yeah, I like that, yeah, I do.

E2: So if we wanted, say it was a function, we know enough to figure out the function, right?

In this discussion the engineers gave a clear description (without drawing any curves on the graph) of how they could graphically obtain the derivative as a function of both F_x and y (which they called L_2). The engineers continued to further discuss how they could obtain the slope from the graphical data by performing a curve fit.

3. Mathematicians

The mathematicians spent much longer than the physicists or engineers before finding an answer to the prompt that they were satisfied with. In the process, they used physical manipulation of the machine to reach several conclusions regarding properties of x .

Very early on, the mathematicians grappled with identifying the arguments of the function x . This was in strong contrast to the physicists and engineers, who did not talk about x as a function until after having experimentally found the partial derivative. In Excerpt 16, the mathematicians used physical reasoning to conclude that x is a function of F_x and F_y . This discussion followed considerable manipulation of the machine.

EXCERPT 16 11:56

M2: So, yeah, so x is our position x [points to x position marker], like if we decide that x is some position on here [points to measuring tape on x], um, that's going to be a function of the weight we have here [points to F_x weight], right?

M1: Um hmm.

M2: It seems like x has to be a function of something, it's not just... It's not going to be constant, right?

M1: Right, right.

M2: It's going to be, it's going to depend, but doesn't it depend on both of these things [pulls on both F_x and F_y weights], right? Because I can leave this constant here [lets go of F_x], but then this is gonna... If I move, if I add weight here [pulls on F_y], then...

M1: Right, right. Then if we like tie this off [clamps y string], then maybe x really does just depend on this guy [pulls F_x weight], whereas if it's here [unclamps y string] and on both...

The mathematicians have concluded that by fixing y , x becomes independent of F_y . This reasoning probably reflects an interpretation of F_y as the mass on the hanger, rather than the tension in the string—an interpretation that is entirely consistent with the information they were given, although this was not the interpretation we intended.

A few minutes later, in Excerpt 17, the mathematicians discussed the derivative as “rate of change” and addressed how their understanding of x as a function of the two forces relates to the partial derivative they were asked to find.

EXCERPT 17 14:26

M2: Well so [points to $\frac{\partial x}{\partial F_x}$ on whiteboard], um, so you know, think about derivatives as rates of change, right? So, if you think about like in this $\left[\frac{\partial x}{\partial F_x}\right]$, how I usually look, you know, if we were looking at this lowercase f of x , y [points to $f(x, y)$], the derivative of f with respect of x , that partial derivative is like the rate of change with respect to x . But now x is gonna be dependent on these two forces and so it's

like we're finding out the rate of change in x with regard to this particular force [points to $\frac{\partial x}{\partial F_x}$]. So originally that notation is like why, 'these physicists are doing it all wrong, they're putting it all in the wrong place!' But now I understand that, now I understand the notation. Does that make sense?

M1: Uh huh.

M2: So if x is dependent on these two forces [pulls on F_x and F_y], then, right, which we're figuring that out, um, then this notation makes sense.

M1: Mhmm. Mhmm.

After this, the mathematicians began discussing how to address the prompt. This led to a discussion of what would happen if they fixed the y string, which appears in Excerpt 28 in the following section.

In the middle of the interview, the mathematicians began discussing what it is that they are being asked. In Excerpt 18 the interviewer responded by explicitly asking what they think their task is, and clarified that they were not actually asked for a symbolic expression for the partial derivative.

EXCERPT 18 28:00

INT: So what do you think your assigned task is?

M2: To find an expression that represents... [points to the $\partial x/\partial F_x$ on the board]

INT: I never said to find an expression, I just said to find this derivative.

M2: Find that derivative, okay.

M1: And so... I mean I guess this is a dumb question, but I'm asking it. So like what's the nature of what we're trying to find, like is it a number, is it a function, is it a, an expression?

M2: Right.

This discussion highlighted their confusion over what it was they were being asked to find. They continued with a discussion of how a derivative could both have a numerical value and be a function at the same time. A few minutes later, they addressed this question directly in Excerpt 19, in which M2 remembers from calculus texts how the position of a ball can both be a number and a function.

EXCERPT 19 43:55

M2: But if position is just like a number, like forty five or forty four, forty six or forty seven, then any derivative of the numbers is just going to be zero.

M1: Yeah, well on the position is, I mean, yeah, I see what you're saying.

M2: But then, you know, our calculus text, our calculus texts talk about a position function where a ball is flying through the air and position is a quadratic function, so it's just two times something... Right? Um, or negative, depending on the coefficient of x squared, so...

Throughout the interview, the mathematicians returned to speculation as to the functional form of x . In some

cases, they used hypothetical symbolic expressions to reason about possible behavior of x . As the end of the time allocated for the interview approached, the interviewer began pressing the mathematicians for an explicit answer to the prompt. In Excerpt 20 one mathematician replied jokingly with a guess as to a symbolic expression. When pressed by the interviewer, the mathematicians ended up describing a process by which they could find a numeric answer from the slope of a graph of their data.

EXCERPT 20 44:54

M2: It's one over x ! [Laughs]

INT: Okay, so you don't know, you don't know a functional relationship.

M2: No.

INT: So, what else could you do?

M2: I don't know... Take a partial derivative to find that expression? [references $\partial x/\partial F_x$]... Um, well I know that when, you know, like if I were just, if it was... I mean to find the slope, right, between two points to approximate the derivative, yeah?

INT: I'm giving you my blank interviewer face! [joking]

M2: I know! [Laughter and inaudible joking]

M2: Um, right so, if we didn't know the function and one way you approximate the tangent, right, or the slope at a particular point which is a rate of change, you just find two points close together and find slope. So, if we find, if we know change in the force and the change in the x [points to $\frac{\partial x}{\partial F_x}$], like we can take some of our ordered, we can consider these to be like ordered pairs [referencing table of values for F_x , F_y , x , and y] and then just approximate... Uh, you know come up with a numerical value.

The mathematicians proceeded to write ordered pairs of numbers on their board (using data that they had previously collected), rather than drawing the graph they described. Like the engineers and physicists, soon after they recognized the existence of a numerical solution, they began considering whether those changes were small enough. In Excerpt 21, the mathematicians discussed the accuracy of their approximation, and concluded that they could improve it by adding smaller weights.

EXCERPT 21 47:30

M1: Okay so we have those differences, and so these are, we can think of them as just being points on our function like whatever that function is.

M2: Actually a really rough approximation...

M1: An approximation. And so, we can just sort of consider the slope of those.

M2: Mhmm.

M1: Right? And that would be a reasonable approximation.

M2: Mhmm. Well, I don't know about reasonable. [Laughs]

M1: Yeah, but I mean that would be something that could approximate that...

M2: Cause I mean you know you might want to think about getting [points at an ordered pair] more fine grained slope by like just increasing by, you know, what are these, grams? [looks at weights]

After this, they spent some time adding small weights and measuring changes in x —although they never did actually perform a division or write down a ratio.

C. What should be held fixed, and does it matter?

A defining feature of a partial derivative is that some other variables must be held fixed. As discussed in Section VB, this choice of what to hold fixed has important implications in thermodynamics, and we were very interested to see how our experts treated this question. We therefore believed that the prompt to find $\partial x/\partial F_x$ was ambiguous, and were interested to see how our experts responded to this ambiguity. We expected that at least some experts would point out the ambiguity and ask us which property to hold fixed. Our interview had other results: all our experts assumed that since we were taking a partial derivative with respect to one force, then the other force must be held constant. In this respect, the PDM differs from thermodynamics.

In further discussion, each group addressed the question of what would happen if they fixed y instead of F_y . Our experts felt that clamping the y string would change either the system or the function x , and thus the derivative would change. The mathematicians measured how x changed with and without y fixed, and concluded that it moved less when y was fixed, although they did not discuss this result in terms of derivatives.

Also related to this question is whether the derivative depends on the value of F_y . This is a further level of the concept of *function* than we explored in the previous section. Ordinary derivatives are a function of the single independent variable, but partial derivatives are functions of multiple independent variables.

1. Physicists

One physicist decided that F_y must be held fixed when measuring $\partial x/\partial F_x$, because they were taking a derivative with respect to F_x . Thus if they took the inverse derivative $\partial F_x/\partial x$, then y would have been held fixed. He stated in Excerpt 24 that this is “what they taught me.” When he considered fixing the thumb nut, he stated that this (physical act?) would change x to be a different function, a function of F_x and y .

In Excerpt 22, before starting to take data the physicists began discussing the meaning of the partial derivative, and what to hold fixed.

EXCERPT 22

10:04

P1: Yeah, is ∂x over ∂F_x ? If it was ∂F_x over ∂x , it would be different.

P2: Right.

P1: Because we are keeping it constant, two different things.

P2: Right.

P1: So if it is ∂x over ∂F_x , the dependent variable is F_x , so we need to keep constant F_y .

P2: Right.

P1: Which is simple, it means we assume there is weight on each side. Then we completely ignore that part.

P2: Right. What if we just clamp it here though? In this case there's no, there's no, there's no F_y because there's no, I mean this is constant, right? So there's no...

At this stage, they spend a minute clarifying which string was x and which was y . Then in Excerpt 23, they continued their discussion of what would happen if they clamp the y string.

EXCERPT 23

11:17

P2: Okay, right. So we need F_y to be constant, so should we clamp this [points to y clamp]?

P1: Should we clamp... No, I'm going to say no because if we clamp it [points to y string], then we do not hold the tension constant. If we clamp it, we keep y constant, not F_y .

P2: Right.

P1: But we need to keep F_y constant, so I would let it move and just make sure we don't change the weight here [points to F_y].

Thus the physicists concluded that they should fix F_y in order to measure the partial derivative with which they were prompted.

Later in the interview, the interviewer prompted the physicists by asking them to measure the derivative

$$\left(\frac{\partial x}{\partial F_x}\right)_y.$$

In Excerpt 24, the physicists correctly explained how to measure this derivative by clamping the other string. They also explained about why they had assumed that F_y should be held fixed, and went on to discuss how a partial derivative is like a derivative along a path in a multidimensional space. They end by suggesting that clamping the y string changes the system itself, creating a different function.

EXCERPT 24

24:27

P1: Now we can clamp that [indicates y string].

P2: Okay... So now...

P1: But that's the natural expectation if you don't specify what you're holding constant. You're holding constant the other. That's what they taught me.

INT: That's what...

P1: They taught me. Like when I was a student.

INT: So tell me exactly, what exactly did they tell you when you were a student?

P1: That when you do a partial derivative of a function that is a function of, um, more than one independent variable, you take, you do the incremental [pointing to $\frac{\Delta x}{\Delta F}$ on whiteboard], keeping all the independent variables constant.

INT: Okay.

P1: Now if you're doing the second thing, it's real, funny because you are telling me to keep a constant, a function constant. Because y [points to y string] is always a function of the two forces [points to weights].

P2: Right, so it's...

P1: So it's not something that...

P2: So you take the derivative in a different direction in F_y and F_x plane then you would take it if you were to keep F_y constant. So the partial derivative, right, is in the specific direction, so it depends on the direction.

P1: It seems to me something like a partial derivative along a path, more than along an axis.

P2: Yeah, exactly. So you do it in a different, along a different path or a different...

P1: Along a certain path that is not, I don't know what to call it.

P2: Right, yeah.

P1: Let's do it.

INT: But y is not separately independent?

P1: y ... well you can if you want, I guess. Say that, I mean the moment you clamp it [points at clamp on y side], you change your system. And you can say now x is not a function of F_x and F_y , it's a function of F_x and y . That is a different function.

2. Engineers

At a point in the middle of the interview, E2 pointed out that they had been instructed that they could fix y using the clamp, leading to a discussion of what that would mean and how it would affect their results.

EXCERPT 25 27:00

E2: They said we could stop [inaudible, points at the thumb nut for fixing y]. Does that help us at all?

E3: That would be crazy! [jokingly]

After this the engineers spent a couple minutes discussing what they would find by performing the same derivative measurement while the thumb screw holds y fixed. After a bit of discussion involving how to perform the measurements, E1 concluded in Excerpt 26 that they are now holding something different constant.

EXCERPT 26 29:06

E1: We're holding something different constant, and we could think about, if, once we look at the numbers and see how they define that.

E3: But it's sort of like you're saying, certainly a different problem holding that [points to y string] constant.

E1: Yep.

At this point the engineers proceeded to take some data with y fixed. Then the engineers discussed how to notate what they just measured with y fixed. E1 (who teaches thermodynamics) suggests that they use the subscript convention to notate which quantity is held fixed.

EXCERPT 27 35:58

E1: So that would be a partial derivative, I think that's what they asked for.

E3: So that way we want to write...

E1: Partial of x , yeah, with respect to F_x and then put parentheses around that whole big thing.

E3: This thing?

E1: Yeah... And then write at the bottom, subscript L_2 .

E3: I'm not very good at subscripts [jokingly], I've heard at this meeting earlier... [continuing earlier joke] [E3 writes on board:]

$$\left(\frac{\partial x}{\partial F_x} \right)_{L_2}$$

E1: That's awesome.

The engineers at this stage were using the symbol L_2 for the distance we call y . Thus this notation is in complete accord with our understanding of the problem. We find this unsurprising, given that E1 is entirely familiar with thermodynamics. E3, who was writing on the board, seemed considerably less comfortable with this notation.

3. Mathematicians

We saw in Excerpt 16 a mathematician said that fixing y would change the system by making x no longer a function of F_y . This would make x a function of only one variable, since they did not talk about the value of y as a variable. A few minutes later, in Excerpt 28, the mathematicians explored what happens if they fix y by clamping its string. They observed a smaller change in x , and decided that this made sense, since the clamp impedes the motion.

EXCERPT 28 17:48

M1: But like if I... if we turn this off [clamps y]...

M2: Oh yeah.

M1: Does the same thing happen... [adds weight to F_x] So it moved a lot less, but, I mean... Okay, well, let's just see what we get, so this is like [takes x measurement] a little over forty four. [a smaller change in x]

M1: Which makes sense because... Well, I mean, this [touches y clamp] seems harder, like this is actually not allowing it to move at all.

After this, they spent some time discussing what value of F_x to use for their derivative, not being sure what to focus on. We believe based on the above quote that they understood this to mean that the derivative itself was smaller if y were fixed, although they did not explicitly state this.

D. How many independent variables are there?

As discussed in Section IV B, the number of independent variables—or the number of degrees of freedom—is a critical property of a physical system. It determines the number of parameters we must control in order to determine the state of that system, and at the same time limits the number of parameters we can fix when finding a partial derivative. The PDM has two degrees of freedom, as discussed in Section II.

The existence of two degrees of freedom means that the derivative $\partial x/\partial F_x$ (and the variable x itself) should be viewed as a function of two variables. This is an expansion of the layer of *function* into multiple dimensions. In this section, we discuss how our experts interpreted the number of degrees of freedom of the system.

The question of how many independent variables were present arose in each interview. Both the physicists and engineers treated this question explicitly (the physicists, at the prompting of the interviewer), and went through a stage of talking of x being a function of the remaining three variables. They then concluded that one could eliminate one of those three, and that only two were independent. The mathematicians were not asked this question explicitly, but addressed the question during their discussion of the meaning of the partial derivative.

All of our experts were able to discern the number of degrees of freedom present in the system, but we were surprised at how long it took the physicists and engineers to agree upon and express the fact of the interdependence among the controllable quantities.

1. Physicists

When asked how many independent variables there were in Excerpt 29 below, the physicists recognized that only two variables were independent, because they could independently control the two forces.

EXCERPT 29 17:21

INT: How many independent variables are there?

P2: So the way that you defined it, it looks like, so this is the question, you know, what exactly is x and what is y , right? So if this is, if you're saying that this is y [points to y side] and this is x [points to

x side] and there's only F_x and there's only F_y , you, you know, I mean... Uh, otherwise, you know, there's also this spring to take into account, so...

INT: So talk to your partner.

P1: Well I would say can could do a way independently on both sides.

P2: Right.

P1: So F_x and F_y are independent.

P2: Right.

P1: Anything else we can change.

P2: Right, my only question is this spring [points to spring], right? Because, I mean, there's also a force on this side [points to y side], right? So the total F_x is also has a contribution from the spring [points to spring], in principle.

P1: Umm... No, because F_x [points towards F_x] is the tension of this string.

P2: Right, but...

P1: So no matter how much weight I put here [pulling on weights], the tension on the spring is the same. Now I can change x [points to x position marker] by changing F_y [points to y string], um, but then that's because F and y are both functions of the variables.

After some more discussion, they agreed that the derivative must be a function of the two independent variables F_x and F_y .

A bit later in Excerpt 30, the physicists pondered how to understand the possibility of directly controlling y , suggesting that x depends on three variables each of which they could control, but they could not control all three independently. Based on their gestures leading up to this and the discussion in Excerpt 29 above, we believe they arrived at this conclusion based on “physical” reasoning.

EXCERPT 30 27:05

P1: Can we say that x is actually a function of y , F_x , and F_y , but then these three are not independent. [writes $x(y, F_x, F_y)$ on whiteboard]

P2: Exactly.

2. Engineers

Towards the end of their interview (without interviewer prompting), the engineers discussed their concept image of a partial derivative. In Excerpt 31 below, the senior engineer E1 decided to ask his two partners how they understand the concept of a partial derivative. This led to a discussion of how many independent variables were present in the system. At this stage we believe E1 knew the answers to his questions, and had taken on the role of interviewer of his younger colleagues, who were less certain as to how to treat this multidimensional system. Discussion after the interview confirmed that E1 had been envisioning the use of the PDM for this purpose in the context of a class in thermodynamics.

EXCERPT 31 42:19

- E1: So, let's take a step back, what's your conceptualization of the partial derivative?
- E3: Well it's a great question, E2.
- E2: I just had to ask E3 about it, uh... It's, I think of it as a, in a multidimensional system watching how one, um, dimension changes when the others are fixed.
- E1: Okay, so... How does that apply to this then? What dimensions do we have?
- E2: Two... Well we sort of have four.
- E1: Yeah you kind of do, kind of don't, right?
- E2: Yeah, I don't know if we're going to go relative or...
- E1: Well, I mean if you didn't know the statics [points to spring system] you could say you have four, but you really don't have four because how, how this side [points to x side] behaves depends on how that side's fixed [points to y side]. And that's what E3 was talking about too, right?
- E2: Umm.
- E1: So if we're just doing it empirically, we would just say we would say we have the length [points to x string] of this the F_x of this, the length of that [points to y string] the F , the tension of that, right?
- E2: Mhmm.
- E1: So then, what would the partial derivative be?
- E2: The partial derivative of the length with respect to force [E1 points at plot of x and F_x], well that's helping us.
- E1: The partial derivative of the length with respect to the force, we have, since there's four parameters, right? We have our multivariable space.
- E3: I mean...
- E2: So I guess, well, the problem is adding them up that... when I think of it as an equation, I see more clearly how you can manipulate it to more of like some cover, the situation. [E3 begins writing equation on board] But when we're just making measurements, it seems much more incremental.
- E1: What if we said, so the x is, um, the dependent variable, right? So what if we said, what if we just stated x as a function? And we would say x here [points to x string] is a function of F_x , F_2 [i.e. F_y], and L_2 [points to y string].

$$x(F_x, F_2, L_2)$$

The engineers (like the physicists) concluded by discussing the quantity x as a function of the three other variables, and recognized that those three variables are not themselves independent.

3. Mathematicians

As we described above, the mathematicians were very quick (once they knew what the variables meant) to recognize that x depends on F_x and F_y (see Excerpt 16), indicating that they recognized two independent degrees

of freedom. Moreover, they found in Excerpt 28 that the derivative $\partial x/\partial F_x$ was smaller if they fixed y rather than fixing F_y , indicating that fixing the third quantity y had some effect. They did not, however, count the independent degrees of freedom, nor did we prompt them to do so.

The mathematicians began discussing how to treat the second side of the system in Excerpt 32. M2 expressed a tension between a “mathy” understanding and their physical experimentation which led them to believe that the partial derivative depended on what was done on the y side of the PDM.

EXCERPT 32 32:17

- M2: But like, ah, I just keep going back to the, you know, more mathy idea [points to an illegible derivative on whiteboard], like this idea of a partial derivative where, you know, when I'm taking the partial derivative of that with respect to x , you know y is just... But then we see this like physical thing where this side does matter [points to y side], so then...
- M1: Well do you want to, I mean, can we... Should we just focus on that question and just ask ourselves, like convince ourselves one way or the other?
- M2: Whether the y matters.

After agreeing to explore the dependence on the y side of the system, in Excerpt 33, the mathematicians concluded based on experimentation that x is indeed affected by the value of the other force F_y , confirming that the derivative is a function of two variables. We note that their language is at times confusing, as they often use x and y to refer to the two independent variables (which we call F_x and F_y), and had written $f(x, y)$ on the board to refer to a generic function of two dimensions.

EXCERPT 33 32:58

- M1: Okay, so what would the y mattering look like for us?
- M2: So, I guess, the, if the y mattered, um, the force in y [F_y] would have an impact, would somehow have an impact on our x position, you know, as we... Here [references $\partial x/\partial F_x$], we're trying, we're changing the force in x , so with that regard, does this force here [points to y side] still have an impact when we're just seeing what the change is with regard to that force.

Thus the mathematicians concluded that the partial derivative was a function of the two forces.

VII. CONCLUSIONS

In this section, we summarize the experts' concept images and definitions for partial derivatives, propose the need for an extended theoretical framework, describe

differences in the concept of limit between mathematics/theory and experiment, and discuss the limitations of our study and the prospects for future work.

A. Concept images of derivative

All three groups of experts assumed that when taking a partial derivative with respect to F_x , F_y can be assumed to be held fixed. This finding was unexpected, particularly with regard to those experts who work with thermodynamics in their research: both physicists and one engineer. This question merits further study, particularly to probe when and how experts approach problems in which there is ambiguity in the choice of “independent” degrees of freedom.

1. Physicists

At first, the physicists thought that they were being asked to find (the inverse of) the spring constant. Once they recognized that the spring constant would be represented by a total derivative rather than a partial derivative, they spent some time exploring other possible meanings for the interview prompt. After they were told how to interpret the prompt, they moved immediately to describing the partial derivative as a ratio of small numerical changes, collecting data, and calculating a number. They spent some time establishing that their ratio was accurate enough and attended to determining how many independent variables were in the system and which of them to fix. While they acknowledged that the derivative was a function, they did not try to evaluate this function in any way. They did not mention slope at all, nor did they try to express the relationship between position and tension analytically so that they could take a symbolic derivative.

2. Engineers

Like the physicists, the engineers thought of the partial derivative as something that could be approximated experimentally, and spent a lot of time collecting data so that they could represent the partial derivative as a ratio of small numerical changes. They mentioned that the derivative could be found as a slope and that they could therefore determine the derivative by graphing their data, although they did not pursue this approach. They further noted that their experimentally-determined function should match some sort of theoretical equation. The engineers were the only ones to state a concept definition, albeit incomplete and bordering on inaccurate. They defined a partial derivative as “how one dimension in a multidimensional system changes when the other dimensions are fixed.”

3. Mathematicians

The mathematicians repeatedly expressed their interest in determining an algebraic expression for $\partial x/\partial F_x$; they seemed to interpret the task statement as directing them to find an explicit function definition. Once they had collected data, they talked about the derivative as a slope, as a ratio of small changes, and as a rate of change. Like the physicists, they recognized the need to fix one quantity, which for them was connected with the idea of holding a variable constant when using symbolic differentiation rules for a multivariable function.

B. Need for an Extended Framework

In Section III B, we gave a brief summary of Zandieh’s framework for the concept image of derivative. In Section IV B, we gave our own list of five different ways to understand and think about the concept of derivative, which has substantial overlap with Zandieh’s framework, but included a specific numerical category. Throughout our analysis, our interdisciplinary team found it useful to use elements of both of these descriptions. In particular, we found ourselves frequently referring to Zandieh’s choice of process–object pairs, and describing our own list in similar terms.

However, we had deliberately made the choice to embed our interviews in a task in which the interviewees had easy access to numerical data, but not to an analytic expression for the relationship between the physical quantities. This is, after all, the environment in which most experimentalists typically find themselves. For this reason, our analysis focused on a numerical representation within the concept image of derivative.

We were surprised that Zandieh’s framework did not include a numerical representation, even though her introduction made it clear that she was aware of this possibility. A careful reading of her paper shows several examples of students using numbers and these discussions are included in her analysis. This suggests that she considers numerical evaluation as a later step in each of her representations rather than as a representation in its own right. In our own list of five categories, we are now finding the possible need to distinguish between *two* numerical representations: *exact* numerical work (such as would happen when plugging particular values for the independent variable into an explicit analytical model for a physical process) and *experimental* numerical work (such as data from an experiment, including its experimental uncertainties). We are currently working on a paper which further elucidates these issues.³¹

We suspect that the lack of an explicit experimental representation in previous frameworks reflects a more general absence of this representation in lower-division mathematics courses, a view which is supported by the mathematicians’ efforts to reframe the problem in terms of an analytic representation. It seems clear that the

interpretation of experimental data is a topic requiring instructional intervention; how best to accomplish this, and to what extent this should be done by mathematicians or within experimental disciplines themselves, are topics worthy of for further discussion.

It will also be necessary to extend any resulting theoretical framework for the concept image of derivative to elements of the concept of partial derivative that are distinct from derivative. In particular, we see the need to add a layer for what is held fixed when finding a particular partial derivative.

C. Limits and the Real World

There were striking differences between the mathematicians and the physicists and engineers. The physicists and engineers were relatively quick to find the derivative at a point as the numerical ratio of small changes. In this process, they were so casual with approximations that they were willing to give this ratio as their *final answer* to the interview prompt asking for a partial derivative without qualifying their answer with the word approximation. They did signal that they understood that their answer was an approximation by checking that the ratio they calculated was “good enough” by either decreasing (physicists) or increasing (engineers) the change in the independent variable. They also demonstrated an understanding of the function layer but did not bother to find the value of the derivative at more than one point. On the other hand, the mathematicians were firmly embedded in the function layer, pursuing the desire to find an analytic expression until they were explicitly asked by the interviewer “What else could you do?” Subsequently, when they were discussing finding a slope from their numerical data, they continued to doubt the reasonableness of the resulting approximation (see Excerpt 21).

We note that, in the idealized world of pure mathematics (and theoretical physics), the need to approximate rarely arises, whereas, for experimentalists, a ratio of small numerical changes is often the best answer that they have for a derivative, particularly in the absence of a theoretical model for the process they are studying. So the shorthand of calling this ratio “the derivative” rather than the more cumbersome “an approximation to the derivative” makes cultural sense. Experimentalists can always hope to design a better measuring apparatus or at least imagine a *gedanken* experiment to improve their approximation.

In the extreme case, physicists, in particular, are likely aware that continuity itself and therefore the ability to take formal limits in derivatives, are properties of the continuous models that they use for the physical world and not properties of the real world itself on atomic scales and below. The real world imposes a fundamental limitation on the concept of limit.

D. Limitations of this Work

We recognize that we cannot use a single interview with two or three content experts to make conclusions about every expert in a given field, nor can we come to definite conclusions about what these experts may or may not have been able to do in other settings or contexts. We plan to interview different types of mathematicians (especially computational and applied mathematicians who may have more experience with approximation), different types of physicists (including experimentalists and non-computational theorists), and different types of engineers. We would also like to interview experts in such fields as economics or oceanography, whose mathematical cultures may differ substantially from those already considered (and from each other). In particular, we hypothesize that thermodynamics experts from a variety of fields will behave more similarly on this task than non-thermodynamics experts within a single field.

Another possible limitation of this work arose due to the relative linearity of the PDM system in the regime explored by our experts. While they recognized that the system was nonlinear, this was a small effect that was easily dismissed. It would be interesting to examine expert responses with the machine configured to operate in a more nonlinear regime. This would require the experts to take greater care in their treatment of the limit layer, and could trigger experts to go ahead and explore the function layer of the derivative.

E. Other Future Directions

It would be interesting to ask interviewees how they would report their data for the partial derivative (assuming they were to take lots of data) if they were to publish this result in a paper. We are interested to see (a) if they address both dimensions of Zandieh’s representation and (b) how they would represent the result: tables, single or multiple one-dimensional plots, three-dimensional surfaces, contours, etc.

We would also be interested to explicitly prompt experts to give us a concept definition, to see how their concept image (as determined from their approach to this novel task) relates to their concept definition.

Every group spent some time exploring the physical system of the PDM. In future, it would be interesting to explore this aspect of the interviews and to think about the pedagogical and research implications of this type of play as an aspect of expert reasoning. In particular, the mathematicians spent more time than other groups exploring the covariation of the physical quantities by pulling on various strings and making expert-like observations of how the system responded.

The notational confusion we observed suggests deep cultural differences between disciplines, warranting further study. Although the mathematicians’ specific confusion regarding F_x is not likely to be present in physics stu-

dents, the use of x and y as the two independent position variables—particularly in variables that are not spatially orthogonal—has consistently created confusion both in experts and novices. However, in order to focus more directly on expert concept images of (partial) derivatives, in future we intend to name our four variables x_1 , x_2 , F_1 , and F_2 .

Finally, we are interested in pursuing the pedagogical consequences of this study for classroom learning trajectories. For example, physics students have traditionally been exposed to the discreteness of data and the necessity of addressing experimental uncertainty in their lower division laboratories. But this exposure is disappearing with the near ubiquitous use of motion sensors and computer interfacing that blur the distinction between dis-

crete data and continuous plots. What other physics experiences (research, advanced laboratory courses, etc.) will be needed to reinforce these aspects of the concept image of derivative?

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