Spin Correlation and Bell's Inequality

1. Correlations in Spin-Singlet States

electron 1

Spin-singlet state of an electron pair: $$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big)$$ Two electron wave vector

electron 2

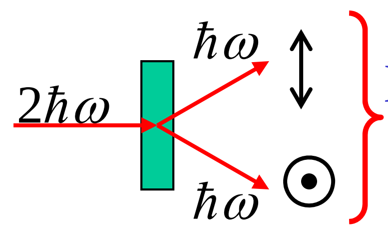
- $P_1(\uparrow) = P_1(\downarrow) = \dfrac{1}{2}, \quad P_2(\uparrow) = P_2(\downarrow) = \dfrac{1}{2}$
- Measurement of electron 1 spin: $\begin{cases} \uparrow & \Rightarrow & \downarrow \\ \downarrow & \Rightarrow & \uparrow \end{cases}$ electron 2 spin state

Nonlocality: The spin correlation persists even if the two particles are well separated and have no interaction.

Example:

(i) Decay of η meson into muon pair: $\eta \rightarrow \mu^+ + \mu^-$

(ii) Parametric down conversion in nonlinear crystal



Spin-Singlet States

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\Big)$$

$$= \frac{1}{\sqrt{2}}\left[\boxed{\frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_x + |\downarrow\rangle_x\Big)} \otimes \boxed{\frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_x - |\downarrow\rangle_x\Big)} - \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_x - |\downarrow\rangle_x\Big) \otimes \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_x + |\downarrow\rangle_x\Big)\right]$$

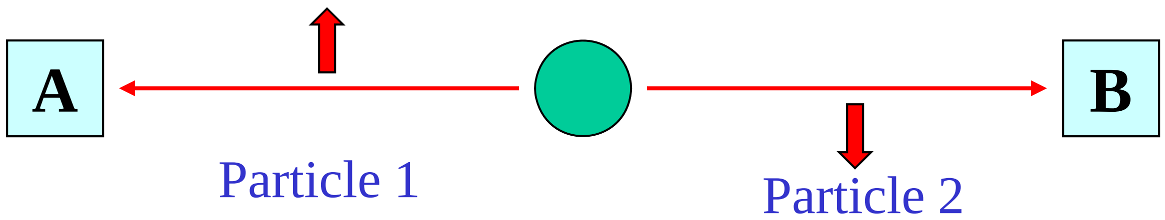
$$= \frac{1}{\sqrt{2}}\left[\frac{1}{2}\Big(|\uparrow\rangle_x \otimes |\uparrow\rangle_x - |\uparrow\rangle_x \otimes |\downarrow\rangle_x + |\downarrow\rangle_x \otimes |\uparrow\rangle_x - |\downarrow\rangle_x \otimes |\downarrow\rangle_x\Big)\right.$$

$$\left. - \frac{1}{2}\Big(|\uparrow\rangle_x \otimes |\uparrow\rangle_x + |\uparrow\rangle_x \otimes |\downarrow\rangle_x - |\downarrow\rangle_x \otimes |\uparrow\rangle_x - |\downarrow\rangle_x \otimes |\downarrow\rangle_x\Big)\right]$$

$$= -\frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_x \otimes |\downarrow\rangle_x - |\downarrow\rangle_x \otimes |\uparrow\rangle_x\Big) = \frac{e^{i\pi}}{\sqrt{2}}\Big(|\uparrow\rangle_x \otimes |\downarrow\rangle_x - |\downarrow\rangle_x \otimes |\uparrow\rangle_x\Big)$$

$$= \frac{e^{i\pi}}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle_x - |\downarrow\uparrow\rangle_x\Big) \quad = \boxed{\frac{e^{i\phi_n}}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle_n - |\downarrow\uparrow\rangle_n\Big)}$$

Spin correlation in a spin-singlet state



(i) Measurement of particle 1 spin, $S_{1,z}$

$\Rightarrow$ wave vector collapse $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \;\Rightarrow\; |\uparrow\downarrow\rangle$

$\Rightarrow$ Determine particle 2 spin state instantly even if two particles are macroscopically separated.

(ii) Measurement of S_z and S_x $\quad |\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle_x - |\downarrow\uparrow\rangle_x\right)$

- A measures S_z and B measures S_x : completely random correlation
- A measures S_x and B measures S_x : 100 % (opposite sign) correlation
- A makes no measurement $\rightarrow$ B's measurements are random.

Measurement even in local system determines state of whole system.

2. Einstein's locality principle and Bell's inequality

Einstein's locality principle

The real factual situation of the system S_2 is independent of what is done with the system S_1, which is spatially separated from the former.

"Einstein-Podolsky-Rosen (EPR) paradox"

Model to explain the spin-correlation measurement w/o violating the locality principle

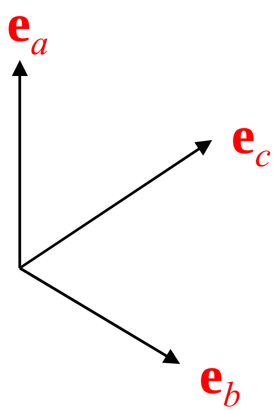
A: particle 1		B: particle 2	
$(z+, x+)$	$\leftrightarrow$	$(z-, x-)$	25%
$(z+, x-)$	$\leftrightarrow$	$(z-, x+)$	25%
$(z-, x+)$	$\leftrightarrow$	$(z+, x-)$	25%
$(z-, x-)$	$\leftrightarrow$	$(z+, x+)$	25%

- When we measure S_z, we do not measure S_x, and vice versa.

$\Rightarrow$ Impossible to determine S_z and S_x simultaneously

B's result is predetermined independently of A's choice as to what to measure.

Bell's inequality: spin measurement on direction of three unit vectors



$(x, y) \to (a, b, c)$

Population	A: particle 1		B: particle 2
N_1	$(a+,b+,c+)$	$\leftrightarrow$	$(a-,b-,c-)$
N_2	$(a+,b+,c-)$	$\leftrightarrow$	$(a-,b-,c+)$
N_3	$(a+,b-,c+)$	$\leftrightarrow$	$(a-,b+,c-)$
N_4	$(a+,b-,c-)$	$\leftrightarrow$	$(a-,b+,c+)$
N_5	$(a-,b+,c+)$	$\leftrightarrow$	$(a+,b-,c-)$
N_6	$(a-,b+,c-)$	$\leftrightarrow$	$(a+,b-,c+)$
N_7	$(a-,b-,c+)$	$\leftrightarrow$	$(a+,b+,c-)$
N_8	$(a-,b-,c-)$	$\leftrightarrow$	$(a+,b+,c+)$

If A measures $(a+, b-)$, number of particles $N_3 + N_4$

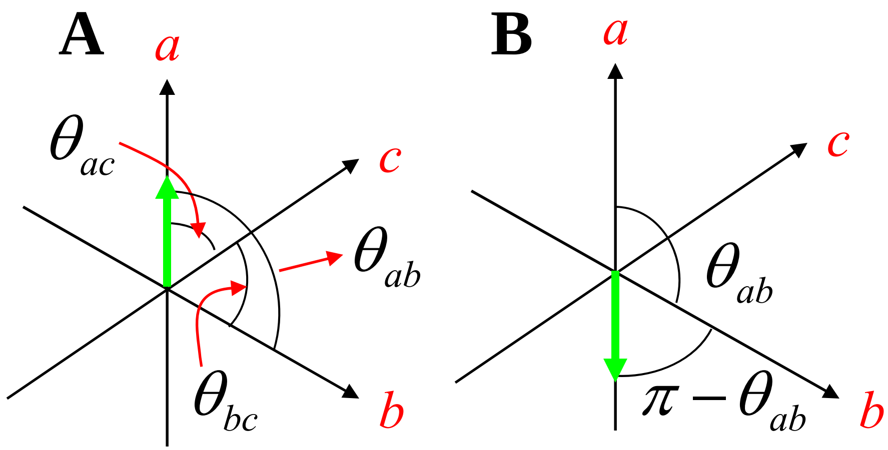
Since $N_i \geq 0$, $N_3 + N_4 \leq (N_2 + N_4) + (N_3 + N_7)$

$P(a+ ; b+)$: probability that observer A measures $a+$ and observer B measures $b+$

$$P(a+;b+) = \frac{N_3 + N_4}{\sum_{i=1}^{8} N_i}, \quad P(a+;c+) = \frac{N_2 + N_4}{\sum_{i=1}^{8} N_i}, \quad P(c+;b+) = \frac{N_3 + N_7}{\sum_{i=1}^{8} N_i}$$

$\Rightarrow$ $\boxed{P(a+;b+) \leq P(a+;c+) + P(c+;b+)}$ **Bell's inequality** (from locality principle)

Quantum mechanics and Bell's inequality



A (particle 1) $a+$ $\Rightarrow$ B (particle 2) $a-$

$$P(a+;b+)=\frac{1}{2}\cos^2\left(\frac{\pi-\theta_{ab}}{2}\right)=\frac{1}{2}\sin^2\left(\frac{\theta_{ab}}{2}\right)$$

Bell's inequality

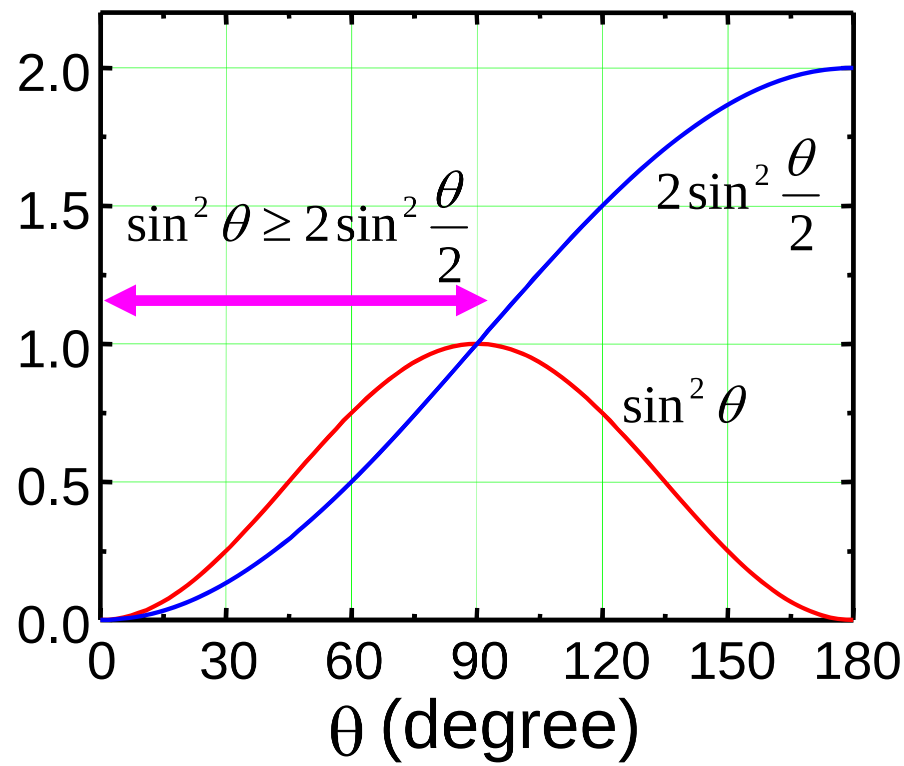
$$P(a+;b+)\le P(a+;c+)+P(c+;b+)$$

$$\Rightarrow \quad \frac{1}{2}\sin^2\frac{\theta_{ab}}{2}\le\frac{1}{2}\sin^2\frac{\theta_{ac}}{2}+\frac{1}{2}\sin^2\frac{\theta_{cb}}{2}$$

Assume $\theta_{ab}=2\theta, \theta_{ac}=\theta_{cb}=\theta$

$$\Rightarrow \quad \sin^2\theta\le 2\sin^2\frac{\theta}{2}$$

not true for 0 < θ < π/2 !!!



Quantum Theory vs. Locality Principle

- QM predictions are not compatible with Bell’s inequality.

- Experiments confirms that Bell’s inequality can be violated in such a way that QM predictions are correct.

- Despite the nonlocality, no transmission of useful information by spin-correlation:
  No violation of principles of relativity.

- Nonlocality of spin-correlation (entanglement):
  Quantum information theory