What Sort of a Beast is it?

Students need to be able to identify the objects they work with. Is it a vector or a scalar? Finite or infinitesimal? What units does it have? This ability is essential when setting up a problem, especially word problems. It is also a useful technique for checking whether the answer makes sense. Are both sides vectors? Infinitesimals? Do the units match?

1 Vectors and Scalars

The fundamental objects in vector calculus are *vector fields*, such as the velocity field of a fluid, and *scalar fields*, such as the density of chocolate on a pretzel. Students must be able to tell which is which.

This starts with notation. We *never* write vectors as pairs or triples of numbers; this notation is reserved for the coordinates of points, a quite different concept. The symbols we use for vectors have arrows on them [1] (to match what we write by hand) as well as being bold-faced (to match the notation usually used in textbooks).

Checking whether both sides of an equation are vectors is then easy. In the first instance, are there arrows on both sides? One must of course learn the rules for multiplying vectors, namely that the dot product yields a scalar, while the cross product yields a vector. [2]

Students often have trouble distinguishing between scalar and vector line integrals, especially in word problems. Given a problem involving the (lin-

[1] We usually put hats on unit vectors; hats count as arrows for this purpose.

[2] It is in part for this reason that we categorically reject the concept of the "scalar cross product", occasionally introduced as a name for the $\hat{\boldsymbol{k}}$-component of the cross product of two vectors lying in the xy-plane.

ear) density λ (of chocolate, say), and asked to find the total amount (of chocolate), many students will write things like $\int \lambda \cdot d\vec{\boldsymbol{r}}$ or $\int \vec{\boldsymbol{\lambda}} \cdot dr$, neither of which makes sense, as well as $\int \vec{\boldsymbol{\lambda}} \cdot d\vec{\boldsymbol{r}}$, which makes formal sense, but which turns the scalar λ into a vector. [3]

2 Small and Large

One of the most common errors students make when dealing with differentials is to confuse them with derivatives, writing things like $d(x^2) = 2x$, rather than $d(x^2) = 2x\,dx$. The easiest way to avoid such errors is to emphasize that both $d(x^2)$ and dx are really, really small, in fact, *infinitesimal*, whereas derivatives are the *ratios* of differentials, which can be large. For instance, $\frac{d(x^2)}{dx} = 2x$, which can be as large as you like, depending on the value of x. An easy check is to verify that the "powers of d" are the same on each side of an equation, or more simply that either both sides are small, or both are large.

An extreme example of this problem occurs in line integrals, where some students might write something like $d\vec{\boldsymbol{r}} = (\hat{\boldsymbol{\imath}} + 2x\,\hat{\boldsymbol{\jmath}})$, leaving out a factor of dx, which in turn leads to expressions like $\int 3x^2$ when evaluating $\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$. This error is often self-correcting, in that "obviously" $\int 3x^2 = x^3$, although this leads to errors if the element of integration is something other than dx. But some students will take seriously the lack of dx, and not integrate at all!

3 Dimensions and Units

What does the equation $y = x^2$ mean? This depends on whom you ask. To a mathematician, this is simply the equation of a parabola. Yet x and y typically have dimensions of length, in which case the above equation is nonsense; the dimensions don't match! Similar problems arise with expressions like $\sin(x)$, which only makes sense if x is dimensionless.

For this reason, other scientists and engineers are careful to insert constants carrying appropriate dimensions into such expressions. The parabola would take the form $y = ax^2$, with a have the dimensions of inverse length;

[3]Some students will try to resolve this problem by computing $\int \vec{\nabla}\lambda \cdot d\vec{\boldsymbol{r}}$; these students are usually not sophisticated enough to be disturbed when this yields 0 for a closed path. (What? No chocolate?)

trig functions typically take the form $\sin(\omega t)$, with t having dimensions of time, and where the constant ω denotes frequency, with dimensions of inverse time.

Students in mathematics classes not only don't get practice in using dimensions in this way, they are deprived of an important way of checking their work, namely checking whether the units match. The total amount of chocolate had better have dimensions of mass; if a student gets mass/length, say, he or she immediately knows there's a problem. So when graphing a function, it is important to realize that the dimensions along the vertical axis may not (and in fact usually are not) the same as those along the horizontal axis or axes.

There is also an important distinction between dimensions (length, say) and the units used to actually measure things (such as meters and feet). Dimensions must balance in an equation, but this is not necessarily true of units. Isn't

$$1 \text{ foot} = 12 \text{ inches} \tag{1}$$

a perfectly valid equation?

This leads to a subtle problem when using hills and topographic maps to introduce functions of two variables, as this is one of the few examples where the dimensions actually do match. We emphasize the distinction by measuring distance on the map in kilometers, but height in feet — different units, but not different dimensions.

Another subtlety arises when measuring angles. What are the units? Surely an angle is a pure number – the ratio of two lengths, expressed in the same units. We measure angles in radians for convenience, and need a way to distinguish radians from degrees. But radians are special, since they are dimensionless; we call them a *geometric unit*. It is *not* necessary for geometric units to balance, as witness the defining equation for radian measure in terms of arc length, $s = r\theta$.

But the argument of a trig function had better be an angle in radians — not a length, or a time. Similarly, the argument of a logarithm or exponential function must be dimensionless, as must the parameter in any power series expansion!

4 In the Classroom

We recommend explicitly discussing the process used to decide what sort of integral needs to be done. In the above example it goes something like this:

> "I want to add up λ, which is a scalar, so I need to multiply it by a scalar, ds, and do a scalar line integral."
>
> "The linear density is λ, so the (small) amount on a small piece of the curve is λ times the length, ds, of the small piece."
>
> "The density is λ, with dimensions of mass/length, so to get the mass I need to multiply it by a length, ds."

We also recommend asking at each step of a calculation, "What sort of a beast is it?" Do the arrows match? Do the differentials match? Do the dimensions match?

Bibliography

[1] Tevian Dray and Corinne A. Manogue, *Spherical Coordinates*, College Math. J. **34**, 168–169 (2003).

[2] Robert Osserman, **Two-Dimensional Calculus**, Harcourt, Brace, and World, New York, 1968.

[3] Tevian Dray & Corinne A. Manogue, *The Murder Mystery Method for Determining Whether a Vector Field is Conservative*, College Math. J. **34**, 238–241 (2003).