Magnetic Field for Ring - Instructor's Guide

Keywords: Upper-division, E and M, Magnetic Field, Symmetry, Ring

Highlights of the activity

Working in small groups students are asked to consider a ring with charge Q, and radius R rotating about its axis with period T and create an integral expression for the magnetic field caused by this ring everywhere in space.

Reasons to spend class time on this activity

This is designed as the culminating activity for this unit which allows students to connect much prior learning in a single problem. Prior to upper division physics courses, students have little experience in dealing with anything involving the synthesizing or "pulling together" of so many things simultaneously. Students need to use symmetry and geometric understanding to be able to construct the integral using the Biot-Savart Law. For many students, this will be the messiest integral they have ever had to face (see Eq. 6 in the solutions for this activity). Successfully unpacking (dealing with) this integral requires that students first believe that they are capable of tackling something like this.

The primary new piece to the problem is the vector cross product in the numerator of the integrand. Although students have done vector cross products in math classes, students will need to realize that they can apply vector cross products to this context and that doing so will help make the problem more manageable. There is ample opportunity for algebraic errors while taking the cross product, including sign errors, losing track of ϕ vs ϕ', and failure to recognize the trigonometric identity $\cos\phi\cos\phi' + \sin\phi\sin\phi' = \cos(\phi - \phi')$. However, students should be encouraged to recognize that they have all the fundamental pieces to understanding this problem along with the ability to put them together, and that they simply need to work carefully in order to obtain a correct solution.

Solution for magnetic field in all space due to a ring with total charge Q and radius R rotating with a period T

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \int\limits_{\text{ring}} \frac{\vec{\boldsymbol{I}}(\vec{\boldsymbol{r}}') \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}')\, dl'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|^3} \tag{1}$$

where $\vec{\boldsymbol{r}}$ denotes the position in space at which the magnetic field is measured and $\vec{\boldsymbol{r}}'$ denotes the position of the current segment. As described in previous solutions,

$$\begin{aligned}
dl' &= R\, d\phi' && (2)\\
\vec{\boldsymbol{I}}(\vec{\boldsymbol{r}}') &= \frac{QR}{T}(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}}) && (3)\\
\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}' &= (r\cos\phi - R\cos\phi')\hat{\boldsymbol{\imath}} + (r\sin\phi - R\sin\phi')\hat{\boldsymbol{\jmath}} + (z - z')\hat{\boldsymbol{k}} && (4)\\
|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'| &= \sqrt{r^2 - 2rR\cos(\phi - \phi') + R^2 + z^2} && (5)
\end{aligned}$$

Thus $\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) =$

$$\frac{\mu_0}{4\pi}\frac{QR^2}{T}\int_0^{2\pi}\frac{(-\sin\phi'\hat{\mathbf{i}}+\cos\phi'\hat{\mathbf{j}})\times[(r\cos\phi-R\cos\phi')\hat{\boldsymbol{\imath}}+(r\sin\phi-R\sin\phi')\hat{\boldsymbol{\jmath}}+z\hat{\boldsymbol{k}}]d\phi'}{\sqrt{r^2-2rR\cos(\phi-\phi')+R^2+z^2}} \tag{6}$$

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\frac{QR^2}{T}\int_0^{2\pi}\frac{(z\sin\phi'\hat{\boldsymbol{\imath}}+z\cos\phi'\hat{\boldsymbol{\jmath}}+[R+\cos(\phi-\phi')]\hat{\mathbf{k}})d\phi'}{\sqrt{r^2-2rR\cos(\phi-\phi')+R^2+z^2}} \tag{7}$$

1 The z axis

For points on the z axis, $r=0$ and ϕ can be arbitrarily taken as zero. Thus, the integral simplifies to

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\frac{QR^2}{T}\int_0^{2\pi}\frac{[z\sin\phi'\hat{\boldsymbol{\imath}}+z\cos\phi'\hat{\boldsymbol{\jmath}}+(R+\cos\phi')\hat{\mathbf{k}}]d\phi'}{\sqrt{R^2+z^2}} \tag{8}$$

Doing the integral results in

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\frac{QR^2}{T}\frac{2\pi R}{\sqrt{R^2+z^2}} \tag{9}$$

2 The x axis

For points on the x axis, $z=0$ and $\phi=0$, so the integral simplifies to

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\frac{QR^2}{T}\int_0^{2\pi}\frac{[z\sin\phi'\hat{\boldsymbol{\imath}}+z\cos\phi'\hat{\boldsymbol{\jmath}}+(R+\cos\phi')\hat{\mathbf{k}}]d\phi'}{\sqrt{r^2-2rR\cos\phi'+R^2}} \tag{10}$$

using the same process as the previous two solutions, the $\hat{\boldsymbol{\imath}}$ and the component disappears and the remaining elliptic integral is

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi}\frac{QR^2}{T}\int_0^{2\pi}\frac{[z\cos\phi'\hat{\boldsymbol{\jmath}}+(R+\cos\phi')\hat{\mathbf{k}}]d\phi'}{\sqrt{r^2-2rR\cos\phi'+R^2}} \tag{11}$$