A Learning Progression for Partial Derivatives

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A Story about Differentials

Differentials underpin the entire learning progression

- physicists reason about small quantities, which is not how derivatives are commonly taught in math courses
- differentials represent small "enough" quantities
- differentials linearize complicated equations
- differentials can be manipulated algebraically

PRIMUS 2017 – Dray *et al.*

Example Concept Image – Tables and Contour Maps

Differentials describe the small changes necessary to define the derivative from representations of data.

Derivatives can be thought of as "ratios of small changes" using a variety of representations

- tables with two columns (ordinary)
- tables with three columns (partial)
- contour maps (partial)

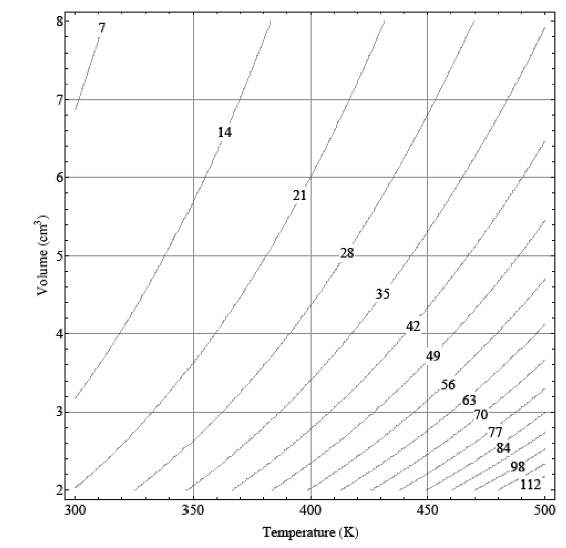
A concept image should emphasize coordination and translation between representations (*e.g.,* transferring information from a table to a contour map).

PR-PER in progress – Bajracharya *et al.*

Evaluate $\left(\frac{\partial U}{\partial T}\right)_P$ at $P = 10$ atm., $T = 410$K using the information below.

P(atm.)	T(K)	V(cm^3)
10	300	1.32
10	310	1.44
10	320	1.57
10	330	1.71
10	340	1.85
10	350	2.00
10	360	2.15
10	370	2.32
10	380	2.49
10	390	2.67
10	400	2.86
10	410	3.05
10	420	3.25
10	430	3.47
10	440	3.69
10	450	3.91
10	460	4.15
10	470	4.40

Pressure P, Temperature T, and Volume



Internal Energy U (T, V).

Anchor – Expert Concept Image

Experts' concept images of partial derivatives

- involves many solution paths
- is strengthened by sense-making
- varies from person to person and from field to field (math experts and physics experts are not the same!)

Based on task-based interviews with experts
PERC 2012 – Kustusch *et al.*
PRST-PER 2014 – Kustusch *et al.*

Anchor – Novice Understanding

Students are good at

- thinking of the derivative as a slope
- relating the derivative to changes
- computing derivatives algebraically

Students need more practice

- viewing the derivative as a ratio
- thinking about small changes and when a change is "small enough"

Based on research surveys
PERC 2017 – Emigh *et al.*

Multivariable Calculus Intermediate Mechanics Vector Calculus Upper-level Thermodynamics Upper-level Electromagnetism

Name the Experiment!
AJP 2014 – Roundy *et al.*

Partial derivatives machine
PERC 2017 – Vignal *et al.*
PRST-PER 2015 – Roundy *et al.*
PERC 2014 – Roundy *et al.*
PERC 2013 – Sherer *et al.*

Plastic surfaces

raisingcalculus.winona.edu

Example Task Analysis – Identifying Partial Derivatives from Total Differentials

Differentials equations can be manipulated to solve many thermodynamics problems

- total differentials are always linear
- substitution is an effective strategy, even for complicated equations
- chain rules can be determined quickly
- Legendre transformations commonly use differentials
- coefficients that are equivalent to partial derivatives can often be identified as physical quantities

PERC 2017 – Vignal *et al.* and Founds *et al.*

$$dU = TdS - pdV$$
$$H = U + pV$$
$$dH = TdS - pdV + pdV + Vdp$$
$$dH = TdS + Vdp$$
$$dH = \left(\frac{\partial H}{\partial S}\right)_p dS + \left(\frac{\partial H}{\partial p}\right)_S dp$$
$$T = \left(\frac{\partial H}{\partial S}\right)_p \qquad V = \left(\frac{\partial H}{\partial p}\right)_S$$

Example Representations – Algebraic, Graphical, Tangible, Embodied

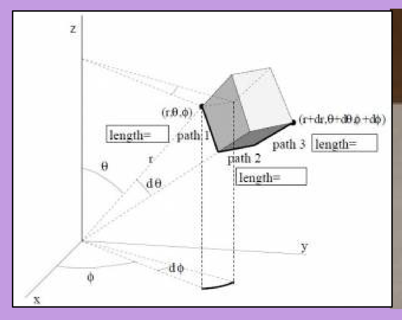
Students pointing in the direction of the gradient

Differentials in spatial settings can be combined into the algebraic vector differential, which describes all directions at once. The multivariable chain rule becomes a statement about the gradient

$$df = \vec{\nabla} f \cdot d\vec{r} \qquad (1)$$

Equation (1) can be used to find:

- any directional derivative
- the total derivative with respect to a parameter like time
- properties of the gradient (*e.g.,* it is perpendicular to the level curves and its magnitude is the slope in the steepest direction)



Representations for the vector differential and volume element in spherical coordinates

Coll. Math. J. 2010 – Dray and Manogue
Coll. Math. J. 2003 – Dray and Manogue

Acknowledgements: We would like to thank the members of the Paradigms research project over many years. This work was supported in part by NSF grant DUE-1323800.

Learning Progression

Learning progressions include:

- upper and lower anchors
- hypotheses about learning
- descriptions of how students might develop ways of thinking
- empirically tested curricular elements

LPs in Science 2012 – Plummer
CPRE 2009 – Corcoran *et al.*

Concept Image

A concept image includes the set of all mental pictures, associated properties and processes associated with a concept. Middle-division students need rapidly to develop rich concept images.

Students should be able to go back and forth between various elements.

CBMS 2000 – Zandieh,
Ed. Stud. Math. 1981 – Tall & Vinner

Task Analysis

Cognitive task analysis triangulates information from informal/formal research, existing PER, classroom work, feedback from teachers, TAs, student focus groups, etc.

Discussion identifies themes that recur and lead to rich concept images.

AJP 2009 – Redish & Hammer
PERC 2009 – Manogue & Gire

Representations

Different representations have material features that support different kinds of reasoning and problem-solving.

We use and develop representations that relate different elements of a rich concept image and/or task.

Cognition in the Wild 1995 – Hutchins
PERC 2012 – Gire & Price

Research

It is not possible to validate an entire learning progression that spans four undergraduate years and multiple math and physics subdisciplines. Instead, we empirically validate in snapshots across the curriculum using a wide variety of formal and informal methods and perspectives.

PERC 2017 – Manogue *et al.*