# Billy Narrative: Understanding Operators in Quantum Measurements

Gire (Gire & Manogue, PERC 2011) conducted semi-structured clinical interviews with n=14 junior physics majors enrolled in the Paradigms in Physics program. The interviews took place three weeks after the final exam for Central Forces, the third Paradigm course that includes quantum mechanics.

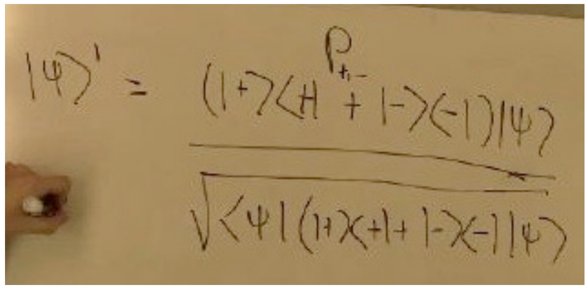
These semi-structured interviews were conducted to explore how these students understood operators in quantum mechanics and the role of operators in problems about quantum measurements. Students were asked to (1) reflect on their experiences in their quantum courses, (2) describe how they would explain to a friend or roommate what an operator is and how it is used in quantum mechanics, (3) think aloud while solving a problem related to sequential measurements on identically prepared hydrogen atoms, and (4) consider seven statements about operators and quantum measurement and discuss the whether they agree the these statements.

The students were instructed to talk aloud while solving the sequential measurement problems and while considering the Agree/Disagree statements. Students wrote on tabletop whiteboards and were video and audio recorded. The interview with Billy is a particularly illustrative case of a student who used an incorrect approach to solve the sequential measurement problem based on his understanding of eigenvalue equations.

Billy describes operators as acting on states/vectors and producing a new state/vectors.

> "Um, I would say an operator is what acts - so mathematically, an operator is what acts on some state, on some eigenvector. And you can either represent that eigenvector as either a wavefunction, if you aren't using the discrete case. And, the way I've been thinking about it is, when you ah, like the classical example is, you're given some operator, like some angular momentum operator, and then you see how that acts on your state and you see what comes out of that state. And so with the spins, it's kinda like, you have some like projection operator and you have some state and you project onto that state with your operator. And then you see what comes out of that operator. So, kinda like the Stern-Gerlach experiment, where you have a spin up and then you send it through some operator, either some mixed state or not, then you see what comes out of that operator, or out of that, basically out of that projection. And so, if I were to describe it to a friend in mathematics, so I'm both a math and physics major. So to a math major I'd be like 'Oh, linear algebra, matrix acting on a vector, you're going to get some new vector.' Physics major, it's like 'Ok, you have some state which you can do all these calculations with, and you project it' and I'd talk about the math with that. 'Project it onto another state and you can do some more calculations with that. And in quantum mechanics, you happen to just take probabilities of stuff.'

He mentions projections several times in this statement, and it's clear that he understands measurements to be closely associated with projections. Projection operators are among the first operators he mentions as examples he's used in quantum mechanics, and using a projection operator is the first example he gives of a

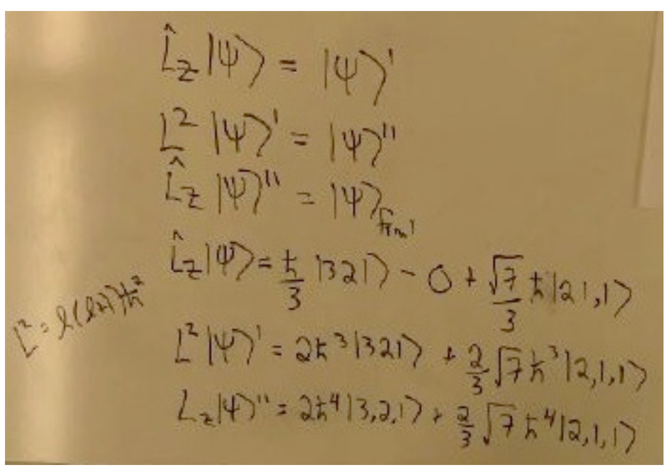



computation in quantum mechanics that involves operators.

When he begin thinking about the Sequential Hydrogen Measurement problem, the first thing he wants to do is to see what the Lz operator does to the initial state.

> "So, I would do, I'd first see how L_z acts on Psi. And then you'd get some new state, essentially, and I this is -iħ ∂/∂ϕ. And, then, I guess the way I'd first do it, because I'm not exactly sure how it looks in just ket notation, is I'd do the long route, when you actually have to do the derivatives of the continuous form. From there you get some state."

He confirms that the $\Psi$ he's talking about is the initial state of the particle. He writes on his board $L_z$ acting on $\Psi$ to yield a new state, $\Psi'$. Then, Billy has takes this new state and let's $L^2$ act on it to yield a second new state, $\Psi''$. Finally, he let's $L_z$ act on $\Psi''$ to yield $\Psi_{final}$. He describes this sequence of operators as the sequence of measurements. He is unable to proceed with his calculation until the interviewer reminds him of the eigenstates and eigenvalues of $L_z$. He then performs the calculation, carrying the




eigenvalues through each transformation so that he ends up with an $\hbar^4$ in his final state.

He comments that this is weird. "And from here, we just get more ħ's. Something's weird. Well, I mean, granted we never actually did like, oh, do it, you know, go one after the other in our actual courses, but I'm not used to seeing $\hbar^4$ kind of thing." He's troubled by these factors in his final state and comments that while this seems unfamiliar, it may just be unfamiliar because he hasn't done a similar repeated measurement calculation like this before.

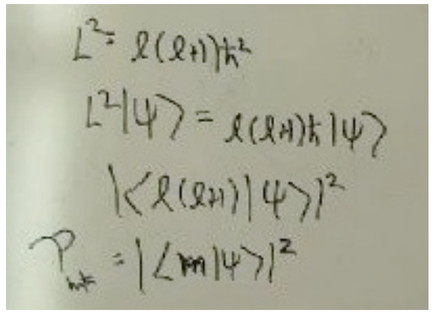
At this point he says "My first, actually, my first thought before I even started looking at the problem was, oh, if we prepare it in this state, kinda like projections, how like when you prepare a state and you project it into the plus state, you get that plus state out. Oh, we'll get this state and then something weird will happen here. But, actually doing it all the way through, um," It seems here that he initially considered doing a projections approach, but his phrase "doing it all the way through" may be indicating that this operator approach may be a more careful or rigorous way to proceed.

Interestingly, when asked what the state of the particle would be after the first measurement $L_z$, he says that the state would be your m value times ħ times your state back. This appears that this is consistent, in his mind, with the computation he just performed by having the $L_z$ operator act on the initial state $\Psi$. Billy indicates that in this initial state, the m is 1,0, and 1 and so you just get ħ $|\Psi>$ back. The interviewer asks what happens to the m=0, and Billy says, "Mathematically, it's zero, if we just use the definition. But now I'm thinking it seems like it'd be weird if it was, um, if you could have another state that was not one, but so you'd have like, ah, you know 1 ħ, 1 ħ, with, I don't know, I don't know if it's, I don't know if |3,2,2> is actually allowed or if it's prohibited. I think it is. Then you'd have 2ħ. And that seems kind of wild, because it seems like your angular momentum should be a discrete value." Billy doesn't resolve this issue in the interview. "But, I might be thinking about that wrong. Ok, so, if I was just doing the problem out, this is how I would do it though, like in the homework. The first step." In this discussion, seems to be thinking that his calculation is deterministic - having the operator act on the state should tell you which eigenvalue is going to be measured. When he imagines a different case that would result in different (non-zero) eigenvalues showing up in different terms, Billy seems troubled by this.

The interviewer then asks Billy to calculate the probability of measuring $L^2$ to be $2ħ^2$ if the $L^2$ measurement was done first. Billy tries to calculate this probability by taking the inner product of the initial state with a ket that is labeled with $l(l+1)$ and taking the norm squared. He generally describes the process that would happen (a lot of cancellation due to orthogonality) but then he admits that he cannot do this calculation because he can't remember which state to do the inner product with the initial state.

> I'm pretty sure you just take your whatever your $l$ value...(mumbles) either it's your $l$ times $l$ +1 and then it's that projected onto this. Like that. Or, it's just $l$. 'Cause I remember with the $L_z$ calculation, you would just do, m like that, and that would be your probability of mħ...Um, and essentially what mathematical operation is you operate, you project your, this, ah, your, essentially your eigenvalue state that's given by your particular mħ, I guess, by your $L_z$ or your $L^2$. You project it onto your state and then you're going to get a lot of cancellation. Like, if this was, back to the regular old spins case, if it was, if you have some basis vector "plus" and you

operated that on your state and that just had one plus, then those would just collapse, you'd just get, basically it's an inner product. Then you, it's the magnitude of the inner product...So, it's like, I'm not positive how it actually, like in principle this is the method you do, but I can't remember what, if it's, like what state you put here. Because putting a number here




just seems wrong to me.

When considering the statement "A acting on Psi is not a statement about the measurement of A", Billy tentatively disagrees with this statement. "I would say that is False - it may not, it's not going to give you an observable, necessarily, but when you measure something you are acting on it. Whenever you do an experiment you act on that state which changes the state, which is why quantum mechanics is weird. I would say this is false, but, you don't get any, like ah, the energy value or your eigen...or angular momentum value." Here, Billy is anchoring his reasoning on the fact that when you making a measurement in quantum mechanics, the state changes - "you're acting on it". He also states that "it", the operator acting on $\Psi$, is not going to yield an eigenvalue, which he refers to as an "observable".

When considering the statement "The operator $\hat{A}$ acting on the wavefunction $\Psi$ is: $\hat{A}\Psi = \Psi'$", Billy interprets this statement as saying,

"To me it's saying that like if you have $L_z$ and you operate on to your state vector, your given state vector for instance $L_z$ is the operator, then you'd get whatever that $L_z$ pulls out, which is some eigenvalue times that state back. So actually. And I guess the way I'm interpreting the $\Psi'$ is that, it's really the same state but with some new constants in front now. Although, in good old linear algebra that would, that could drastically, you wouldn't necessarily just get that back. Like I, the way we've been viewing these is just the eigenvalue equation. That's how we've always been, that's how we've been interpreting all, whenever we do ah, operator acting on some wave function or a state vector. So, how we, the way that looks you just get, you have your state and the states remain the same but now you have an eigenvalue multiplied by your state. So I would say, it's not like, it's not like your whole state is drastically changed but now there's just some scalar multiple of that state... Yes, although I would [agree with this statement], like, like $\Psi'$, I wouldn't call it like some drastically new wavefunction. I would say it's a wavefunction but then with some scalars and whatnot. Um, I think, cause then I'm thinking now there's some operators that does do derivatives of your wavefunction which would be kinda different."

This discussion reveals that Billy is using the eigenvalue equation in his reasoning about measurements. As soon as he mentions it's the same state with constants out in front, he starts to consider whether that really is the same state. He refers to "good old linear algebra," trying to think what the pure math is telling him and realizes that that change could be "drastic". Then he reverts to thinking about the interpretation of the eigenvalue equation. He is clearly struggling in his own mind with how much the state has changes.

When considering the statement "When an operator $\hat{A}$ corresponding to an observable A acts on a wavefunction $\Psi$, it corresponds to a measurement of that observable", Billy emphatically agrees with this statement.

"I believe this is true. (Reading) 'When an operator A corresponding to a physical observer (sic) A , acts on a wavefunction, corresponds to a measurement of that observable.' That's like um again its back to the angular momentum case, where you operate with some operator L_z. You get your eigenvalue which is your observable. And I would say that when you operate on some wavefunction, then you are "measuring" (makes air quotations) and you get some observable out. So I would agree with this."