SELF-SIMILARITY IN LEVEL SET TREES OF GEOMETRIC RANDOM WALKS

ANNA-SOPHIA HIRST, RACHEL LINDER, AND DANIEL MALMUTH

ADVISOR: YEVGENIY KOVCHEGOV
OREGON STATE UNIVERSITY

ABSTRACT. Level set trees provide insight into the topology of a function's relative extrema. We consider a random walk where the displacement between successive states is determined by a mix of geometric variables, and calculate how the parameters of the transition kernel evolve under the pruning operation. We find that the level set tree of the geometric random walk does not have Horton or Tokunaga self-symmetry, but does have asymptotic Horton self-symmetry.

1. INTRODUCTION

There has been substantial work into studying and understanding self-similarity of random walks that produce binary trees, and Horton and Tokunaga laws provide a convenient way to do just that. However, there is no comparable work done for understanding self-similarity in non-binary trees. Within this paper we explore the self-similarity of discrete random walks. The major motivation for this summer's project was a paper coauthored by Dr. Kovchegov. The major result was a proof that the level set tree of a finite symmetric homogeneous Markov chain has Horton self similarity with exponent R = 4, and Tokunaga self-similarity with parameters (a, c) = (1, 2). Based upon this previous work we decided to look at random walk excursions generated by discrete random walks rather than continuous random walks, which left us to deal with n-ary trees because of the discrete state space. This is due to the fact that in a continuous state space the probability of returning to a previous state is 0. It is clear that this is not the case in a discrete state space in which there is a chance of returning to a previous state or staying at the current state.

2. TERMINOLOGY

We introduce here Horton-Strahler ordering, Horton laws, Tokunaga self-similarity, trees, level set trees, and pruning.

2.1. **Trees and Hierarchical Ordering.** A *tree* T, is a connected simple non-cyclic graph with vertices, V and edges E, with T = (V,E) such that $N_E = N_V - 1$. In a *rooted tree*, one node is designated as the root, which gives the tree a direction of edges, as well as creating a parent-child relationship within the tree. If $v = \langle i_1, ..., i_n \rangle \in T$ then $u = \langle i_1, ..., i_{n-1} \rangle \in T$ is called the child of v, and v is the parent of u. A leaf is a vertex that has no children. It is possible to represent the

Date: 12 August 2016.

This work was done during the Summer 2016 REU program in Mathematics at Oregon State University with support from National Science Foundation Grant DMS-1359173.

planar tree T (which is rooted at ϕ) by a bijection between the set of vertices V and the set of finite integer-valued sequences $\langle i_1,...,i_n\rangle \in$ T, such that

(1) $\phi = \langle \emptyset \rangle$,
(2) if $\langle i_1,...,i_n\rangle \in$ T then $\langle i_1,...,i_k\rangle \in$ T $\quad \forall 1 \geq k \geq n$ and
(3) if $\langle i_1,...,i_n\rangle \in$ T then $\langle i_1,...,i_{n-1},j\rangle \in$ T $\quad \forall 1 \geq j \geq i_n$.

The *Horton-Strahler ordering* of the vertices of a finite, rooted labeled tree is performed in the following order, from leaves to the root: (i) each leaf has order r(leaf) =1; (ii) the maximum order r of the children c_1, c_2,..., c_n of a parent vertex p is assigned to the parent vertex if the number of children with the maximum order of r is 1; (iii) when more than one child is of order r, then the parent vertex is assigned the order M+1. Figure 1 below illustrates this.

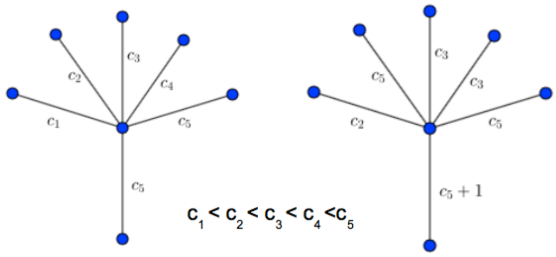

FIGURE 1. An example of Horton-Strahler ordering for a non-binary tree is represented.

Formally,

$$\text{Let M} = \max\{r(c_1),(c_2),...,(c_n)\} \quad \text{and let } \mu = \{c : r(c) = M\}$$

$$r(p) = \begin{cases} M & \text{if } |\mu| = 1 \\ M+1 & \text{if } |\mu| > 1 \end{cases}$$

An extension of Horton-Strahler ordering is Tokunaga indexing which focuses on side branching and can be useful when dealing with incomplete trees. A tree T is a *complete tree* if each branch of T has a single vertex, meaning that there is no side branching. All other trees are considered incomplete trees. Thus *incomplete trees* have *side branching* which is when branches of differing orders merge together. For instance, an order one branch merging into an order three branch. Side branching often takes place in natural hierarchies and Tokunaga indexing is well equipped to address this issue. Let $\tau^k_{ij}, 1 \leq k \leq N_j, 1 \leq i < j \leq \Omega$ be the number of branches with order i which join in the non-terminal vertices of the k-th branch of order j. Then $N_i j = \sum_k \tau^k_{ij}, j > i$ is the total number of such branches in a tree T. The *Tokunaga index* T_{ij} is the average number of branches of order $i < j$ per branch of order j in a finite tree of order $\Omega \geq j$:

$$T_{ij} = \frac{N_{ij}}{N_j}.$$

2.2. **Pruning and Self-Similarity.** *Self similar trees* are statistical structures that are preserved under the operation of pruning. *Pruning* of a finite, rooted tree T cuts its leaves (vertices of degree one) and their parental edges, and removes the resulting chains of degree two vertices and their parental edges, also called series reduction. The Horton-Strahler order k of a vertex v is the minimal number of prunings that are necessary in order to eliminate the sub-tree rooted at v.

A sequence of probability laws $\{P_N\}_{N\in\mathbb{N}}$ is said to have *well-defined asymptotic Horton-Strahler orders* for $k \in \mathbb{N}$, with random variables

$$\frac{N_k^{(P_N)}}{N} \to \mathcal{N}_k \quad \text{in probability as} \quad N \to \infty,$$

with the quantity $\mathcal{N}_{\Downarrow}$ is called the asymptotic ratio of the branches of order k.

2.3. **Horton and Tokunaga branching laws.** The Horton and Tokunaga branching laws allow for the studying of self-similarity in random trees. Horton laws acknowledge the principal branching that takes place in a tree. The *Horton law* states

$$\frac{N_r}{N_{r+1}} = R_B, \quad \frac{M_{r+1}}{M_r} = R_M, \quad R_B, R_M > 0, \quad r \geq 1,$$

such that N_r, M_r are the total number and average mass of branches of order r in a finite tree of order Ω.

Looking at a sequence $\{P_N\}_{N\in\mathbb{N}}$ of probability laws over binary trees with well-defined asymptotic Horton-Strahler orders will follow the Horton self-similarity law if and only if at least one of the following limits below exists and the limit is (i)finite and (ii) positive:

(1) root law: $\lim_{k\to\infty} \left(\mathcal{N}_k\right)^{-\frac{1}{k}} = R, \quad R > 0,$
(2) ratio law: $\lim_{k\to\infty} \mathcal{N}_k / \mathcal{N}_{k+1} = R, \quad R > 0,$
(3) geometric law: $\lim_{k\to\infty} \mathcal{N}_k * R^k = N_0, \quad N_0 > 0,$

with the constant R being the *Horton exponent*.

The *Tokunaga laws* allow for the studying of side branching within a tree. Within a deterministic setting, a tree T of order Ω is a self-similar tree if its side-branching structure (i) is the same for all branches of a given order:

$$\tau_{ij}^k =: \tau_{ij}, \quad 1 \leq k \leq N_j, \quad 1 \leq i < j \leq \Omega,$$

and (ii) is invariant with respect to the branch order:

$$\tau_{i(i+k)} \equiv T_{i(i+k)} =: T_k \quad for 2 \leq i+k \leq \Omega.$$

Tokunaga self-similar trees obey an additional constraint of

$$\frac{T_{k+1}}{T_k} = c \iff T_k = ac^{k-1} \quad a, c > 0, 1 \leq k \leq \Omega - 1.$$

In a random setting, a tree T of order Ω is self-similar if $\mathbf{E}(\tau_{i(i+k)}^j) =: T_k$ for $1 \leq j \leq N_i + k, 2 \leq i + k \leq \Omega$; and furthermore, it is Tokunaga self similar if the additional constraint above holds. [1]

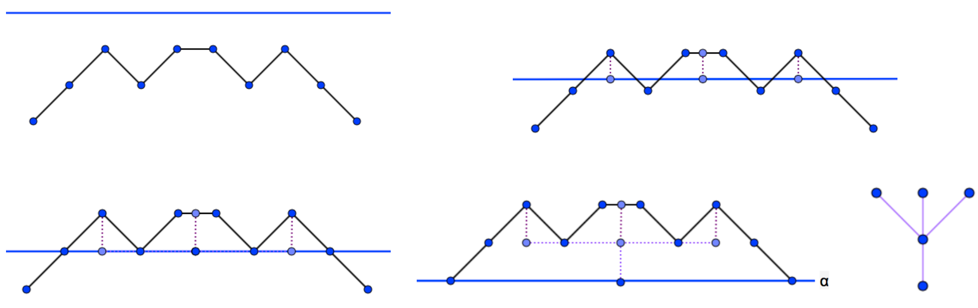

FIGURE 2. The level set tree representation of a continuous function captures the topology of its local extrema.

2.4. **Tree Representation of Functions.** A level set tree is a topological structure of the level sets of a time series or a real function and gives a visual representation of when the level set intervals merge together. [1] Starting with a water level of α that initially covers the entire excursion. Then allowing the water to start to recede, parts of the graph will slowly emerge, similar to islands in an ocean. As more water recedes the island eventually merge together. The point at which the islands merge together creates a vertex. This is illustrated below.

More formally, the level set $L_\alpha(X_t)$ is defined as the pre-image of the function values above α:

$$L_\alpha(X_t) = t : x \geq \alpha$$

3. RESULTS

This section presents the main theorems resulting from our work. Before proceeding, we shall formalize our treatment of the geometric random walk. For parameters $\{p_1, p_2, r_1, r_2\}$ such that $0 < p_1 + p_2 \leq 1$ and $0 < r_1, r_2 \leq 1$, we define the process so that (i) there is probability p_1 of an upward step where the step size is determined by a geometric variable with parameter r_1, (ii) there is probability p_2 of a downward step where the step size is determined by a geometric variable with parameter r_2, and (iii) there is probability $1 - p_1 - p_2$ of a flat step.

Definition 3.1. *A* geometric random walk*, more precisely called a* geometric homogeneous Markov chain (GHMC)*, is a probabilistic process whose transition kernel $K(x)$ is a weighted mix of geometric random variables. Specifically,*

$$K(x) = p_1 g_1(x) + (1 - p_1 - p_2)\delta_0 + p_2 g_2(-x),$$

where

$$g_i(x) = \begin{cases} r_i(1-r_i)^{x-1} & x = 1,2,3,\ldots \\ 0 & o.w. \end{cases} \qquad i = 1,2$$

and δ_0 is a degenerate random variable with support $\{0\}$.

Note that the time and state spaces for this process are both discrete. We fix the state $X_t = 0$ at time $t = 0$. The state is integer-valued, and the time is a nonnegative integer.

3.1. **Effect of Pruning on Geometric Random Walk.**

Theorem 3.2. *The local minima of a geometric homogenous Markov chain form a GHMC with parameters*

$$r_1^* = \frac{p_2 r_1}{p_1+p_2}$$
$$r_2^* = \frac{p_1 r_2}{p_1+p_2}$$
$$p_1^* = \frac{-p_1^2(p_1p_2r_1+p_1p_2r_2-p_1p_2r_1r_2+p_1^2r_2+p_2^2r_1+p_2r_1r_2-p_1r_2-p_2r_1-p_2r_2)}{(p_1+p_2)(p_1-1)(p_2-1)(p_1p_2r_1+p_1p_2r_2-p_1p_2r_1r_2+p_1^2r_2+p_2^2r_1)}$$
$$p_2^* = \frac{-p_2^2(p_1p_2r_1+p_1p_2r_2-p_1p_2r_1r_2+p_1^2r_2+p_2^2r_1+p_1r_1r_2-p_1r_2-p_2r_1-p_1r_1)}{(p_1+p_2)(p_1-1)(p_2-1)(p_1p_2r_1+p_1p_2r_2-p_1p_2r_1r_2+p_1^2r_2+p_2^2r_1)}$$

Proof. We calculate the characteristic function of the difference between successive local minima and set it equal to the characteristic function of the GHMC kernel. The difference between minima is

$$d_j = \sum_{i=1}^{\xi^+} Y_i - \sum_{i=1}^{\xi^-} Z_i.$$

The first sum represents upward or flat steps until the local maximum between the two minima is reached. The second sum represents downward or flat steps from the local maximum down to the following local minimum. Since the first sum includes upward and flat increments, ξ^+ is a geometric random variable with parameter p_2, which is the probability of a downward step. Similarly, ξ^- is a geometric variable with parameter p_1. The variables Y_i and Z_i are weighted mixes of a degenerate variable and a geometric variable. Specifically,

$$Y_i \sim \left(\frac{1-p_1-p_2}{1-p_2}\right)\delta_0 + \left(\frac{p_1}{1-p_2}\right)g_1(x)$$

$$Z_i \sim \left(\frac{1-p_1-p_2}{1-p_1}\right)\delta_0 + \left(\frac{p_2}{1-p_1}\right)g_2(x).$$

For Y_i, the first weight represents the probability that a step is flat given that it is not downward, and the second weight represents the probability that a step is upward given that it is not downward. The expression for Z_i is constructed in a similar fashion.

The characteristic function of d_j is

$$(1)\quad \left(\frac{p_2(p_1r_1+(1-p_1-p_2)(e^{-is}-1+r_1))}{(1-p_2)(e^{-is}-1+r_1-p_1r_1-(1-p_1-p_2)(e^{-is}-1+r_1))}\right)\times \left(\frac{p_1(p_2r_2+(1-p_1-p_2)(e^{is}-1+r_2))}{(1-p_1)(e^{is}-1+r_2-p_2r_2-(1-p_1-p_2)(e^{is}-1+r_2))}\right).$$

The characteristic function of the kernel $K^*(x)$ of the pruned walk is

$$(2)\quad \frac{p_1^*r_1^*(e^{is}-1+r_2^*)+(1-p_1^*-p_2^*)(e^{is}-1+r_2^*)(e^{-is}-1+r_1^*)+p_2^*r_2^*(e^{-is}-1+r_1^*)}{(e^{-is}-1+r_1^*)(e^{is}-1+r_2^*)}.$$

Setting expressions 1 and 2 equal and solving in Maple yields two solutions. One solution was not consistent with the restrictions on the parameters and was ignored, leaving the solution presented in theorem 3.2. These calculations are described in greater detail in Appendix A. □

Solutions to more specific cases of the geometric walk can be derived from theorem 3.2. By defining $p = p_1$ and $1-p = p_2$, we obtain the result of pruning a geometric walk that does not have plateaus.

Corollary 3.3. *The local minima of a GHMC with nonzero displacement between successive points form a GHMC with parameters*

$$\begin{aligned} r_1^* &= r_1(1-p), \\ r_2^* &= r_2 p, \\ p_1^* &= \frac{p^2 r_1 r_2 - p r_1 r_2 + p r_2}{p^2 r_1 r_2 - p r_1 r_2 - p r_1 + p r_2 + r_1}, \\ p_2^* &= \frac{p^2 r_1 r_2 - p r_1 r_2 - p r_1 + r_1}{p^2 r_1 r_2 - p r_1 r_2 - p r_1 + p r_2 + r_1}. \end{aligned}$$

We can also look at the case of a symmetric random walk, where we let $r = r_1 = r_2$ and $\alpha = 1 - p_1 - p_2 = 1 - 2p_1$.

Corollary 3.4. *The local minima of a symmetric GHMC with parameters $\{r, \alpha\}$ form a symmetric GHMC with parameters*

$$\alpha^* = \frac{\alpha^2(r-4) - r}{(\alpha+1)^2(r-4)}, \quad r^* = \frac{r}{2}.$$

Corollary 3.5. *The local minima of a symmetric GHMC with $\alpha = 0$ form a symmetric GHMC with*

$$\alpha^* = \frac{r}{4-r}, \quad r^* = \frac{r}{2}.$$

3.2. **Asymptotic Horton Self-Similarity.** We begin by establishing some lemmas that are relevant to the main theorem of this section.

Lemma 3.6. *For a random walk with probability p_1 of a step up and probability p_2 of a step down, the expected proportion of local maxima is $\frac{p_1 p_2}{p_1 + p_2}$. For a symmetric GHMC, this value simplifies to $(1-\alpha)/4$.*

Proof. The probability that a point in the random walk is a local maximum (or is the left edge of a plateau) is equivalent to the probability that a point is higher than its left neighbor, followed by 0 or more flat segments, and higher than the first point to its right that is not at the same level. This value is $p_1 p_2 (1 - p_1 - p_2)^n$, where n is the number of flat segments. Adding up over all possible n yields a geometric series with first term $p_1 p_2$ and ratio $1 - p_1 - p_2$, and its sum is

$$\frac{p_1 p_2}{1 - (1 - p_1 - p_2)} = \frac{p_1 p_2}{p_1 + p_2}.$$

Letting $p_1 = p_2 = (1-\alpha)/2$, which is the case for the symmetric GHMC, reduces this expression to $(1-\alpha)/4$. □

For the remainder of this section, the notation $\{\alpha_{(k)}, r_{(k)}\}$ shall denote the parameters of a symmetric GHMC after k prunings.

Lemma 3.7. *There is no symmetric GHMC where* $\alpha = \alpha_{(1)} = \alpha_{(2)}$. *Equivalently,* α *cannot stay fixed for more than one iteration of pruning.*

Proof. We attempt to find parameters $\{\alpha, r\}$ such that $\alpha_{(1)} = \alpha$. This is satisfied by $\{\alpha, r\}$ where

$$\begin{aligned}\alpha &= \frac{\alpha^2(r-4)-r}{(\alpha+1)^2(p-4)}\\ &\implies \alpha(\alpha+1)^2(r-4) = \alpha^2(r-1)-r\\ &\implies \alpha^3 r + 2\alpha^2 r + \alpha r - 4\alpha^3 - 8\alpha^2 - 4\alpha = \alpha^2 r - 4\alpha^2 - r\\ &\implies \alpha^3 r + \alpha^2 r + \alpha r - 4\alpha^3 - 4\alpha^2 - 4\alpha = -r\\ &\implies (\alpha^3+\alpha^2+\alpha)(r-4) = -r\\ &\implies \alpha^3+\alpha^2+\alpha = \frac{r}{4-r}.\end{aligned}$$

It follows that for $\alpha_{(2)} = \alpha_{(1)}$, $\{\alpha_{(1)}, r_{(1)}\}$ must satisfy the relationship

$$\alpha_{(1)}^3 + \alpha_{(1)}^2 + \alpha_{(1)} = \frac{r_{(1)}}{4-r_{(1)}}.$$

If we assume $\alpha = \alpha_{(1)}$ and use the fact that $r_{(1)} = r/2$, this relationship becomes

$$\alpha^3+\alpha^2+\alpha = \frac{r/2}{4-r/2} = \frac{r}{8-r}.$$

This implies that

$$\frac{r}{4-r} = \frac{r}{8-r},$$

which is a contradiction because r cannot be 0. Thus, $\alpha = \alpha_{(1)}$ precludes $\alpha_{(2)} = \alpha_{(1)}$, proving the lemma. □

Lemma 3.8. *For any symmetric GHMC, the parameters* $\{\alpha_{(k)}, r_{(k)}\}$ *both tend to zero as the number of prunings* k *tends to infinity.*

Proof. It is trivial to show that $r_{(k)} \to 0$ as $k \to \infty$, because $r_{(k+1)} = r_{(k)}/2$.

The expression for $\alpha_{(k+1)}$ in terms of $\alpha_{(k)}$ and $r_{(k)}$ is

$$\alpha_{(k+1)} = \frac{\alpha_{(k)}^2(r_{(k)}-4)-r_{(k)}}{(\alpha_{(k)}+1)^2(r_{(k)}-4)} = \left(\frac{\alpha_{(k)}}{\alpha_{(k)}+1}\right)^2 - \frac{r_{(k)}}{(\alpha_{(k)}+1)^2(r_{(k)}-4)}$$

Since $r_{(k)} \to 0$ and neither term in the denominator can get arbitrarily close to 0, we have

$$\alpha_{(k+1)} \approx \left(\frac{\alpha_{(k)}}{\alpha_{(k)}+1}\right)^2$$

for large k. Now assume $\alpha_{(k+1)} > \alpha_{(k)}/2$. Then

$$\begin{aligned}
\left(\frac{\alpha_{(k)}}{\alpha_{(k)}+1}\right)^2 &> \alpha_{(k)}/2 \\
&\implies 2\alpha_{(k)}^2 > \alpha_{(k)}(\alpha_{(k)}+1)^2 \\
&\implies 2\alpha_{(k)} > (\alpha_{(k)}+1)^2 \\
&\implies 2\alpha_{(k)} > \alpha_{(k)}^2 + 2\alpha_{(k)} + 1 \\
&\implies 0 > \alpha_{(k)}^2 + 1,
\end{aligned}$$

which is impossible. So for sufficiently large k, $\alpha_{(k+1)} \leq \alpha_{(k)}/2$, meaning $\alpha_{(k)} \to 0$ as $k \to \infty$. □

Theorem 3.9. *The level set tree of a GHMC has ratio asymptotic Horton self-similarity with exponent 4. However, it does not have Horton self-similarity or Tokunaga self-similarity.*

Proof. Applying Lemma 3.6 in the context of a symmetric GHMC, we see that $N_{k+1}/N_k \to (1-\alpha_{(k)})/4$ as the tree's order $\Omega \to \infty$. Lemma 3.7 shows that $\alpha_{(k)}$ cannot be fixed for all k, meaning that the expected value of N_{k+1}/N_k cannot be the same for all k, which is required for Horton self-similarity. Zaliapin [2] proved that a tree with Tokunaga self-similarity must have Horton self-similarity, so the lack of Horton similarity here implies a lack of Tokunaga similarity. To show asymptotic self-similarity, we combine the results of Lemmas 3.6 and 3.8. Since $N_{k+1}/N_k \to (1-\alpha_{(k)})/4$ and $\alpha_{(k)} \to 0$, we conclude that $N_{k+1}/N_k \to 1/4$ as $k \to \infty$. □

The following definition and conjectures attempt to extend the result of theorem 3.9 to more general situations.

Definition 3.10. *A GHMC is* mean-zero *when the expected value of the kernel $K(x)$ is 0. Since the expected value of the GHMC kernel is $p_1/r_1 - p_2/r_2$, it is mean-zero precisely when $p_1/p_2 = r_1/r_2$.*

Conjecture 3.11. *A GHMC has ratio asymptotic Horton self-similarity with exponent 4 if and only if it is mean-zero.*

Conjecture 3.12. *A GHMC never has true Horton or Tokunaga self-similarity.*

4. Future Work

In the future, we hope to build upon this work by formally proving the conjectures and extending certain results. We would like to investigate random walks where the type of probability distribution is not known, in order to see what general conclusions can be made about its self-similarity properties. We also want to study side-branching statistics in GHMCs, as this paper focuses on principal branching statistics. We hope to find a methodology for labeling branches in non-binary trees, since side-branching and merging of multiple maxima orders can both occur at the same parental vertex. We plan to frame this methodology in a way that is consistent with our previous work and helps us come to stronger conclusions about the self-similarity.

APPENDIX A. CALCULATIONS RELATED TO THEOREM 3.2

This section explains the proof of theorem 3.2 in greater detail, particularly with regard to deriving the characteristic functions.

Some important facts needed for this section are the characteristic function of a geometric variable, probability generating function of a geometric variable, and the characteristic function of a degenerate random variable. In the following expressions, let A denote a geometric variable with parameter a, and B denote a degenerate variable with parameter b. The notation $\varphi_X(s)$ shall denote the characteristic function of a random variable X, and $G_X(z)$ shall represent the probability generating function of X.

$$\varphi_A(s) = \frac{a}{e^{-is} - (1-a)} \quad G_A(z) = \frac{za}{1 - z(1-a)} \quad \varphi_B(s) = e^{isb}$$

We start by deriving the kernel of the pruned GHMC:

$$\begin{aligned}
\widehat{K}^*(s) &= p_1^* \widehat{g_1^*}(s) + (1 - p_1^* - p_2^*)\widehat{\delta_0} + p_2^* \widehat{g_2^*}(-s) \\
&= p_1^* \left(\frac{r_1^*}{e^{-is} - (1 - r_1^*)} \right) + (1 - p_1^* - p_2^*) + p_2^* \left(\frac{r_2^*}{e^{is} - (1 - r_2^*)} \right) \\
&= \frac{p_1^* r_1^* (e^{is} - 1 + r_2^*) + (1 - p_1^* - p_2^*)(e^{is} - 1 + r_2^*)(e^{-is} - 1 + r_1^*) + p_2^* r_2^* (e^{-is} - 1 + r_1^*)}{(e^{-is} - 1 + r_1^*)(e^{is} - 1 + r_2^*)},
\end{aligned}$$

which is expression 2.

Deriving the characteristic function of d_j, the difference between successive local minima in a GHMC, is more complicated. Recall that

$$d_j = \sum_{i=1}^{\xi^+} Y_i - \sum_{i=1}^{\xi^-} Z_i = \sum_{i=1}^{\xi^+} Y_i + \left(-\sum_{i=1}^{\xi^-} Z_i \right)$$

where

$$\begin{aligned}
\xi^+ &\sim \mathrm{Geom}(p_2) \\
\xi^- &\sim \mathrm{Geom}(p_1) \\
Y_i &\sim \left(\frac{1 - p_1 - p_2}{1 - p_2} \right) \delta_0 + \left(\frac{p_1}{1 - p_2} \right) g_1(x) \\
Z_i &\sim \left(\frac{1 - p_1 - p_2}{1 - p_1} \right) \delta_0 + \left(\frac{p_2}{1 - p_1} \right) g_2(x).
\end{aligned}$$

The characteristic function of Y_i for any i is

$$\begin{aligned}
\varphi_Y(s) &= \left(\frac{1 - p_1 - p_2}{1 - p_2} \right) \widehat{\delta_0} + \left(\frac{p_1}{1 - p_2} \right) \widehat{g_1}(s) \\
&= \left(\frac{1 - p_1 - p_2}{1 - p_2} \right) + \left(\frac{p_1}{1 - p_2} \right) \left(\frac{r_1}{e^{-is} - 1 + r_1} \right). \qquad (3) \\
&= \frac{(1 - p_1 - p_2)(e^{-is} - 1 + r_1) + p_1 r_1}{(1 - p_2)(e^{-is} - 1 + r_1)}
\end{aligned}$$

The probability generating function of ξ^+ is

$$G_{\xi^+}(z) = \frac{zp_2}{1-z(1-p_2)} = \frac{p_2}{z^{-1}-(1-p_2)}. \tag{4}$$

The characteristic function of the sum $\sum_{i=1}^{\xi^+} Y_i$ is calculated by plugging in expression 3 for z in expression 4, which yields

$$\frac{p_2}{\frac{(1-p_2)(e^{-is}-1+r_1)}{(1-p_1-p_2)(e^{-is}-1+r_1)+p_1r_1}-(1-p_2)} = \frac{p_2(p_1r_1+(1-p_1-p_2)(e^{-is}-1+r_1))}{(1-p_2)(e^{-is}-1+r_1-p_1r_1-(1-p_1-p_2)(e^{-is}-1+r_1))}. \tag{5}$$

The derivation of the characteristic function for $\sum_{i=1}^{\xi^-} Z_i$ is very similar, and turns out to be the same as expression 5 except that p_1 and p_2 switch, as do r_1 and r_2. Multiplying this function by -1 changes the s variables to $-s$, so the characteristic function of $-\sum_{i=1}^{\xi^-} Z_i$ is

$$\frac{p_1(p_2r_2+(1-p_1-p_2)(e^{is}-1+r_2))}{(1-p_1)(e^{is}-1+r_2-p_2r_2-(1-p_1-p_2)(e^{is}-1+r_2))}. \tag{6}$$

Due to the independence of the two sums in d_j, the characteristic function of d_j is simply the product of expressions 5 and 6, which forms expression 1.

Expressions 1 and 2 are set equal to one another, and we wish to solve for the starred variables in terms of their unstarred counterparts. We cross multiply the expressions to remove the fractions, and subtract one side from the other to form a large expression set equal to zero:

$$\begin{aligned} &p_2(p_1r_1+(1-p_1-p_2)(e^{-is}-1+r_1))p_1(p_2r_2+(1-p_1-p_2)(e^{is}-1+r_2))\times \\ &(e^{-is}-1+r_1^*)(e^{is}-1+r_2^*)- \\ &(1-p_2)(e^{-is}-1+r_1-p_1r_1-(1-p_1-p_2)(e^{-is}-1+r_1))\times \\ &(1-p_1)(e^{is}-1+r_2-p_2r_2-(1-p_1-p_2)(e^{is}-1+r_2))\times \\ &(p_1^*r_1^*(e^{is}-1+r_2^*)+(1-p_1^*-p_2^*)(e^{is}-1+r_2^*)(e^{-is}-1+r_1^*)+p_2^*r_2^*(e^{-is}-1+r_1^*))=0. \end{aligned} \tag{7}$$

Expanding this equation and grouping terms by e^{is} produces an equation

$$Ae^{-2is}+Be^{-is}+C+De^{is}+Ee^{2is}=0,$$

where A,B,C,D,E are functions of $p_1,p_2,r_1,r_2,p_1^*,p_2^*,r_1^*,r_2^*$.

Solving the system $A=B=C=D=E=0$ produces two solutions:

$$\begin{cases} r_1^* = \frac{p_2r_1}{p_1+p_2} \\ r_2^* = \frac{p_1r_2}{p_1+p_2} \\ p_1^* = \frac{-p_1^2(p_1p_2r_1+p_1p_2r_2-p_1p_2r_1r_2+p_1^2r_2+p_2^2r_1+p_2r_1r_2-p_1r_2-p_2r_1-p_2r_2)}{(p_1+p_2)(p_1-1)(p_2-1)(p_1p_2r_1+p_1p_2r_2-p_1p_2r_1r_2+p_1^2r_2+p_2^2r_1)} \\ p_2^* = \frac{-p_2^2(p_1p_2r_1+p_1p_2r_2-p_1p_2r_1r_2+p_1^2r_2+p_2^2r_1+p_1r_1r_2-p_1r_2-p_2r_1-p_1r_1)}{(p_1+p_2)(p_1-1)(p_2-1)(p_1p_2r_1+p_1p_2r_2-p_1p_2r_1r_2+p_1^2r_2+p_2^2r_1)} \end{cases},$$

and

$$\begin{cases} r_1^* = \frac{p_1 r_2}{p_1 r_2 - p_1 - p_2} \\ r_2^* = \frac{p_2 r_1}{p_2 r_1 - p_1 - p_2} \\ p_1^* = \frac{-p_1^2(-p_1 p_2 r_1 r_2 + p_1^2 r_2 + p_1 p_2 r_1 + p_1 p_2 r_2 + p_2^2 r_1 + p_2 r_1 r_2 - p_1 r_2 - p_2 r_1 - p_2 r_2)}{(-p_1 p_2 r_1 r_2 + p_1^2 r_2 + p_1 p_2 r_1 + p_1 p_2 r_2 + p_2^2 r_1)(-p_2 r_1 + p_1 + p_2)(p_2 - 1)(p_1 - 1)} \\ p_2^* = \frac{p_2^2(-p_1 p_2 r_1 r_2 + p_1^2 r_2 + p_1 p_2 r_1 + p_1 p_2 r_2 + p_2^2 r_1 + p_1 r_1 r_2 - p_1 r_2 - p_2 r_1 - p_1 r_1)}{(-p_1 p_2 r_1 r_2 + p_1^2 r_2 + p_1 p_2 r_1 + p_1 p_2 r_2 + p_2^2 r_1)(p_1 r_2 - p_1 - p_2)(p_2 - 1)(p_1 - 1)} \end{cases} .$$

In the second "solution", it can be seen that plugging in valid $\{p_1, p_2, r_1, r_2\}$ leads to invalid $\{p_1^*, p_2^*, r_1^*, r_2^*\}$. As a particular example, the expression for r_1^* evaluates as negative for any valid input. The numerator is positive because it is the product of two positive parameters. The denominator is negative because due to the constraints on the terms, $p_1 r_2 < p_1$, implying $p_1 r_2 - p_1 - p_2 < 0$. Accordingly, we accept the first solution as the explanation of how the GHMC parameters are affected by pruning.

References

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California State, Dominguez Hills
E-mail address: ahirst1@toromail.csudh.edu

University of Maryland, Baltimore County
E-mail address: rlinder1@umbc.edu

Oregon State University
E-mail address: malmuthd@oregonstate.edu