A PROOF OF MAO'S CONJECTURES ON PARTITION RANK INEQUALITIES AND AN INVESTIGATION OF RANK DIFFERENCE FUNCTIONS

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ABSTRACT. Rank differences functions were studied by Atkin and Swinnerton-Dyer [3], who gave a combinatorial explanation for the famous Ramanujan congruences modulo 5 and 7. Recently, Mao proved several rank difference identities for the Dyson rank of the partition function modulo 10 [10], and for the M_2 rank for partitions without repeated odd parts modulo 6 and 10 [11]. Additionally, Mao proved a number of rank inequalities, leaving some to conjecture in each case. In this paper, we prove several of Mao's inequality conjectures involving rank difference functions and investigate Mao's method to prove rank difference functions for partition restrictions.

1. INTRODUCTION AND STATEMENT OF RESULTS

A *partition* of n is a non-increasing sequence of positive integers that sum to n, where each summand is called a *part*. The partition function $p(n)$ is defined to count the number of partitions of n. For example, we see that

$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1,$$

give all of the partitions of $n=4$, so $p(4)=5$. Ramanujan discovered and proved the following congruences involving the function $p(n)$:

$$p(5n+4) \equiv 0 \pmod 5$$
$$p(7n+5) \equiv 0 \pmod 7$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

In addition, Ahlgren and Boylan proved in 2003 that, in fact, $\ell = 5, 7,$ and 11 were the only primes for which $p(\ell n + d) \equiv 0 \pmod{\ell}$ is true [1].

The *rank* of a partition λ was defined by Dyson to be the largest part of λ, $l(\lambda)$, minus the number of parts, $n(\lambda)$. For example, if

$$\lambda = 5+4+4+4+2+1+1+1,$$

then we have

$$\text{Dyson rank}(\lambda) = 5 - 8 = -3.$$

Throughout, we let $N(s,m,n)$ denote the number of partitions of n with rank congruent to s modulo m.

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1.1. **Work of Atkin and Swinnerton-Dyer.** In response to a conjecture of Dyson, Atkin and Swinnerton-Dyer [3] found a number of elegant formulas, in terms of modular functions and generalized Lambert series, for the generating functions for rank differences of the form

$$N(r,\ell,\ell n+d)-N(s,\ell,\ell n+d)$$

for the cases $\ell=5$ and $\ell=7$. Their work showed that the rank provides a combinatorial explanation for the celebrated Ramanujan congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7. \end{aligned}$$

For example, they found that for numbers of the form $5n+4 \pmod 5$ and $7n+5 \pmod 7$, the rank difference functions are equal to 0. For example, they showed that for $\ell=5$,

$$\sum_{n\geq 0}\left(N(0,5,5n+4)-N(2,5,5n+4)\right)q^n=0.$$

Additionally, for other residues modulo 5 and 7, they proved identities for rank difference functions in terms of infinite products and modular forms, along with proving a number of inequalities involving the rank difference functions. For example, they proved that

$$\sum_{n\geq 0}\left(N(0,5,5n+1)-N(2,5,5n+1)\right)q^n=\frac{(q^{25};q^{25})_\infty}{(q^5;q^{25})_\infty(q^{20};q^{25})_\infty}.$$

In order to understand their results (along with many results and ideas discussed in this paper), we must first introduce some notation that we will utilize throughout. We define the following q-hypergeometric series notation for the q-rising factorial. For any $x_i\in\mathbb{C}$, define

$$\begin{aligned} (x_1,x_2,\ldots,x_k;q)_m &:= \prod_{n=0}^{m-1}(1-x_1q^n)(1-x_2q^n)\cdots(1-x_jq^n), \\ (x_1,x_2,\ldots,x_k;q)_\infty &:= \prod_{n=0}^{\infty}(1-x_1q^n)(1-x_2q^n)\cdots(1-x_jq^n). \end{aligned}$$

1.2. **Overpartition Setting.** Lovejoy and Osburn then took the work of Atkin and Swinnerton-Dyer and translated it to the setting of overpartitions. An *overpartition* of n is a non-increasing sequence of positive integers that sum to n in which the first occurrence of each part may or may not be overlined. Thus, the overlined parts form a partition into distinct parts, while the non-overlined parts form an unrestricted partition. Therefore, an overpartition can be interpreted as a pair of partitions, one unrestricted and one into distinct parts, which sum to n.

For example, consider the overpartition

$$\lambda=\overline{6}+5+5+\overline{4}+4+\overline{3}+2+2+\overline{1}.$$

Then we have the pair of partitions

$$\{(\overline{6}+\overline{4}+\overline{3}+\overline{1}),(5+5+4+2+2)\}$$

which sum to $n=31$. Conveniently, the Dyson rank of a partition generalizes naturally to overpartitions. The Dyson rank of an overpartition λ is defined to be $l(\lambda)-n(\lambda)$.

Let $\overline{N}(s,m,n)$ denote the number of overpartitions of n with rank congruent to s modulo m. In [7], Lovejoy and Osburn found formulas for the generating functions for rank differences of the form $\overline{N}(r,\ell,\ell n+d)-\overline{N}(r,\ell,\ell n+d)$ for the cases $\ell=3$ and $\ell=5$. For example, Lovejoy and Osburn proved that

$$\sum_{n\geq 0}(\overline{N}(0,3,3n)-\overline{N}(1,3,3n))=-1+\frac{(q^3;q^3)_\infty^2(-q;q)_\infty}{(q;q)_\infty(-q^3;q^3)_\infty^2}.$$

In addition to the Dyson rank, we define the M_2 rank of an overpartition λ to be

$$\overline{M_2}(\lambda):=\left\lceil\frac{l(\lambda)}{2}\right\rceil-n(\lambda)+n(\lambda_0)-\chi(\lambda),$$

where λ_0 is the subpartition of λ consisting of the odd non-overlined parts of λ, and $\chi(\lambda)=1$ if the largest part of λ is odd and non-overlined and $\chi(\lambda)=0$ otherwise. For example, if λ is the overpartition of $n=15$ defined by

$$\lambda=6+\overline{3}+3+\overline{2}+1,$$

then

$$\overline{M_2}(\lambda)=\left\lceil\frac{6}{2}\right\rceil-5+2+0=0.$$

Let $\overline{N}_2(s,m,n)$ denote the number of overpartitions of n with M_2 rank congruent to s modulo m. In [9], Lovejoy and Osburn found formulas for the generating functions for rank differences of the form $\overline{N}_2(r,\ell,\ell n+d)-\overline{N}(r,\ell,\ell n+d)$ for the cases $\ell=3$ and $\ell=5$. For example, Lovejoy and Osburn proved that

$$\sum_{n\geq 0}(\overline{N_2}(0,3,3n+1)-\overline{N_2}(1,3,3n+1))=\frac{2(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q;q)_\infty}.$$

1.3. **Work of Mao.** More recently, Mao [10] modified the methods of Lovejoy and Osburn to prove identities for rank difference functions modulo 10 in the unrestricted partition setting. Before looking at some of Mao's results, we need to again introduce some notation. For $a,b\in\mathbb{Q}$, define

$$J_b:=(q^b;q^b)_\infty$$
$$J_{a,b}:=(q^a,q^{b-a},q^b;q^b)_\infty.$$

In the unrestricted partition setting, Mao found similar infinite products and modular forms for rank differences modulo 10, which he states in the following theorem.

Theorem 1.1. [10] *We have*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\big(N(0,10,n)+N(1,10,n)-N(4,10,n)-N(5,10,n)\big)q^n\\
&=\left(\frac{J_{25}J_{20,50}^2J_{50}^5}{J_{15,50}^3J_{10,40}^4}+\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2+5}}{1+q^{25n+5}}\right)\\
&\quad+q\left(\frac{J_{25}J_{50}^5}{J_{5,50}J_{10,50}^2J_{15,50}^2}\right)+q^2\left(\frac{J_{25}J_{50}^5}{J_{15,50}J_{5,50}^2J_{20,50}^2}\right)\\
&\quad+q^3\left(\frac{J_{25}J_{10,50}^2J_{50}^5}{J_{5,50}^3J_{20,50}^4}-\frac{1}{J_{25}}\sum_{-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2+5}}{1+q^{25n+10}}\right)\\
&\quad+q^4\left(\frac{2J_{50}^6}{J_{25}J_{5,50}J_{10,50}J_{15,50}J_{20,50}}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty}\big(N(1,10,n)+N(2,10,n)-N(3,10,n)-N(4,10,n)\big)q^n\\
&=\left(\frac{2q^5J_{50}^6}{J_{25}J_{10,50}^2J_{15,50}^2}-\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2+5}}{1+q^{25n+5}}\right)\\
&\quad+q\left(\frac{2q^5J_{50}^6}{J_{25}J_{5,50}J_{15,50}J_{20,50}^2}\right)+q^2\left(\frac{J_{25}J_{20,50}J_{50}^5}{J_{10,50}^3J_{15,50}^3}\right)\\
&\quad+q^3\left(\frac{J_{25}J_{50}^5}{J_{5,50}J_{10,50}J_{20,50}J_{15,50}^2}\right)\\
&\quad+q^4\left(\frac{J_{25}J_{25,50}J_{20,50}^2J_{50}^5}{2q^5J_{10,50}^4J_{15,50}^4}-\frac{1}{q^5J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(75n^2+25n)/2}}{1+q^{25n}}\right).
\end{aligned}$$

Additionally, Mao gave the following conjectures in [10].

Conjecture 1.2. *Computational evidence suggests that*

$$N(0,10,5n)+N(1,10,5n)>N(4,10,5n)+N(5,10,5n)\ \textit{for}\ n\geq 0, \tag{1}$$

$$N(1,10,5n)+N(2,10,5n)\geq N(3,10,5n)+N(4,10,5n)\ \textit{for}\ n\geq 1. \tag{2}$$

Furthermore, Mao studied rank differences for a different rank definition for partitions with distinct odd parts, which are also of combinatorial interest. The M_2 rank of a partition λ with distinct odd parts is defined by

$$M_2(\lambda)=\left\lceil\frac{l(\lambda)}{2}\right\rceil-n(\lambda),$$

where again $l(\lambda)$ is the largest part of λ and $n(\lambda)$ is the number of parts of λ. Let $N_2(s,m,n)$ denote the number of partitions with distinct odd parts with M_2 rank congruent to s modulo m. In [8], Lovejoy and Osburn found formulas for the generating functions for rank differences of the form $N_2(r,\ell,\ell n+d)-N_2(s,\ell,\ell n+d)$ for the cases $\ell=3$ and $\ell=5$. For example, Lovejoy and Osburn proved that

$$N_2(0,3,3n+2)-N_2(1,3,3n+2)=\frac{(q^3;q^3)_\infty(-q^6;q^6)_\infty}{(q,q^5;q^6)_\infty(q^4,q^8;q^12)_\infty}.$$

Mao [11] proved formulas for M_2 rank differences modulo 6 and 10. For example, Mao proved the following theorem for the modulo 6 setting.

Theorem 1.3. [10] *For* $\ell=6$*,*

$$\begin{aligned}&\sum_{n=0}^{\infty}\big(N_2(0,6,n)+N_2(1,6,n)-N_2(2,6,n)-N_2(3,6,n)\big)q^n\\ &=\left(\frac{1}{J_{9,36}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{18n^2+9n}}{1-q^{18n+3}}\right)+q\left(\frac{J_{18,36}J_{6,36}^2J_{36}^3}{J_{9,36}J_{3,36}^2J_{15,36}^2}\right)\\ &\quad+q^2\left(\frac{J_{6,36}J_{18,36}J_{36}^3}{2q^3J_{9,36}J_{3,36}^2J_{15,36}^2}-\frac{1}{q^3J_{9,36}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{18n^2+9n}}{1+q^{18n}}\right).\end{aligned}$$

Moreover, Mao proved the following in the modulo 10 setting.

Theorem 1.4. [10] *For* $\ell=10$*,*

$$\begin{aligned}&\sum_{n=0}^{\infty}\big(N_2(0,10,n)+N_2(1,10,n)-N_2(4,10,n)-N_2(5,10,n)\big)q^n\\ &=\frac{2q^5J_{10,100}J_{50,100}J_{100}^{15}}{J_{30,100}J_{15,100}^2J_{20,100}^2J_{35,100}^2J_{5,100}^3J_{25,100}^3J_{45,100}^3}+\frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{50n^2+25n}}{1+q^{50n+10}}\\ &\quad+q\frac{J_{20,100}J_{50,100}J_{30,100}^2J_{100}^{15}}{J_{25,100}J_{5,100}^2J_{10,100}^2J_{45,100}^2J_{40,100}^3J_{15,100}^4J_{35,100}^4}\\ &\quad+q^2\frac{J_{50,100}J_{100}^{15}}{J_{20,100}J_{25,100}J_{40,100}J_{5,100}^3J_{15,100}^3J_{35,100}^3J_{45,100}^3}\\ &\quad+q^3\frac{J_{40,100}J_{50,100}J_{10,100}^2J_{100}^{15}}{J_{25,100}J_{15,100}^2J_{30,100}^2J_{35,100}^2J_{20,100}^3J_{5,100}^4J_{45,100}^4}\\ &\quad+q^4\left(\frac{2J_{30,100}J_{50,100}J_{100}^{15}}{J_{10,100}J_{5,100}^2J_{40,100}^2J_{45,100}^2J_{15,100}^3J_{25,100}^3J_{35,100}^3}+\frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{50n^2+75n+20}}{1+q^{50n+30}}\right),\end{aligned}$$

also

$$\begin{aligned}
&\sum_{n=0}^{\infty}\big(N_2(1,10,n)+N_2(2,10,n)-N_2(3,10,n)-N_2(4,10,n)\big)q^n\\
=&\frac{J_{30,100}J_{50,100}J_{100}^{15}}{J_{10,100}J_{25,100}J_{40,100}^2J_{5,100}^3J_{15,100}^3J_{35,100}^3J_{45,100}^3}\\
&-2q^5\frac{J_{10,100}J_{50,100}J_{100}^{15}}{J_{30,100}J_{15,100}^2J_{20,100}^2J_{35,100}^2J_{5,100}^3J_{25,100}^3J_{45,100}^3}-\frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{50n^2+25n}}{1+q^{50n+10}}\\
&+q\frac{2q^5J_{50,100}J_{100}^{15}}{J_{20,100}J_{40,100}J_{5,100}^2J_{45,100}^2J_{15,100}^3J_{25,100}^3J_{35,100}^3}\\
&+q^2\Big(\frac{2q^5J_{40,100}J_{50,100}J_{10,100}^2J_{100}^{15}}{J_{15,200}^2J_{30,100}^2J_{35,100}^2J_{5,100}^3J_{20,100}^3J_{25,100}^3J_{45,100}^3}\\
&+\frac{J_{20,100}J_{30,100}J_{40,100}J_{10,100}^2J_{15,100}^2J_{35,100}^2J_{50,100}^2}{2q^5J_{25,100}J_{100}^9}+\frac{1}{q^5J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{50n^2+25n}}{1+q^{50n}}\Big)\\
&+q^3\frac{J_{30,100}J_{50,100}J_{100}^{15}}{J_{10,100}J_{25,100}J_{5,100}^2J_{40,100}^2J_{45,100}^2J_{15,100}^4J_{35,100}^4}\\
&+q^4\frac{J_{10,100}J_{50,100}J_{100}^{15}}{J_{25,100}J_{30,100}J_{20,100}^2J_{5,100}^3J_{15,100}^3J_{35,100}^3J_{45,100}^3}.
\end{aligned}$$

Additionally, Mao made the following conjectures.

Conjecture 1.5. *Computational evidence suggests that*

$$N_2(0,6,3n+2)+N_2(1,6,5n)>N_2(2,6,3n+2)+N_2(3,6,3n+2)\ \textit{for}\ n\geq 0, \tag{3}$$

$$N_2(0,10,5n)+N_2(1,10,5n)>N_2(4,10,5n)+N_2(5,10,5n)\ \textit{for}\ n\geq 0, \tag{4}$$

$$N_2(0,10,5n+4)+N_2(1,10,5n+4)>N_2(4,10,5n+4)+N_2(5,10,5n+4)\ \textit{for}\ n\geq 0. \tag{5}$$

$$N_2(1,10,5n)+N_2(2,10,5n)>N_2(3,10,)+N_2(4,10,5n)\ \textit{for}\ n\geq 1. \tag{6}$$

$$N_2(1,10,5n+2)+N_2(2,10,5n+2)>N_2(3,10,5n+2)+N_2(4,10,5n+2)\ \textit{for}\ n\geq 1. \tag{7}$$

1.4. **Results.** In this paper we prove the following theorem:

Theorem 1.6. *Mao's conjectures* (1), (2), (4), *and* (5) *are true.*

Additionally, we investigate the following rank difference identity. As in [11], we define for a positive integer t,

$$L_t(q^2):=\sum_{n=0}^{\infty}\frac{q^{2n^2}(-q^2;q^4)_n}{(\zeta_tq^4,\zeta^{-1}q^4;q^4)_n}. \tag{8}$$

Observe that taking $t = 6$ in (8), we define $N_4(m,n)$ by

$$\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty} N_4(m,n)z^m q^n = L_6(q^2) = \frac{(-q^2;q^4)_\infty}{(q^4;q^4)_\infty}\left[\sum_{n\in\mathbb{Z}}\frac{(-1)^n(1-z^{-1})(1-z)q^{4n^2+2n}}{(1-zq^{4n})(1-z^{-1}q^{4n})}\right] \tag{9}$$

where $N_4(m,n)$ counts the number of partitions of n that contain even parts occurring with even multiplicity with a type of rank equal to m. We have not explicitly determined this rank,

Theorem 1.7. *We have the following identity:*

$$\begin{aligned}\sum_{n\geq 0}(N_4(0,6,n)+N_4(1,6,n)-N_4(2,6,n)-N_4(3,6,n))q^n &= L_6(q^2)\\ &= \frac{1}{J_{18,72}}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{36n^2+18n}}{1+q^{36n+6}} + q^2\frac{J_{12,72}^2 J_{36,72} J_{72}^3}{J_{6,72}^2 J_{18,72} J_{30,72}^2}\\ &\quad + \frac{J_{12,72}J_{36,72}^2 J_{72}^3}{2q^2 J_{6,72}^2 J_{18,72} J_{30,72}^2} - \frac{1}{J_{18,72}}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{36n^2+18n-2}}{1+q^{36n}}.\end{aligned}$$

This paper is organized in the following manner. First, in Section 2, we gather some definitions, notation, and lemmas that we will utilize in order to prove our results. Then, in Section 3, we prove Theorem 1.6. In Section 4, we review Mao's method of proof in [11] to prove Theorem 1.7. Finally, we conclude in Section 5 with some observations and ideas for future research.

2. Preliminaries

Before moving into the proofs of Theorems 1.6 and 1.7, we begin with some important lemmas that will be utilized later.

Define

$$\begin{aligned} V_{2,0} &:= \frac{[q^6,q^{12},q^{12};q^{36}]_\infty (q^{36};q^{36})_\infty^2}{[-1,-q^{12},-q^{12,}-q^6;q^{36}]_\infty} = \frac{J_{6,72}^2 J_{12,72}^3 J_{24,72}^2 J_{30,72}^2 J_{36,72}^2}{2J_{72}^9}\\ V_{2,1} &:= \frac{[q^18,q^{12},q^{12};q^{36}]_\infty (q^{36};q^{36})_\infty^2}{q^2[-1,-q^12,-q^12,-q^6;q^{36}]_\infty} = \frac{J_{6,72}J_{12,72}^3 J_{18,72}^2 J_{24,72}^2 J_{30,72}J_{36,72}}{2q^2 J_{72}^9}.\end{aligned}$$

The following lemmas follow directly by substituting $q = q^2$ in the corresponding Lemmas 3.1 and 3.2, respectively, in [11].

Lemma 2.1. *We have*

$$\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{4n^2+2n}}{1+q^{12n}} = V_{2,0} - \frac{(q^4;q^4)_\infty}{(q^18,q^54,q^72;q^72)_\infty(-q^2;q^4)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{36n^2+54n+18}}{1+q^{36n+24}}$$

and

$$\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{4n^2+6n}}{1+q^{12n}} = V_{2,1} - \frac{(q^4;q^4)_\infty}{(q^18,q^54,q^72;q^72)_\infty(-q^2;q^4)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{36n^2+18n-2}}{1+q^{36n}}.$$

Furthermore,

$$V_{2,0}+V_{2,1}=\frac{(q^4;q^4)_\infty}{(-q^2;q^4)_\infty}\cdot\left[\frac{J_{12,72}J_{36,72}^2J_{72}^3}{J_{6,72}J_{18,72}^3J_{30,72}}+q^2\frac{J_{12,72}^2J_{36,72}J_{72}^3}{J_{6,72}^2J_{18,72}J_{30,72}^2}+\frac{J_{12,72}J_{36,72}^2J_{72}^3}{2q^2J_{6,72}^2J_{18,72}J_{30,72}^2}\right]. \tag{10}$$

We will make use of the following lemma of Chan, which appears as Equation (2.1) in [4]. Here we use the usual notation

$$F(b_1,b_2,\dots,b_m)+idem(b_1,b_2,\dots,b_m):=F(b_1,b_2,\dots,b_m)+F(b_2,b_1,\dots,b_m)+\cdots+F(b_m,b_2,\dots,b_1).$$

Lemma 2.2. *We have*

$$\begin{aligned}&\frac{[a_1,\dots,a_r]_\infty(q;q)_\infty^2}{[b_1,\dots,b_s]_\infty}\\&=\frac{[a_1/b_1,\dots,a_r/b_1]_\infty(q;q)_\infty^2}{[b_2/b_1,\dots,b_s/b_1]_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^{s-r}q^{(s-r)n(n+1)/2}}{1-b_1q^n}\cdot\left(\frac{a_1\cdots a_rb_1^{s-r-1}}{b_2\cdots b_s}\right)^n\\&+idem(b_1;b_2,..,b_s).\end{aligned}$$

We will require the following lemma of Mao [10] regarding the nonnegativity of the coefficients of certain q-series. First, we define

$$L_{p,r}(q):=\sum_{n=0}^{\infty}b_{p,r}(n)q^n:=\frac{(q^p;q^p)_\infty}{(q^r;q^p)_\infty(q^{p-r};q^p)_\infty}$$

and

$$L_{p,r}(q)+q^p:=\sum_{n=0}^{\infty}c_{p,r}(n)q^n:=\sum_0+\sum_1+\cdots+\sum_{r-1},$$

where

$$\sum_i=\sum_{n=0}^{\infty}c_{p,r}(nr+i)q^{nr+i}.$$

Lemma 2.3. *If p and r are positive integers with $p\geq 2$ and $r<p$ and $L_{p,r}(q)$ is defined as above, then $b_{p,r}(n)\geq 0$ for all n. Moreover, for each i, the sequence $\{c_{p,r}(nr+i)\}_{n\geq 0}$ in $L_{p,r}(q)+q^p$ is non-decreasing.*

We will also require the following lemma regarding the nonnegativity of the coefficients of certain power series in q.

Lemma 2.4. *A formal power series of the form*

$$\sum_{n=a}^{\infty}\frac{q^{P(n)}}{1-q^{Q(n)}},$$

where $a\geq 0$ is an integer and $P(n)$ and $Q(n)$ are polynomials in n with integer coefficients, has strictly nonnegative q-series coeffiicients for $n\geq 0$.

Proof. Expanding $\frac{1}{1-q^{Q(n)}}$ as a geometric series, we find that the above sum is equal to

$$\sum_{n=a}^{\infty} \frac{q^{P(n)}}{1-q^{Q(n)}} = \sum_{n=a}^{\infty} q^{P(n)}(1+q^{Q(n)}+q^{2Q(n)}+\cdots).$$

The coefficients of the each term of the above sum are visibly nonnegative. $\square$

We also require the Jacobi triple product identity.

Proposition 2.5 ([2] Thm. 2.8)**.**

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = (-zq,-q/z,q^2;q^2)_\infty. \tag{11}$$

Now we record some identities involving infinite products.

2.1. **Infinite Product Identities.** We use the following lemma implicitly in our calculations.

Lemma 2.6. *For all $m \in \mathbb{Z}$, we have*

$$\frac{(q^{2m};q^{2m})_\infty}{(q^m;q^m)_\infty} = (-q^m;q^m)_\infty.$$

Proof. We factor each of the factors appearing in $(q^{2m};q^{2m})_\infty$ as a difference of two perfect squares to obtain:

$$\frac{(q^{2m};q^{2m})_\infty}{(q^m;q^m)_\infty} = \prod_{n=1}^{\infty} \frac{1-q^{2mn}}{1-q^{mn}} = \prod_{n=1}^{\infty} \frac{(1-q^{mn})(1+q^{mn})}{1-q^{mn}} = \prod_{n=1}^{\infty}(1+q^{mn}) = (-q^m;q^m)_\infty.$$

$\square$

We are now in a position to prove Theorems 1.6 and 1.7.

3. Proof of Theorem 1.6

Here we prove Theorem 1.6 by obtaining bounds on the size of the coefficients in the generating functions for rank differences of the form

$$N(s_1,m,\ell n+d)+N(s_1,m,\ell n+d)-N(t_1,m,\ell n+d)-N(t_1,m,\ell n+d).$$

Recall that Mao proved generating functions of the form

$$\begin{aligned}&\sum_{n=0}^{\infty}(N(s_1,m,n)+N(s_2,mn)-N(t_1,m,n)-N(t_2,m,n))\,q^n\\ &=F_0(q^\ell)+qF_1(q^\ell)+q^2F_2(q^\ell)+\cdots+q^{\ell-1}F_{\ell-1}(q^\ell),\end{aligned}$$

where each $F_i(q^\ell)$ is a power series in q^ℓ. Then the generating functions relevant to the proof of Theorem 1.6 are

$$\sum_{n=0}^{\infty}(N(s_1,m,\ell n+d)+N(s_2,m,\ell n+d)-N(t_1,m,\ell n+d)-N(t_2,m,\ell n+d))\,q^n = F_d(q),$$

since $q^dF_d(q^\ell)$ has nonzero coefficients only for powers of q congruent to d modulo ℓ.

3.1. **Proof of (1).** .

In order to prove (1), we show that

$$\frac{J_5 J_{4,10}^2 J_{10}^5}{J_{3,10}^3 J_{2,10}^4} + \frac{1}{J_5} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} = \frac{1}{J_5} \left(\frac{J_5^2 J_{4,10}^2 J_{10}^5}{J_{3,10}^3 J_{2,10}^4} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} \right)$$

has strictly positive q-series coefficients for $n \geq 0$. Since $\frac{1}{J_5}$ has all nonnegative coefficients and a constant term of 1, it suffices to show that the term inside the parentheses has all positive coefficients. We begin by examining the sum. We have

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} + \sum_{n=-\infty}^{-1} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{(15n^2-5n)/2}}{1+q^{5n-1}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}(1-q^{5n+1})}{1-q^{10n+2}} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{(15n^2-5n)/2}(1-q^{5n-1})}{1-q^{10n-2}}.
\end{aligned}$$

Splitting each of the two series above into two series according to the summation index n modulo 2, we find that

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} \\
&= \sum_{n=0}^{\infty} \frac{q^{(15(2n)^2+15(2n))/2+1}(1-q^{5(2n)+1})}{1-q^{10(2n)+2}} - \sum_{n=0}^{\infty} \frac{q^{(15(2n+1)^2+15(2n+1))/2+1}(1-q^{5(2n+1)+1})}{1-q^{10(2n+1)+2}} \\
&\quad + \sum_{n=1}^{\infty} \frac{q^{(15(2n)^2-5(2n))/2}(1-q^{5(2n)-1})}{1-q^{10(2n)-2}} - \sum_{n=1}^{\infty} \frac{q^{(15(2n+1)^2-5(2n+1))/2}(1-q^{5(2n+1)-1})}{1-q^{10(2n+1)-2}}.
\end{aligned}$$

By distributing across the factor in the numerator of each sum, we find that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} = S_1 + S_2 + S_3 + S_4 - T_1 - T_2 - T_3 - T_4,$$

where

$$S_1 := \sum_{n=0}^{\infty} \frac{q^{(15(2n)^2+15(2n))/2+1}}{1-q^{10(2n)+2}},$$
$$S_2 := \sum_{n=0}^{\infty} \frac{q^{(15(2n+1)^2+25(2n+1))/2+2}}{1-q^{10(2n+1)+2}},$$
$$S_3 := \sum_{n=1}^{\infty} \frac{q^{(15(2n)^2-5(2n))/2}}{1-q^{10(2n)-2}},$$
$$S_4 := \sum_{n=1}^{\infty} \frac{q^{(15(2n+1)^2+5(2n+1))/2-1}}{1-q^{10(2n+1)-2}},$$
$$T_1 := \sum_{n=0}^{\infty} \frac{q^{(15(2n)^2+25(2n))/2+2}}{1-q^{10(2n)+2}},$$
$$T_2 := \sum_{n=0}^{\infty} \frac{q^{(15(2n+1)^2+15(2n+1))/2+1}}{1-q^{10(2n+1)+2}},$$
$$T_3 := \sum_{n=1}^{\infty} \frac{q^{(15(2n)^2+5(2n))/2-1}}{1-q^{10(2n)-2}},$$
$$T_4 := \sum_{n=1}^{\infty} \frac{q^{(15(2n+1)^2-5(2n+1))/2}}{1-q^{10(2n+1)-2}}.$$

By Lemma (2.4), each of the series S_i and T_i has all nonnegative q-series coefficients for $n \geq 0$. In order to prove (1), it thus suffices to show that the function

$$\frac{J_5^2 J_{4,10}^2 J_{10}^5}{J_{3,10}^3 J_{2,10}^4} - T_1 - T_2 - T_3 - T_4$$

has all positive coefficients.

We now find upper bounds on the size of the coefficients of $T_1 + T_2 + T_3 + T_4$. Let N be a nonnegative integer. Define $a_i(N)$ to be the coefficient of q^N in T_i. By expanding $\frac{1}{1-q^{10(2n)-2}}$ in each term of T_3 as a geometric series, we see that

$$T_3 = \sum_{N=1}^{\infty} a_3(N) q^N = \sum_{n=1}^{\infty} q^{(15(2n)^2+5(2n))/2-1}(1 + q^{10(2n)-2} + q^{2(10(2n)-2)} + \cdots).$$

We see that $a_3(N)$ is equal to the number of ordered pairs (n,k) of integers such that $n \geq 0$ and $k \geq 1$ satisfying the equation

$$\frac{15(2n)^2+5(2n)}{2} - 1 + k(10(2n)-2)) = N.$$

Simplifying the above expression, we find that the ratio $\frac{N}{n}$ is equal to

$$\frac{N}{n} = \frac{30n^2+5n-1+k(20n-2)}{n}.$$

We note that the above expression strictly increases as n or k increases. It follows that for $n \geq 1$, the ratio $\frac{N}{n}$ is at least 34. Since the summation index in T_3 ranges from 1 to ∞, we may conclude that $a_3(N) \leq \lfloor \frac{N}{34} \rfloor$. Arguing in the same fashion in the cases of the other three sums and taking special care to account for the $n = 0$ term in T_1 and T_2, we find that $a_1(N), a_2(N) \leq \lfloor \frac{N}{34} \rfloor + 1$ and $a_4(N) \leq \lfloor \frac{N}{34} \rfloor$. In order to prove (1), it thus suffices to show that the function

$$\frac{J_5^2 J_{4,10}^2 J_{10}^5}{J_{3,10}^3 J_{2,10}^4} - \sum_{n=0}^{\infty} \left(4 \left\lfloor \frac{n}{34} \right\rfloor + 2\right) q^n = \sum_{n=0}^{\infty} c'(n) q^n \tag{12}$$

has all positive coefficients for sufficiently large $n > B$, and that (1) has positive coefficients for all powers of q up to B. We now examine the product. By using Lemma 2.6, we find that

$$\begin{aligned}
\frac{J_5^2 J_{4,10}^2 J_{10}^5}{J_{3,10}^3 J_{2,10}^4} &= \frac{(q^5;q^5)_\infty^2 (q^4,q^6;q^{10})_\infty^2}{(q^3,q^7;q^{10})_\infty^3 (q^2,q^8;q^{10})_\infty^4} \\
&= \frac{(q^5;q^5)_\infty^2 (-q^2,-q^3,-q^7,-q^8;q^{10})_\infty^2}{(q^3,q^7;q^{10})_\infty (q^2,q^8;q^{10})_\infty^2} \\
&= \frac{(q^5;q^5)_\infty^2 (-q^2,-q^3;q^5)_\infty^2}{(q^3,q^7;q^{10})_\infty (q^2,q^8;q^{10})_\infty^2}.
\end{aligned}$$

Applying (11) twice with $z = q^{1/2}$ and $q = q^{5/2}$, we obtain

$$\frac{J_5^2 J_{4,10}^2 J_{10}^5}{J_{3,10}^3 J_{2,10}^4} = \frac{1}{(q^3,q^7;q^{10})_\infty (q^2,q^8;q^{10})_\infty^2} \left[\sum_{n=-\infty}^{\infty} q^{n(5n+1)/2}\right]^2.$$

Substituting the above expression into (12) and separating out the first factors of $\frac{1}{(q^2,q^3;q^{10})_\infty}$, we obtain

$$\frac{1}{(1-q^2)(1-q^3)} \left\{ \frac{1}{(q^2,q^7,q^{12},q^{13};q^{10})_\infty (q^8;q^{10})_\infty^2} \left[\sum_{n=-\infty}^{\infty} q^{n(5n+1)/2}\right]^2 \right\} - \sum_{n=0}^{\infty} \left(4 \left\lfloor \frac{n}{34} \right\rfloor + 2\right) q^n.$$

Let $c(n)$ be defined by

$$\frac{1}{(1-q^2)(1-q^3)} = (1+q^2+q^4+\cdots)(1+q^3+q^6+\cdots) = \sum_{n=0}^{\infty} c(n) q^n.$$

We see that $c(n)$ is equal to the number of ordered pairs (a,b) of nonnegative integers satisfying the equation $2a+3b = n$. The number of choices of b such that $3b \leq n$ is at least $\lceil \frac{n}{3} \rceil$, and thus the number of choices of b such that $3b \leq n$ and $n - 3b$ is even is at least $\lfloor \lceil \frac{n}{3} \rceil \cdot \frac{1}{2} \rfloor$. We thus conclude that $c(n) \geq \lfloor \frac{n}{6} \rfloor$ for $n \geq 0$. We note that the term inside the braces above has all nonnegative coefficients and has a constant term of 1. Since $\lfloor \frac{N}{6} \rfloor > 4 \lfloor \frac{n}{34} \rfloor + 2$ for $n \geq 42$, it thus suffices to check that the coefficient of q^n in (1) is positive for $0 \leq n \leq 41$. Computation in Maple shows that this is true.

3.2. **Proof of (2).** To prove (2) we show that

$$\frac{2qJ_{10}^6}{J_5J_{2,10}^2J_{3,10}^2}-\frac{1}{J_5}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{15n(n+1)/2+1}}{1+q^{5n+1}}=\frac{1}{J_5}\left(\frac{2qJ_{10}^6}{J_{2,10}^2J_{3,10}^2}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{15n(n+1)/2+1}}{1+q^{5n+1}}\right)$$

has strictly nonnegative q-series coefficients for $n\geq 1$. Since $\frac{1}{J_5}$ has all nonnegative coefficients and a constant term of 1, it suffices to show that the term inside the parentheses has nonnegative coefficients for $n\geq 1$. Arguing as before, we find that it suffices to show that

$$\frac{2qJ_{10}^6}{J_{2,10}^2J_{3,10}^2}-\sum_{n=0}^{\infty}\left(4\left\lfloor\frac{n}{25}\right\rfloor+2\right)q^n=\sum_{n=0}^{\infty}c'(n)q^n \tag{13}$$

has nonnegative coefficients for sufficiently large $n>B$, and that (2) has nonnegative coefficients for all powers of q up to B. We now examine the product. We have

$$\begin{aligned}\frac{2qJ_{10}^6}{J_{2,10}^2J_{3,10}^2}&=\frac{2q(q^{10};q^{10})_\infty^2}{(q^2,q^3,q^7,q^8;q^{10})_\infty^2}\\&=\frac{2q(-q^2,-q^8;q^{10})_\infty(q^{10};q^{10})_\infty^2}{(-q^2,-q^8;q^{10})_\infty(q^2,q^3,q^7,q^8;q^{10})_\infty^2}\\&=\frac{2q(q^{10};q^{10})_\infty}{(q^3,q^7;q^{10})_\infty}\left\{\frac{(-q^2,-q^8;q^{10})_\infty(q^{10};q^{10})_\infty}{(q^4,q^{16};q^{20})_\infty(q^2,q^3,q^7,q^8;q^{10})_\infty}\right\}\\&=2qL_{10,3}\left\{\frac{(-q^2,-q^8;q^{10})_\infty(q^{10};q^{10})_\infty}{(q^4,q^{16};q^{20})_\infty(q^2,q^3,q^7,q^8;q^{10})_\infty}\right\}.\end{aligned}$$

Applying (11) with $z=q^2$ and $q=q^5$, we obtain

$$\frac{2qJ_{10}^6}{J_{2,10}^2J_{3,10}^2}=\frac{2qL_{10,3}}{(q^4,q^{16};q^{20})_\infty(q^2,q^3,q^7,q^8;q^{10})_\infty}\cdot\sum_{n=-\infty}^{\infty}q^{5n^2+2n}.$$

Substituting the product on the right-hand side of the last equality above into (13) and separating out the first factors of $\frac{1}{(q^2,q^3;q^{10})_\infty}$, we obtain

$$\frac{2q}{(1-q^2)(1-q^3)}\left\{\frac{L_{10,3}}{(q^4,q^{16};q^{20})_\infty(q^7,q^8,q^{12},q^{13};q^{10})_\infty}\sum_{n=-\infty}^{\infty}q^{5n^2+2n}\right\}-\sum_{n=0}^{\infty}\left(4\left\lfloor\frac{n}{25}\right\rfloor+2\right)q^n.$$

Arguing as before, we find that the coefficient of q^n in

$$\frac{2q}{(1-q^2)(1-q^3)}=2q(1+q^2+q^4+\cdots)(1+q^3+q^6+\cdots)$$

is at least $2\left\lfloor\frac{n-1}{6}\right\rfloor$ for $n\geq 0$. By Lemma 2.3, $L_{10,3}$ has all nonnegative coefficients. We note that $L_{10,3}$ also has a constant term of 1. It follows that the term inside the braces above has all nonnegative coefficients and has a constant term of 1. Since $2\left\lfloor\frac{n-1}{6}\right\rfloor>4\left\lfloor\frac{n}{25}\right\rfloor+2$ for $n\geq 25$, it thus suffices to check that the coefficient of q^n in (2) is nonnegative for $1 \leq n \leq 24$. Computation in Maple shows that this is true.

3.3. **Proof of (4).** To prove (4), we show that

$$\frac{2qJ_{2,20}J_{10,20}J_{20}^{15}}{J_{6,20}J_{3,20}^2J_{4,20}^2J_{7,20}^2J_{1,20}^3J_{5,20}^3J_{9,20}^3}+\frac{1}{J_{5,20}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{10n^2+5n}}{1+q^{10n+2}}$$
$$=\frac{1}{J_{5,20}}\left(\frac{2qJ_{2,20}J_{10,20}J_{20}^{15}}{J_{6,20}J_{3,20}^2J_{4,20}^2J_{5,20}^2J_{7,20}^2J_{1,20}^3J_{9,20}^3}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{10n^2+5n}}{1+q^{10n+2}}\right)$$

has strictly positive coefficients q-series coefficients for $n \geq 0$. Since $\frac{1}{J_{5,20}}$ has all nonnegative coefficients and a constant term of 1, it suffices to show that the term inside the parentheses has all positive coefficients. Arguing as before, we find that it suffices to show that

$$\frac{2qJ_{2,20}J_{10,20}J_{20}^{15}}{J_{6,20}J_{3,20}^2J_{4,20}^2J_{5,20}^2J_{7,20}^2J_{1,20}^3J_{9,20}^3}-\sum_{n=0}^{\infty}\left(4\left\lfloor\frac{n}{68}\right\rfloor+2\right)q^n=\sum_{n=0}^{\infty}c'(n)q^n$$

has all positive coefficients for sufficiently large $n > B$, and that (4) has positive coefficients for all powers of q up to B. We note that $n \geq 4\left\lfloor\frac{n}{68}\right\rfloor+2$ for $n \geq 2$. It is simple to check that the coefficients of q^0 and q in sum appearing in (4) are 1 and 0, respectively. We may thus instead consider

$$\frac{2qJ_{2,20}J_{10,20}J_{20}^{15}}{J_{6,20}J_{3,20}^2J_{4,20}^2J_{5,20}^2J_{7,20}^2J_{1,20}^3J_{9,20}^3}-\frac{q}{(1-q)^2}. \tag{14}$$

We now examine the product. By using Lemma 2.6, we find that

$$\begin{aligned}
&\frac{2qJ_{2,20}J_{10,20}J_{20}^{15}}{J_{6,20}J_{3,20}^2J_{4,20}^2J_{5,20}^2J_{7,20}^2J_{1,20}^3J_{9,20}^3}\\
&=\frac{2q(q^2,q^{18};q^{20})_\infty(q^{10},q^{20};q^{20})_\infty^2}{(q^6,q^{14};q^{20})_\infty(q^3,q^4,q^5,q^7,q^{13},q^{15},q^{16},q^{17};q^{20})_\infty^2(q,q^9,q^{11},q^{19};q^{20})_\infty^3}\\
&=\frac{2q(-q,-q^9,-q^{11},-q^{19};q^{20})_\infty(-q^5,-q^{15},q^{20};q^{20})_\infty^2}{(q^6,q^{14};q^{20})_\infty(q,q^3,q^4,q^7,q^9,q^{11},q^{13},q^{16},q^{17},q^{19};q^{20})_\infty^2(q^{19};q^{20})_\infty^3}\\
&=\frac{2q(q^{20};q^{20})_\infty^2}{(q^9,q^{11};q^{20})_\infty^2}\left\{\frac{(-q,-q^9,-q^{11},-q^{19};q^{20})_\infty(-q^5,-q^{15};q^{20})_\infty^2}{(q^6,q^{14};q^{20})_\infty(q,q^3,q^4,q^7,q^{13},q^{16},q^{17},q^{19};q^{20})_\infty^2(q^{19};q^{20})_\infty^3}\right\}\\
&=2qL_{20,9}^2\left\{\frac{(-q,-q^9,-q^{11},-q^{19};q^{20})_\infty(-q^5,-q^{15};q^{20})_\infty^2}{(q^6,q^{14};q^{20})_\infty(q,q^3,q^4,q^7,q^{13},q^{16},q^{17},q^{19};q^{20})_\infty^2(q^{19};q^{20})_\infty^3}\right\}.
\end{aligned}$$

Substituting the product on the right-hand side of the last equality above into (14) and separating out the first factor of $\frac{1}{(q;q^{20})_\infty^2}$, we obtain

$$\begin{aligned}
&\frac{1}{(1-q)^2}\left\{\frac{2qL_{20,9}^2(-q,-q^9,-q^{11},-q^{19};q^{20})_\infty(-q^5,-q^{15};q^{20})_\infty^2}{(q^6,q^{14};q^{20})_\infty(q^3,q^4,q^7,q^{13},q^{16},q^{17},q^{19},q^{21};q^{20})_\infty^2(q^{19};q^{20})_\infty^3}\right\}-\frac{q}{(1-q)^2}\\
&=\frac{1}{(1-q)^2}\left\{\frac{2qL_{20,9}^2(-q,-q^9,-q^{11},-q^{19};q^{20})_\infty(-q^5,-q^{15};q^{20})_\infty^2}{(q^6,q^{14};q^{20})_\infty(q^3,q^4,q^7,q^{13},q^{16},q^{17},q^{19},q^{21};q^{20})_\infty^2(q^{19};q^{20})_\infty^3}-q\right\}.
\end{aligned}$$

By Lemma 2.3, $L_{20,9}$ has all non-negative coefficients. We note that $L_{20,9}$ has a constant term of 1. It follows that the coefficient of q in the term inside the braces above is 1. We may thus conclude that the term inside the braces above has all nonnegative coefficients. Since $\frac{1}{(1-q)^2}$ has all positive coefficients when expanded as a geometric series, it follows that the coefficients of the entire expression above are all positive except for the constant term. It thus suffices to show that the constant term of (4) is positive. By computing the $n=0$ term in the sum appearing in (4), we find that the constant term of (4) is 1.

3.4. **Proof of (5).** To show (5), we show that

$$\begin{aligned}
&\frac{2J_{6,20}J_{10,20}J_{20}^{15}}{J_{2,20}J_{1,20}^2J_{8,20}^2J_{9,20}^2J_{5,20}^3J_{3,20}^3J_{7,20}^3}+\frac{1}{J_{5,20}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{10n^2+15n}}{1+q^{10n+6}}\\
&=\frac{1}{J_{5,20}}\left(\frac{2J_{6,20}J_{10,20}J_{20}^{15}}{J_{2,20}J_{1,20}^2J_{5,20}^2J_{8,20}^2J_{9,20}^2J_{3,20}^3J_{7,20}^3}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{10n^2+15n}}{1+q^{10n+6}}\right)
\end{aligned}$$

has strictly positive q-series coefficients for $n\geq 0$. Since $\frac{1}{J_{5,20}}$ has all nonnegative coefficients and a constant term of 1, it suffices to show that the term inside the parentheses has all positive coefficients. Arguing as before, we find that it suffices to show that

$$\frac{2J_{6,20}J_{10,20}J_{20}^{15}}{J_{2,20}J_{1,20}^2J_{5,20}^2J_{8,20}^2J_{9,20}^2J_{3,20}^3J_{7,20}^3}-\sum_{n=0}^{\infty}\left(4\left\lfloor\frac{n}{38}\right\rfloor+2\right)q^n=\sum_{n=0}^{\infty}c'(n)q^n$$

has all positive coefficients for sufficiently large $n>B$, and that (5) has positive coefficients for all powers of q up to B. We note that $n+1\geq 4\left\lfloor\frac{n}{38}\right\rfloor+2$ for $n\geq 1$. By computing the $n=0$ term of the sum appearing in (5), we find that the constant term is 1. We may thus instead consider

$$\frac{2J_{6,20}J_{10,20}J_{20}^{15}}{J_{2,20}J_{1,20}^2J_{5,20}^2J_{8,20}^2J_{9,20}^2J_{3,20}^3J_{7,20}^3}-\frac{1}{(1-q)^2}. \tag{15}$$

We now examine the product. By using Lemma 2.6, we find that

$$
\begin{aligned}
&\frac{2J_{6,20}J_{10,20}J_{100}^{15}}{J_{2,20}J_{1,20}^2J_{5,20}^2J_{8,20}^2J_{9,20}^2J_{3,20}^3J_{7,20}^3}\\
&=\frac{2(q^6,q^{14};q^{20})_\infty(q^{10},q^{20};q^{20})_\infty^2}{(q^2,q^{18};q^{20})_\infty(q,q^5,q^8,q^9,q^{11},q^{12},q^{15},q^{19};q^{20})_\infty^2(q^3,q^7,q^{13},q^{17};q^{20})_\infty^3}\\
&=\frac{2(-q^3,-q^7,-q^{13},-q^{17};q^{20})_\infty(-q^5,-q^{15},q^{20};q^{20})_\infty^2}{(q^2,q^{18};q^{20})_\infty(q,q^3,q^7,q^8,q^9,q^{11},q^{12},q^{13},q^{17},q^{19};q^{20})_\infty^2}\\
&=\frac{2(q^{20};q^{20})_\infty^2}{(q^9,q^{11};q^{20})_\infty^2}\left\{\frac{(-q^3,-q^7,-q^{13},-q^{17};q^{20})_\infty(-q^5,-q^{15};q^{20})_\infty^2}{(q^2,q^{18};q^{20})_\infty(q,q^3,q^7,q^8,q^{12},q^{13},q^{17},q^{19};q^{20})_\infty^2}\right\}\\
&=2L_{20,9}^2\left\{\frac{(-q^3,-q^7,-q^{13},-q^{17};q^{20})_\infty(-q^5,-q^{15};q^{20})_\infty^2}{(q^2,q^{18};q^{20})_\infty(q,q^3,q^7,q^8,q^{12},q^{13},q^{17},q^{19};q^{20})_\infty^2}\right\}.
\end{aligned}
$$

Substituting the product on the right-hand side of the last equality above into (15) and separating out the first factor of $\frac{1}{(q;q^{20})_\infty^2}$, we obtain

$$
\begin{aligned}
&\frac{1}{(1-q)^2}\left\{\frac{2L_{20,9}^2(-q^3,-q^7,-q^{13},-q^{17};q^{20})_\infty(-q^5,-q^{15};q^{20})_\infty^2}{(q^2,q^{18};q^{20})_\infty(q^3,q^7,q^8,q^{12},q^{13},q^{17},q^{19},q^{21};q^{20})_\infty^2}\right\}-\frac{1}{(1-q)^2}\\
&=\frac{1}{(1-q)^2}\left\{\frac{2L_{20,9}^2(-q^3,-q^7,-q^{13},-q^{17};q^{20})_\infty(-q^5,-q^{15};q^{20})_\infty^2}{(q^2,q^{18};q^{20})_\infty(q^3,q^7,q^8,q^{12},q^{13},q^{17},q^{19},q^{21};q^{20})_\infty^2}-1\right\}.
\end{aligned}
$$

By Lemma 2.3, $L_{20,9}$ has all nonnegative coefficients. We note that $L_{20,9}$ has a constant term of 1, implying that the term inside the braces above has a constant term of 1. Since $\frac{1}{(1-q)^2}$ has all positive coefficients when expanded as a geometric series, the entire expression above has all positive coefficients.

4. Proof of Theorem 1.7

We now return our focus to rank differences and prove Theorem 1.7 in the style of Mao. We first replace z with ζ_6 in (9) to obtain the following:

$$
\begin{aligned}
L_6(q^2)&=\frac{(-q^2;q^4)_\infty}{(q^4;q^4)_\infty}\left[\sum_{n\in\mathbb{Z}}\frac{(-1)^n(1-z^{-1})(1-z)q^{4n^2+2n}}{(1-zq^{4n})(1-z^{-1}q^{4n})}\right]\\
&=\frac{(-q^2;q^4)_\infty}{(q^4;q^4)_\infty}\left[\sum_{n\in\mathbb{Z}}\frac{(-1)^n(1-\zeta_6^{-1})(1-\zeta_6)q^{4n^2+2n}}{(1-\zeta_6q^{4n})(1-\zeta_6^{-1}q^{4n})}\right]\\
&=\frac{(-q^2;q^4)_\infty}{(q^4;q^4)_\infty}\left[\sum_{n\in\mathbb{Z}}\frac{(-1)^n(1+\zeta_6^2)(1+\zeta_6)q^{4n^2+2n}}{(1+\zeta_6q^{4n})(1+\zeta_6^2q^{4n})}\right]\\
&=\frac{(-q^2;q^4)_\infty}{(q^4;q^4)_\infty}\left[\sum_{n\in\mathbb{Z}}\frac{(-1)^n(1+q^{4n})q^{4n^2+2n}}{(1+q^{12n})}\right].
\end{aligned}
\tag{16}
$$

Now, we use the identities from Lemma 2.1 and substitute them into (16) to get

$$\frac{(-q^2;q^4)_\infty}{(q^4;q^4)_\infty}\left[\sum_{n\in\mathbb{Z}}\frac{(-1)^n(1+q^{4n})q^{4n^2+2n}}{(1+q^{12n})}\right]=\frac{(-q^2;q^4)_\infty}{(q^4;q^4)_\infty}\cdot(V_{2,0}+V_{2,1})$$
$$-\frac{1}{J_{18,72}}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{36n^2+54n+18}}{1+q^{36n+24}}-\frac{1}{J_{18,72}}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{36n^2+18n-2}}{1+q^{36n}}.$$

Then, we use equation (10) from Lemma 2.1 to obtain

$$L_6(q^2)=\frac{J_{12,72}J_{36,72}^2J_{72}^3}{J_{6,72}J_{18,72}^3J_{30,72}}-\frac{1}{J_{18,72}}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{36n^2+54n+18}}{1+q^{36n+24}}$$
$$+q^2\frac{J_{12,72}^2J_{36,72}J_{72}^3}{J_{6,72}^2J_{18,72}J_{30,72}^2}+\frac{J_{12,72}J_{36,72}^2J_{72}^3}{2q^2J_{6,72}^2J_{18,72}J_{30,72}^2}-\frac{1}{J_{18,72}}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{36n^2+18n-2}}{1+q^{36n}}.$$

Therefore, it only remains to show the following:

$$\frac{J_{12,72}J_{36,72}^2J_{72}^3}{J_{6,72}J_{18,72}^3J_{30,72}}-\frac{1}{J_{18,72}}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{36n^2+54n+18}}{1+q^{36n+24}}=\frac{1}{J_{18,72}}\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{36n^2+18n}}{1-q^{36n+6}}.$$

But this follows from an application of Lemma 2.2, letting $r=0$, $s=2$, $q=q^{36}$, $b_1=-q^{24}$, and $b_2=q^6$.

5. Conclusion

While our original goal was to move towards generalizing the method of finding nice infinite product and modular form identities for different and more general "rank difference"-type generating functions, we found that the methods that Atkin and Swinnerton-Dyer, Lovejoy and Osburn, and Mao employed generally relied on Lemma 2.2 of Chan [4] in order to transform the unwieldy infinite sums into elegant q-series products, which are much easier to work with. We ran into problems with the exponent of q in the numerator of the infinite series in this identity when attempting to work with any multiple of of this exponent other than 1 or 2. Specifically, since this exponent of q is divided by 2, we ended up with a fractional power of q, which caused problems. Additionally, we ran into similar problems when attempting to find a different lemma or a way around using this identity. Thus, we imagine that a further investigation into a generalization of either Mao or Lovejoy and Osburn's method will require a modified version of Chan's identity, or even a new lemma altogether in order to deal with these problems.

Additionally, we failed to prove Mao's conjectures (3), (6), and (7) due to the fact that these conjectured inequalities involved taking the sum of two large infinite products and then determining the positivity of the coefficients, which doesn't yield to our methods. Computation does provide evidence that these inequalities hold, but they remain open.

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