ALTERNATING INHOMOGENEOUS QUANTUM WALKS ON $\mathbb{Z}$

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Abstract. Quantum walks are a powerful tool for developing efficient algorithms in quantum computing. This paper explores two discrete-time one-dimensional quantum walks where the coin operator varies along even and odd positions on the line. We find closed-form expressions for the coefficients of the wave function for both walks and also arrive at a formula for the probability distribution for one of the walks. A significant discovery in this paper is a way to model the well-known Hadamard walk using two alternating coins.

1. Introduction

1.1. Motivation. A quantum walk is an analog of the classical random walk, a stochastic model following the motion of a particle along the number line. Classical walks have been used to design efficient computer algorithms for traditional computers. With the rise of quantum computing, quantum walks have been used to create new algorithms, such as Grover's search algorithm and Shor's algorithm for factoring numbers that provide quadratic and possibly exponential speedup, respectively, over previous classical methods [2].

There exists substantial literature on single-coined quantum walks—in particular, the most well-studied Hadamard walk. However, little light has been shed on two-coined walks. In this paper, we seek to analyze quantum walks where the coin operator differs for even and odd locations on the line. We refer to these types of walks as *alternating space-inhomogeneous* quantum walks. It is worth mentioning that since our choice of initial condition is a pure state, the walks we will explore are also *time-inhomogeneous* [3].

1.2. Background. In the classical discrete random walk, a particle begins at an initial position and moves one unit left or right along the number line according to some probability. However, in quantum walks we use a quantum particle, which can stay in a suspended superposition between multiple positions. If measured or observed, that particle will "behave" and take on only a single position, but all information about its superposition will be lost upon measurement. The trick with quantum walks is that the measurement is not performed until the very end and we only keep track of the probabilities of where the particle may be, in the form of a wave function [3]. As with many types of waves, this can lead to some interesting constructive and destructive interference patterns. This is a main difference between classical and quantum walks.

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1.3. **Dirac Notation.** The *Dirac notation*, sometimes also called the *bra-ket* notation, is the standard notation used in quantum mechanics to describe quantum states. We define this notation thusly:

Definition 1.1. *A* ket *is a column vector:* $|\psi\rangle := (\psi_1, \psi_2, \ldots, \psi_n)^T$.

Definition 1.2. *A* bra *is a row vector that is the conjugate transpose of a* ket*:* $\psi^* = (\bar{\psi}_1, \bar{\psi}_2, \ldots, \bar{\psi}_n)$.

Using Dirac notation, we may conveniently express the inner product over $\mathbb{C}$ as $\langle\psi|\phi\rangle = \psi^*\phi = \sum_{i=1}^{n} \bar{\psi}_i \phi_i$.

1.4. **Mathematically Encoding a Quantum Walk.**

1.4.1. *States.* Quantum walks begin with a particle at an initial state, $|\psi(0)\rangle$. This state encapsulates two pieces of information: the particle's position and its *chirality* or *spin*. The coin space is defined as a Hilbert space, denoted $\mathcal{H}_c$, spanned by the two basis states $\{|\uparrow\rangle, |\downarrow\rangle\}$. That is, the spin is a linear combination of

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The position is recorded as a vector in a Hilbert space called $\mathcal{H}_p = \{|j\rangle ; j \in \mathbb{Z}\}$. The state of the total system is contained in the space $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_p$.

We model the state of the total system using a *wave function.* Hence, we define $|\psi_j(t)\rangle = a_j(t)|\uparrow\rangle + b_j(t)|\downarrow\rangle$ to be the wave function at a given position j and time t with coefficients $a_j(t)$, $b_j(t)$. Then, the total wave function over all integers at time t is

$$|\psi(t)\rangle = \sum_{j\in\mathbb{Z}} |\psi_j(t)\rangle.$$

If we were to measure the particle at time t, then the probability of the particle being at position j with spin up is equal to $|a_j(t)|^2$. Similarly, the probability of being at position j with spin down would be $|b_j(t)|^2$, so the total probability of the particle being at j during time t is

$$P_j(t) = |a_j(t)|^2 + |b_j(t)|^2. \tag{1}$$

1.4.2. *Unitary Transformations.* To perform a quantum walk, we begin with our initial state $|\psi(0)\rangle$ and multiply it by a unitary matrix U. Recall that a matrix U is unitary if and only if $UU^* = U^*U = I$, that is, the inverse of a matrix is its conjugate transpose. So $|\psi(1)\rangle = U|\psi(0)\rangle$. We may continue to apply this transformation, multiplying by U each time, which means that $|\psi(t)\rangle = U^t|\psi(0)\rangle$, where t represents the number of time steps.

The transformation U is defined as

$$U = S(C \otimes I).$$

From right to left, I is the identity matrix in the position space. C represents the *coin operator*, which will alter the spin of the particle, and S is the *shift operator* which shifts the position of the particle by one unit to the left or to the right. If the spin is up, then S will move the particle to the right (from j to $j+1$) and a down spin will lead S to move the particle to the left (from j to $j-1$).

1.4.3. *The Coin and Shift Operators.* Consider the coin operator as a literal coin being flipped, creating a probability of landing heads up or tails down. In this paper, the coin operator will be a unitary 2×2 matrix.

A detailed explanation of the shift operator can be found in Appendix A. In short, the operator increases the position by one if the particle is oriented up and decreases the position by one if the particle is oriented down. That is,

$$|\uparrow\rangle \otimes |j\rangle \mapsto |\uparrow\rangle \otimes |j+1\rangle$$
$$|\downarrow\rangle \otimes |j\rangle \mapsto |\downarrow\rangle \otimes |j-1\rangle .$$

2. The Hadamard Walk

One of the most common coin operators is called the *Hadamard operator*, defined as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

To explore the effects of this coin on the spin, multiply this matrix by the *pure states* $|\uparrow\rangle$ and $|\downarrow\rangle$.

$$H|\uparrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} |\uparrow\rangle + \frac{1}{\sqrt{2}} |\downarrow\rangle \tag{2}$$

$$H|\downarrow\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} |\uparrow\rangle - \frac{1}{\sqrt{2}} |\downarrow\rangle \tag{3}$$

Note that the squared magnitudes of the coefficients are all $1/2$. So when the Hadamard coin operates on a pure state, it creates an equal probability of changing the spin up or down. We will use the simple case of the Hadamard walk as an example of the approach used to analyze quantum walks.

2.1. **Demonstration of the Coin and Shift Operators.**

Let the initial state be $|\psi(0)\rangle = |\downarrow\rangle \otimes |0\rangle$. Since there is only one term and its coefficient is 1, the probability of the particle being at position 0 at time 0 is $P_0(0) = 1$. Now perform the first iteration by applying the transformation U.

$$|\psi(1)\rangle = U|\psi(0)\rangle = S(H \otimes I)(|\downarrow\rangle \otimes |0\rangle)$$
$$= S(H|\downarrow\rangle \otimes I|0\rangle) = S(H|\downarrow\rangle \otimes |0\rangle)$$

This is a case where it is easier to know the mapping of the function rather than to actually multiply the matrices. From above, we know the effects of the Hadamard operator on the pure states $|\uparrow\rangle$ and $|\downarrow\rangle$. Substituting (2) for $H|\uparrow\rangle$ results in:

$$|\psi(1)\rangle = S\left(\left(\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle\right) \otimes |0\rangle\right)$$
$$= S\left(\frac{1}{\sqrt{2}}|\uparrow\rangle \otimes |0\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle \otimes |0\rangle\right).$$

Remember to increase the position if the spin is up and decrease if the spin is down.

$$|\psi(1)\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle \otimes |1\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle \otimes |-1\rangle$$

The probabilities of the particle being at 1 or at -1 are the same: $\left|1/\sqrt{2}\right|^2 = 1/2$.

Continuing the derivation for the wave function at time $t = 2$:

$$\begin{aligned}
|\psi(2)\rangle &= S(H \otimes I)\,|\psi(1)\rangle && \text{Second iteration}\\
&= S(H \otimes I)\left(\frac{1}{\sqrt{2}}|\uparrow\rangle \otimes |1\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle \otimes |-1\rangle\right) && \text{Substitution}\\
&= \frac{1}{\sqrt{2}}S\left(H|\uparrow\rangle \otimes |1\rangle - H|\downarrow\rangle \otimes |-1\rangle\right) && \text{Factor out } \frac{1}{\sqrt{2}}\\
&= \frac{1}{\sqrt{2}}S\left(\left(\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle\right) \otimes |1\rangle - \left(\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle\right) \otimes |-1\rangle\right) && \text{Hadamard}\\
&= \frac{1}{2}S\left(|\uparrow\rangle \otimes |1\rangle + |\downarrow\rangle \otimes |1\rangle - |\uparrow\rangle \otimes |-1\rangle + |\downarrow\rangle \otimes |-1\rangle\right) && \text{Factor out } \frac{1}{\sqrt{2}}\\
&= \frac{1}{2}(|\uparrow\rangle \otimes |2\rangle + |\downarrow\rangle \otimes |0\rangle - |\uparrow\rangle \otimes |0\rangle + |\downarrow\rangle \otimes |-2\rangle) && \text{Shift}
\end{aligned}$$

We now have probabilities $P_{-2}(2) = \frac{1}{4}, P_0(2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, and $P_2(2) = \frac{1}{4}$. It takes one more iteration to depart from classical random walk probabilities.

$$\begin{aligned}
|\psi(3)\rangle &= S(H \otimes I)\,|\psi(2)\rangle\\
&= \frac{1}{2}S(H \otimes I)(|\uparrow\rangle \otimes |2\rangle + |\downarrow\rangle \otimes |0\rangle - |\uparrow\rangle \otimes |0\rangle + |\downarrow\rangle \otimes |-2\rangle)\\
&= \frac{1}{2}S(H|\uparrow\rangle \otimes |2\rangle + H|\downarrow\rangle \otimes |0\rangle - H|\uparrow\rangle \otimes |0\rangle + H|\downarrow\rangle \otimes |-2\rangle)\\
&= \frac{1}{2\sqrt{2}}S((|\uparrow\rangle + |\downarrow\rangle) \otimes |2\rangle + (|\uparrow\rangle - |\downarrow\rangle) \otimes |0\rangle - (|\uparrow\rangle + |\downarrow\rangle) \otimes |0\rangle + (|\uparrow\rangle - |\downarrow\rangle) \otimes |-2\rangle\\
&= \frac{1}{2\sqrt{2}}(|\uparrow\rangle \otimes |3\rangle + |\downarrow\rangle \otimes |1\rangle + \cancel{|\uparrow\rangle \otimes |1\rangle} - \underbrace{|\downarrow\rangle \otimes |-1\rangle} - \cancel{|\uparrow\rangle \otimes |1\rangle} - \underbrace{|\downarrow\rangle \otimes |-1\rangle} + |\uparrow\rangle \otimes |-1\rangle - |\downarrow\rangle \otimes |-3\rangle\\
&= \frac{1}{2\sqrt{2}}(|\uparrow\rangle \otimes |3\rangle + |\downarrow\rangle \otimes |1\rangle - 2|\downarrow\rangle \otimes |-1\rangle + |\uparrow\rangle \otimes |-1\rangle - |\downarrow\rangle \otimes |-3\rangle)
\end{aligned}$$

Now the probabilities differ from the classical random walk since

$$P_{-3}(3) = \frac{1}{8},\; P_{-1}(3) = \frac{1}{2} + \frac{1}{8} = \frac{5}{8},\; P_1(3) = \frac{1}{8},\; \text{and } P_3(3) = \frac{1}{8}.$$

Notice the probabilities are beginning to drift towards the negative side. This stems from using an initial state with down spin. If we had chosen an up spin to begin with, we would be drifting at the same rate in the positive direction. If we began with a balanced initial state such as $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle) \otimes |0\rangle$, the probabilities would have been symmetrical [3].

2.2. **Recursion Formulas.** Recall that for a given position j at a certain time t,

$$|\psi_j(t)\rangle = a_j(t)\,|{\uparrow}\rangle + b_j(t)\,|{\downarrow}\rangle\,,$$

So the $a_j(t)$ coefficient relates to the probability of the particle having up-spin (and thus having come from below) and the $b_j(t)$ coefficient relates to the probability of having down-spin (and thus having come from above). We can deduce the following recursion formulas for finding the coefficients $a_j(t)$ and $b_j(t)$ at the next time step:

$$a_j(t+1) = \frac{1}{\sqrt{2}}a_{j-1}(t) + \frac{1}{\sqrt{2}}b_{j-1}(t) \tag{4}$$

$$b_j(t+1) = \frac{1}{\sqrt{2}}a_{j+1}(t) - \frac{1}{\sqrt{2}}b_{j+1}(t). \tag{5}$$

2.3. **Fourier Transforms.** We define a Fourier transform as below:

$$\hat{f}_t(s) = \sum_j f_j(t)e^{-ijs}, \tag{6}$$

where i is the imaginary unit and s is a real number, summing over all integer positions j. As such, combining (4) and (6) we get

$$\begin{aligned}
\hat{a}_{t+1}(s) &= \sum_j a_j(t+1)e^{-ijs} && \text{Def. of Fourier Transform}\\
&= \sum_j \frac{1}{\sqrt{2}}a_{j-1}(t)e^{-ijs} + \sum_j \frac{1}{\sqrt{2}}b_{j-1}(t)e^{-ijs} && \text{Substitute (4)}\\
&= \frac{1}{\sqrt{2}}\sum_k e^{-i(k+1)s}a_k(t) + \frac{1}{\sqrt{2}}\sum_k e^{-i(k+1)s}b_k(t) && \text{Change of variable: } k=j-1\\
&= \frac{1}{\sqrt{2}}\sum_k e^{-iks-is}a_k(t) + \frac{1}{\sqrt{2}}\sum_k e^{-iks-is}b_k(t) && \text{Distributed } (k+1)\\
&= \frac{1}{\sqrt{2}}\sum_k e^{-iks}e^{-is}a_k(t) + \frac{1}{\sqrt{2}}\sum_k e^{-iks}e^{-is}b_k(t) && \text{Exponent Rules: } e^{a+b}=e^ae^b\\
&= \frac{e^{-is}}{\sqrt{2}}\underbrace{\sum_k e^{-iks}a_k(t)}_{\hat{a_t}(s)} + \frac{e^{-is}}{\sqrt{2}}\underbrace{\sum_k e^{-iks}b_k(t)}_{\hat{b_t}(s)} && \text{Factored } e^{-is}\\
&= \frac{e^{-is}}{\sqrt{2}}\ \hat{a}_t(s) + \frac{e^{-is}}{\sqrt{2}}\ \hat{b}_t(s). && \text{Def. of Fourier Transform} \qquad (7)
\end{aligned}$$

Similarly, we derive

$$\hat{b}_{t+1}(s) = \frac{e^{is}}{\sqrt{2}}\ \hat{a}_t(s) - \frac{e^{is}}{\sqrt{2}}\ \hat{b}_t(s). \tag{8}$$

If we were to create a column vector out of $\hat{a}_{t+1}(s)$ and $\hat{b}_{t+1}(s)$, then we could describe (7) and (8) as a product of matrix multiplication.

$$\begin{pmatrix} \hat{a}_{t+1}(s) \\ \hat{b}_{t+1}(s) \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} e^{-is} & e^{-is} \\ e^{is} & -e^{is} \end{pmatrix}}_{\text{M}} \begin{pmatrix} \hat{a}_t(s) \\ \hat{b}_t(s). \end{pmatrix}$$

So given initial states $\hat{a}_0(s)$ and $\hat{b}_0(s)$ and the matrix M as defined above, then we can define

$$\begin{pmatrix} \hat{a}_t(s) \\ \hat{b}_t(s) \end{pmatrix} = M^t \begin{pmatrix} \hat{a}_0(s) \\ \hat{b}_0(s) \end{pmatrix} \tag{9}$$

for all time steps t.

2.4. **Diagonalization.** To simplify the process of raising the matrix to the power t, we diagonalize the matrix M as

$$M = UDU^{-1} = UDU^*, \tag{10}$$

where U is a matrix whose columns are the eigenvectors of M. Since U is unitary, its inverse is its adjoint. The matrix D is a diagonal matrix composed of the eigenvalues of M. That is,

$$M = \begin{pmatrix} v_1 & v_2 \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \overline{v}_1 & \cdots \\ \overline{v}_2 & \cdots \end{pmatrix}.$$

This way, when we raise M to a power, we see the following for any t.

$$M^t = \begin{pmatrix} v_1 & v_2 \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} {\lambda_1}^t & 0 \\ 0 & {\lambda_2}^t \end{pmatrix} \begin{pmatrix} \overline{v}_1 & \cdots \\ \overline{v}_2 & \cdots \end{pmatrix} \tag{11}$$

Using the quadratic formula to solve $\det(M - \lambda I) = 0$, we obtain the following eigenvalues:

$$\lambda_{1,2} = \pm \underbrace{\frac{\sqrt{1+\cos^2(s)}}{\sqrt{2}}}_{\text{Re}(\lambda_{1,2})} - i \underbrace{\frac{\sin(s)}{\sqrt{2}}}_{\text{Im}(\lambda_{1,2})}.$$

This complex number can be written in polar form. Recall that $e^{-i\theta} = \cos\theta - i\sin\theta$. Thus, we can equate the real part of λ with $\cos\omega_s$ or the imaginary part with $\sin\omega_s$. Choosing the simpler of the two, we define ω_s such that $\sin(\omega_s) = \frac{\sin(s)}{\sqrt{2}}$. This is known as taking the *argument* of a complex number. Therefore, λ_1 expressed in polar form is

$$\lambda_1 = e^{-i\omega_s}.$$

The second eigenvalue is the same imaginary number except the real part is negative. Hence, it becomes

$$\lambda_2 = e^{-i(\pi+\omega_s)}.$$

Substituting these eigenvalues and eigenvectors into (11) and performing the matrix multiplication we arrive at the following results.

$$\hat{a}_t(s) = \frac{e^{-is}}{2\sqrt{1+\cos^2(s)}} \left(e^{-i\omega_s t} - (-1)^t e^{i\omega_s t} \right) \tag{12}$$

$$\hat{b}_t(s) = \frac{1}{2}\left(1 - \frac{\cos(s)}{\sqrt{1+\cos^2(s)}}\right)e^{-i\omega_s t} + \frac{(-1)^t}{2}\left(1 + \frac{\cos(s)}{\sqrt{1+\cos^2(s)}}\right)e^{i\omega_s t} \tag{13}$$

2.5. **Inverse Fourier Transform.** Our main objective is to derive the equations for $a_j(t)$ and $b_j(t)$, so employ the inverse Fourier transform:

$$f_j(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}_t(s)e^{ijs}ds. \tag{14}$$

To do this, manipulate (12) into the form of (14).

$$\begin{aligned}
a_j(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{a}_t(s)e^{ijs}ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-is}}{2\sqrt{1+\cos^2(s)}}\left(e^{-i\omega_s t} - (-1)^t e^{i\omega_s t}\right)e^{ijs}ds \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \frac{e^{-is}}{2\sqrt{1+\cos^2(s)}}e^{-i\omega_s t}e^{ijs}ds - (-1)^t\int_{-2\pi}^{0} \frac{e^{-is}}{2\sqrt{1+\cos^2(s)}}e^{i\omega_s t}e^{ijs}ds\right]
\end{aligned}$$

Now use a change of variable where $s^* = s + \pi$. Then, $ds^* = ds$, and $s = s^* - \pi$. Also, $\omega_s = \omega_{s^*-\pi} = -\omega_{s^*}$ and $\cos(s^* - \pi) = -\cos(s^*)$, which means $\cos^2(s^* - \pi) = \cos^2(s^*)$.

$$\begin{aligned}
a_j(t) &= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \frac{e^{-is}}{2\sqrt{1+\cos^2(s)}}e^{-i\omega_s t}e^{ijs}ds - (-1)^t\int_{-\pi}^{\pi} \frac{e^{-i\left(s^*-\pi\right)}}{2\sqrt{1+\cos^2(s^*-\pi)}}e^{it\omega_{s^*-\pi}}e^{ij\left(s^*-\pi\right)}ds^*\right] \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \frac{e^{-is}}{2\sqrt{1+\cos^2(s)}}e^{-i\omega_s t}e^{ijs}ds - (-1)^t\int_{-\pi}^{\pi} \frac{e^{-i\pi}e^{-is^*}}{2\sqrt{1+\cos^2(s^*)}}e^{-it\omega_{s^*}}e^{ijs^*}e^{-\pi ij}ds^*\right] \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \frac{e^{-is}}{2\sqrt{1+\cos^2(s)}}e^{-i\omega_s t}e^{ijs}ds - (-1)^t\int_{-\pi}^{\pi} \frac{(-1)e^{-is^*}}{2\sqrt{1+\cos^2(s^*)}}e^{-i\left(t\omega_{s^*}+js^*\right)}(-1)^j ds^*\right] \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \frac{e^{-is}}{2\sqrt{1+\cos^2(s)}}e^{i(js-\omega_s t)}ds + (-1)^{j+t}\int_{-\pi}^{\pi} \frac{e^{is^*}}{2\sqrt{1+\cos^2(s^*)}}e^{i(js^*-\omega_{s^*}t)}ds*\right] \\
&= \frac{1+(-1)^{j+t}}{4\pi}\int_{-\pi}^{\pi} \frac{e^{-is}}{\sqrt{1+\cos^2(s)}}e^{i(js-\omega_s t)}ds
\end{aligned} \tag{15}$$

Continuing our derivations, we calculate $b_j(t)$.

$$\begin{aligned}
b_j(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2}\left(1 - \frac{\cos(s)}{\sqrt{1+\cos^2(s)}}\right)e^{-i\omega_s t} + \frac{(-1)^t}{2}\left(1 + \frac{\cos(s)}{\sqrt{1+\cos^2(s)}}\right)e^{i\omega_s t}\right]e^{ijs}ds \\
&= \frac{1}{2\pi}\left[\frac{1}{2}\int_{-\pi}^{\pi}\left(1 - \frac{\cos(s)}{\sqrt{1+\cos^2(s)}}\right)e^{i(js-\omega_s t)}ds + \frac{(-1)^t}{2}\int_{-2\pi}^{0}\left(1 + \frac{\cos(s)}{\sqrt{1+\cos^2(s)}}\right)e^{i\omega_s t}e^{ijs}ds\right]
\end{aligned}$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}\int_{-\pi}^{\pi}\left(1-\frac{\cos(s)}{\sqrt{1+\cos^2(s)}}\right)e^{i(js-\omega_s t)}ds + \frac{(-1)^t}{2}\int_{-\pi}^{\pi}\left(1+\frac{\cos(s^*-\pi)}{\sqrt{1+\cos^2(s^*-\pi)}}\right)e^{i\omega_{s^*-\pi}t}e^{ij(s^*-\pi)}ds^*\right]$$
$$= \frac{1}{2\pi}\left[\frac{1}{2}\int_{-\pi}^{\pi}\left(1-\frac{\cos(s)}{\sqrt{1+\cos^2(s)}}\right)e^{i(js-\omega_s t)}ds + \frac{(-1)^t}{2}\int_{-\pi}^{\pi}\left(1-\frac{\cos(s^*)}{\sqrt{1+\cos^2(s^*)}}\right)e^{-i\omega_{s^*}t}e^{ijs^*}e^{\pi ij}ds^*\right]$$
$$= \frac{1}{2\pi}\left[\frac{1}{2}\int_{-\pi}^{\pi}\left(1-\frac{\cos(s)}{\sqrt{1+\cos^2(s)}}\right)e^{i(js-\omega_s t)}ds + \frac{(-1)^{j+t}}{2}\int_{-\pi}^{\pi}\left(1-\frac{\cos(s^*)}{\sqrt{1+\cos^2(s^*)}}\right)e^{i(js^*-\omega_{s^*}t)}ds^*\right]$$
$$= \frac{1+(-1)^{j+t}}{4\pi}\int_{-\pi}^{\pi}\left(1-\frac{\cos(s)}{\sqrt{1+\cos^2(s)}}\right)e^{i(js-\omega_s t)}ds \tag{16}$$

These two integrals describe the coefficients of the wave function at a position j and time t. Therefore, the probability of being at j at time t can be calculated using (1).

3. Results

An *alternating quantum walk* is a walk that uses different coin operators for even and odd times and/or positions. A *space-inhomogeneous* quantum walk is one for which the coin operator varies with the particle's position in the position space $\mathcal{H}_p$. Likewise, a *time-inhomogeneous* walk is one where different coins are used at different times. For both of the walks we analyze below, we have chosen the initial state $|\uparrow\rangle \otimes |0\rangle$, so the particle will be at even positions at even times and at odd positions at odd times. This makes our walks both space- and time-inhomogeneous, so we refer to them as *inhomogeneous walks.*

3.1. **PQ-Walk.**

The *PQ-walk* is an alternating quantum walk originally defined in [1] that uses the coin operators below for even and odd positions The two parameters are fixed as $0 \le p, q \le 1$.

$$C_{even} = \begin{pmatrix} \sqrt{p} & \sqrt{1-p} \\ \sqrt{1-p} & -\sqrt{p} \end{pmatrix} \tag{17}$$

$$C_{odd} = \begin{pmatrix} \sqrt{q} & \sqrt{1-q} \\ \sqrt{1-q} & -\sqrt{q} \end{pmatrix}. \tag{18}$$

Due to the nature of this walk, there are two separate sets of recursive formulas that determine the coefficients $a_j(t)$ and $b_j(t)$, one for odd values of j and another for even values of j. Since our initial condition starts at an even position, the time and position are either both even or both odd. Thus, we interchange "even/odd positions" with "even/odd times" whenever it is convenient. For example, instead of writing these recursive formulas with respect to even and odd positions, we define them by even and odd times. For even times t, we use (17), giving us

$$a_j(t+1) = \sqrt{p}\; a_{j-1}(t) + \sqrt{1-p}\; b_{j-1}(t) \tag{19}$$
$$b_j(t+1) = \sqrt{1-p}\; a_{j+1}(t) - \sqrt{p}\; b_{j+1}(t). \tag{20}$$

Since t is even, $t+1$ is odd and we use the odd coin, (18). This yields the recursive formulas

$$a_j(t+1) = \sqrt{q}\ a_{j-1}(t+1) + \sqrt{1-q}\ b_{j-1}(t+1) \tag{21}$$
$$b_j(t+1) = \sqrt{1-q}\ a_{j+1}(t+1) - \sqrt{q}\ b_{j+1}(t+1). \tag{22}$$

Working from here, our research determined that the coefficients $a_j(t)$ and $b_j(t)$ are representable as integral equations in terms of p and q.

$$\begin{aligned} a_j(t) = &\frac{1}{2\pi|\vec{v}_1|^2}\int_{-\pi}^{\pi}\left(\sqrt{q(1-p)}e^{-2is} - \sqrt{(1-q)p}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right)e^{i(\omega_s t/2 + js)}ds \\ &+ \frac{(-1)^j}{2\pi|\vec{v}_2|^2}\int_{-\pi}^{\pi}\left(\sqrt{q(1-p)}e^{-2is} - \sqrt{(1-q)p}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right)e^{i(js - \omega_s t/2)}ds \end{aligned}$$

$$\begin{aligned} b_j(t) = &\frac{1}{2\pi|\vec{v}_1|^2}\int_{-\pi}^{\pi}\left(e^{i\omega_s} - \sqrt{qp}\ e^{-2is} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}\ e^{2is} - \sqrt{(1-q)p}\right)e^{i(\omega_s t/2 + js)}ds \\ &+ \frac{(-1)^j}{2\pi|\vec{v}_2|^2}\int_{-\pi}^{\pi}\left(e^{-i\omega_s} - \sqrt{qp}\ e^{-2is} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}\ e^{2is} - \sqrt{(1-q)p}\right)e^{i(js - \omega_s t/2)}ds \end{aligned}$$

When the parameters are chosen such that $p = q = 1/2$, then the PQ-walk simplifies to the Hadamard walk. We used this fact to verify our calculations and results.

3.2. **Rotations Walk.** In this section, we present the results for the alternating quantum walk with coin operators such that $C_{odd} = (C_{even})^{-1}$. More specifically, we define

$$C_{even} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \tag{23}$$

to be a clockwise rotation, and

$$C_{odd} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{24}$$

to be a counter-clockwise rotation, where θ is a fixed constant and $\theta \in [0, 2\pi]$.

Using the wave function and the definition of a quantum walk, we find recursion formulas for both $a_j(t+1)$ and $b_j(t+1)$ from $a_j(t)$ and $b_j(t)$, respectively, for both coin operators.

For C_{even}, this gives:

$$a_j(t+1) = \cos(\theta)a_{j-1}(t) + \sin(\theta)b_{j-1}(t) \tag{25}$$
$$b_j(t+1) = -\sin(\theta)a_{j+1}(t) + \cos(\theta)b_{j+1}(t) \tag{26}$$

for $j \in \mathbb{Z}$.

Similarly, for C_{odd} we get:

$$a_j(t+1) = \cos(\theta)a_{j-1}(t) - \sin(\theta)b_{j-1}(t) \tag{27}$$
$$b_j(t+1) = \sin(\theta)a_{j+1}(t) + \cos(\theta)b_{j+1}(t) \tag{28}$$

for $j \in \mathbb{Z}$.

Similar to the Hadamard walk, we take the Fourier transform of the recursion formulas in order to be able to express the recursion with a 2×2 matrix. We then diagonalize the matrix to obtain its eigenvalues and eigenvectors and plug these values into (11) to get the formulas for $\hat{a}_t(s)$ and $\hat{b}_t(s)$. Finally, we take the inverse Fourier transform to arrive at final closed-form formulas for $a_j(t)$ and $b_j(t)$.

$$a_j(t) = \frac{1+(-1)^j}{2} \int_{-\pi}^{\pi} \frac{ds}{2\pi} \left(1 - \frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right) e^{i(js+\omega_s t/2)} \tag{29}$$

$$b_j(t) = \frac{1+(-1)^j}{2} \int_{-\pi}^{\pi} \frac{ds}{2\pi} \frac{e^{is}\sin(\theta)}{\sqrt{1-A}} e^{i(js+\omega_s t/2)} \tag{30}$$

As the aforementioned integrals cannot be exactly calculated, we must approximate the integrals asymptotically using the method of stationary phase (refer to Appendix C). Applying this method, we find the following asymptotic expressions:

$$\left.\begin{array}{l} a_j(\alpha t, t) \\ \\ b_j(\alpha t, t) \end{array}\right\} \sim \frac{1+(-1)^{\alpha t}}{\sqrt{2\pi t|\omega''_{s_\alpha}|}} \times \left\{\begin{array}{l} (1+\alpha)\cos\left(\phi(s_\alpha,\alpha)\,t + \pi/4\right) \\ \\ -\alpha\tan(\theta)\cos\left(\phi(s_\alpha,\alpha)\,t+\pi/4\right) \\ -\sqrt{1-\alpha^2\sec^2(\theta)}\sin\left(\phi(s_\alpha,\alpha)\,t+\pi/4\right) \end{array}\right.$$

where we substituted $j = \alpha t$ for $\alpha \in [-1, 1]$.

Therefore, we can calculate the probability of observing the particle at any point $j = \alpha t$. The asymptotic distribution for points $\alpha = j/t$ between $-\cos(\theta) + \epsilon$ and $\cos(\theta) - \epsilon$, for any small constant $\epsilon > 0$ is

$$P(\alpha, t) = \left|a(\alpha t, t)\right|^2 + \left|b(\alpha t, t)\right|^2. \tag{31}$$

3.3. **Alternative Representation of the Hadamard Walk.** Our numerical calculations (see Figure 1) suggest that for $\theta = \frac{\pi}{4}$, the alternating Rotations walk collapses exactly into the Hadamard walk. This was a rather unexpected and surprising finding, so we aim to prove it analytically.

First, we begin by slightly modifying the integral forms for $a_j(t)$ and $b_j(t)$ presented in (15) and (16). In our original calculations for (15) and (16) we used an initial condition of $|{\downarrow}\rangle \otimes |0\rangle$, but for our Rotations walk the initial condition is $|{\uparrow}\rangle \otimes |0\rangle$, so we need to convert (15) and (16) to match the initial condition used in the Rotations walk.

To do this, we first substitute $-j$ for j into both (15) and (16) and then take the conjugate of the entire integral expression. We can do this because the amplitudes are real since the entries in the coin operators are real which implies that $\overline{a_j(t)} = a_j(t)$ and $\overline{b_j(t)} = b_j(t)$. After these modifications, the coefficients of the wave function with initial condition of $|{\uparrow}\rangle \otimes |0\rangle$ for the Hadamard walk become:

$$a_j(t) = \frac{1+(-1)^{j+t}}{2} \int_{-\pi}^{\pi} \frac{ds}{2\pi} \left(1 - \frac{\cos(s)}{\sqrt{1+\cos^2(s)}}\right) e^{i(js+\widetilde{\omega_s}t)} \tag{32}$$

$$b_j(t) = \frac{1+(-1)^{j+t}}{2} \int_{-\pi}^{\pi} \frac{ds}{2\pi} \frac{e^{is}}{\sqrt{1+\cos^2(s)}} e^{i(js+\widetilde{\omega_s}t)} \tag{33}$$

where $\sin(\widetilde{\omega_s}) = \frac{-\sin(s)}{\sqrt{2}}$. We use $\widetilde{\omega_s}$ in place of ω_s to distinguish from the ω_s in (29) and (30).

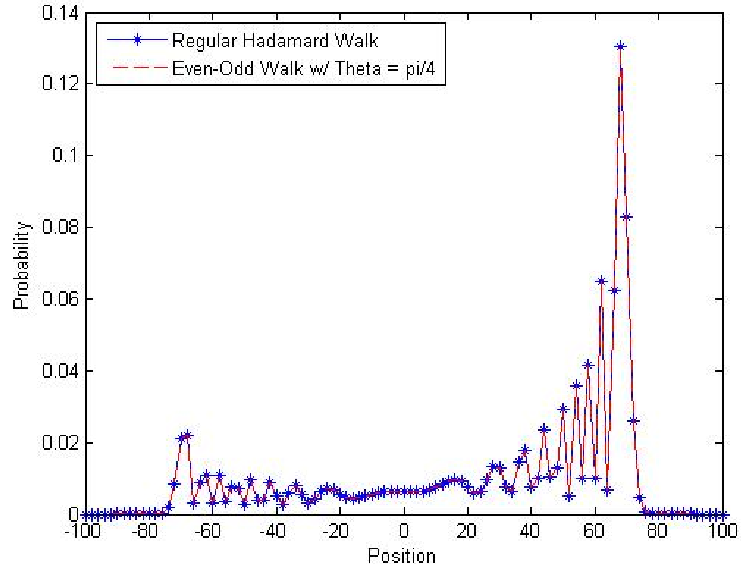


FIGURE 1. The probability distributions of the single-coined Hadamard walk and our double-coined alternating Rotations walks match exactly.

We have defined $\omega_s = 2\arcsin\big(\sin(s)\cos(\theta)\big)$ (see Section 5 for the reasoning behind this choice), so we have $\cos(\omega_s) = 1 - 2\sin^2(s)\cos^2(\theta)$. If we substitute $\theta = \pi/4$ into the expression for ω_s, we get $\cos(\omega_s) = 1 - 2\sin^2(s)\cos^2(\frac{\pi}{4}) = 1 - 2\left(\frac{1}{\sqrt{2}}\right)^2 \sin^2(s) = 1 - \sin^2(s) = \cos^2(s)$. Thus, $\cos(\omega_s) = \cos^2(s)$.

Next, we observe that $\cos(2\widetilde{\omega_s}) = 1 - 2\sin^2(\widetilde{\omega_s}) = 1 - \frac{2\sin^2(s)}{2} = 1 - \sin^2(s) = \cos^2(s)$. Hence, we have that $\cos(\omega_s) = \cos(\widetilde{\omega_s}) \Rightarrow \widetilde{\omega_s} = \frac{\omega_s}{2}$.

Now we substitute $\theta = \pi/4$ into (54) and (55).

$$\begin{aligned}
a_j(t) &= \frac{1+(-1)^j}{2}\int_{-\pi}^{\pi}\frac{ds}{2\pi}\left(1 - \frac{\cos(s)\cos(\frac{\pi}{4})}{\sqrt{1-\sin^2(s)\cos^2(\frac{\pi}{4})}}\right)e^{i(js+\omega_s t/2)} \\
&= \frac{1+(-1)^j}{2}\int_{-\pi}^{\pi}\frac{ds}{2\pi}\left(1 - \frac{\frac{1}{\sqrt{2}}\cos(s)}{\sqrt{1-\frac{1}{2}\sin^2(s)}}\right)e^{i(js+\omega_s t/2)} \\
&= \frac{1+(-1)^j}{2}\int_{-\pi}^{\pi}\frac{ds}{2\pi}\left(1 - \frac{\frac{1}{\sqrt{2}}\cos(s)}{\sqrt{1-\frac{1}{2}\left(1-\cos^2(s)\right)}}\right)e^{i(js+\omega_s t/2)} \\
&= \frac{1+(-1)^j}{2}\int_{-\pi}^{\pi}\frac{ds}{2\pi}\left(1 - \frac{\frac{1}{\sqrt{2}}\cos(s)}{\sqrt{\frac{1}{2}\left(1+\cos^2(s)\right)}}\right)e^{i(js+\omega_s t/2)}
\end{aligned}$$

$$= \frac{1+(-1)^j}{2} \int_{-\pi}^{\pi} \frac{ds}{2\pi} \left(1 - \frac{\cos(s)}{\sqrt{1+\cos^2(s)}}\right) e^{i(js+(\frac{\omega_s}{2})t)}$$

which is exactly (32) when we replace $\frac{\omega_s}{2}$ with $\widetilde{\omega_s}$.

$$\begin{aligned} b_j(t) &= \frac{1+(-1)^j}{2} \int_{-\pi}^{\pi} \frac{ds}{2\pi} \frac{e^{is}\sin(\frac{\pi}{4})}{\sqrt{1-\sin^2(s)\cos^2(\frac{\pi}{4})}} e^{i(js+\omega_s t/2)} \\ &= \frac{1+(-1)^j}{2} \int_{-\pi}^{\pi} \frac{ds}{2\pi} \frac{\frac{1}{\sqrt{2}}e^{is}}{\sqrt{\frac{1}{2}\left(1+\cos^2(s)\right)}} e^{i(js+\omega_s t/2)} \\ &= \frac{1+(-1)^j}{2} \int_{-\pi}^{\pi} \frac{ds}{2\pi} \frac{e^{is}}{\sqrt{1+\cos^2(s)}} e^{i(js+(\frac{\omega_s}{2})t)} \end{aligned}$$

Again. this is exactly (33) after substituting $\frac{\omega_s}{2}$ with $\widetilde{\omega_s}$.

Therefore, we have shown that the Hadamard walk, a walk with one coin operator, can be replicated with two coins when $\theta = \frac{\pi}{4}$ in the Rotations walk.

4. PQ-Walk Analysis

Consider the alternating quantum walk with an even coin defined by (17) and an odd coin defined as (18).

4.1. **Fourier Transformation.** To analyze the results of this walk, we will describe its recursive nature. Let t be even. After taking the Fourier transform of (19) and (20), combine them into the matrix equation

$$\begin{pmatrix} \hat{a}_{t+1}(s) \\ \hat{b}_{t+1}(s) \end{pmatrix} = \underbrace{\begin{pmatrix} \sqrt{p}\ e^{-is} & \sqrt{1-p}\ e^{-is} \\ \sqrt{1-p}\ e^{is} & -\sqrt{p}\ e^{is} \end{pmatrix}}_{M_e} \begin{pmatrix} \hat{a}_t(s) \\ \hat{a}_t(s) \end{pmatrix}. \tag{34}$$

Since $t+1$ is odd, we use the odd recursive equations. Writing the Fourier transforms of (21) and (22) (in the form of matrix multiplication) yields

$$\begin{pmatrix} \hat{a}_{t+2}(s) \\ \hat{b}_{t+2}(s) \end{pmatrix} = \underbrace{\begin{pmatrix} \sqrt{q}\ e^{-is} & \sqrt{1-q}\ e^{-is} \\ \sqrt{1-q}\ e^{is} & -\sqrt{q}\ e^{is} \end{pmatrix}}_{M_o} \begin{pmatrix} \hat{a}_{t+1}(s) \\ \hat{b}_{t+1}(s) \end{pmatrix}. \tag{35}$$

Substituting (34) into (35), a recursive formula appears for any even time $t+2$.

$$\begin{pmatrix} \hat{a}_{t+2}(s) \\ \hat{b}_{t+2}(s) \end{pmatrix} = \begin{pmatrix} \sqrt{q}\ e^{-is} & \sqrt{1-q}\ e^{-is} \\ \sqrt{1-q}\ e^{is} & -\sqrt{q}\ e^{is} \end{pmatrix} \begin{pmatrix} \sqrt{p}\ e^{-is} & \sqrt{1-p}\ e^{-is} \\ \sqrt{1-p}\ e^{is} & -\sqrt{p}\ e^{is} \end{pmatrix} \begin{pmatrix} \hat{a}_t(s) \\ \hat{b}_t(s) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \sqrt{qp}\; e^{-2is} + \sqrt{(1-q)(1-p)} & \sqrt{q(1-p)}e^{-2is} - \sqrt{(1-q)p} \\ \sqrt{(1-q)p} - \sqrt{q(1-p)}\; e^{2is} & \sqrt{(1-q)(1-p)} + \sqrt{qp}\; e^{2is} \end{pmatrix}}_{M} \begin{pmatrix} \hat{a}_t(s) \\ \hat{b}_t(s) \end{pmatrix} \tag{36}$$

This reduces down to an equation for any even time step t from initial conditions $\hat{a}_0(s)$ and $\hat{b}_0(s)$.

$$\begin{pmatrix} \hat{a}_t(s) \\ \hat{b}_t(s) \end{pmatrix} = M^{t/2} \begin{pmatrix} \hat{a}_0(s) \\ \hat{b}_0(s) \end{pmatrix} \tag{37}$$

Now we once again need to diagonalize M to be able to efficiently raise it to a power. For simplicity's sake, we will focus only on the even time values.

4.2. **Diagonalization.** First we solve for eigenvalues and normalized eigenvectors of M. The derivations of both are included in Appendix B. To summarize, the eigenvalues of M are

$$\lambda_{1,2} = \underbrace{\sqrt{qp}\cos(2s) + \sqrt{(1-q)(1-p)}}_{Re(\lambda)} \pm i \underbrace{\sqrt{1 - \left(\sqrt{qp}\cos(2s) + \sqrt{(1-q)(1-p)}\right)^2}}_{Im(\lambda)}.$$

Then the eigenvalues for M can be written as complex numbers in polar form: $e^{i\omega_s}$ and $e^{-i\omega_s}$, where $\cos(\omega_s) = \sqrt{qp}\cos(2s) + \sqrt{(1-q)(1-p)}$. The corresponding eigenvectors are then

$$\vec{v}_{1,2} = \begin{pmatrix} \sqrt{q(1-p)}\; e^{-2is} - \sqrt{(1-q)p} \\ \\ e^{\pm i\omega_s} - \sqrt{qp}\; e^{-2is} - \sqrt{(1-q)(1-p)} \end{pmatrix},$$

which can both be normalized by dividing by their magnitudes, $|\vec{v}_1|$ and $|\vec{v}_2|$, respectively.

$$\left|\vec{v}_{1,2}\right|^2 = 2 - 2\sqrt{qp}\cos(2s)\cos(\omega_s) - 2\sqrt{(1-q)(1-p)}\cos(\omega_s) \pm 2\sqrt{qp}\sin(2s)\sin(\omega_s)$$

Now we may diagonalize the matrix and raise it to the $t/2$-th power.

$$\begin{aligned} M^t \begin{pmatrix} \hat{a}_0(s) \\ \hat{b}_0(s) \end{pmatrix} &= \underbrace{\begin{pmatrix} v_1 & v_2 \\ \vdots & \vdots \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \lambda_1^{t/2} & 0 \\ 0 & \lambda_2^{t/2} \end{pmatrix}}_{D^t} \underbrace{\begin{pmatrix} \bar{v_1} & \cdots \\ \bar{v_2} & \cdots \end{pmatrix}}_{U^*} \begin{pmatrix} \hat{a}_0(s) \\ \hat{b}_0(s) \end{pmatrix} \\ &= UD^{t/2} \begin{pmatrix} \frac{1}{|\vec{v}_1|}\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right) & \frac{1}{|\vec{v}_1|}\left(e^{-i\omega_s} - \sqrt{qp}e^{2is} - \sqrt{(1-q)(1-p)}\right) \\ \frac{1}{|\vec{v}_2|}\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right) & \frac{1}{|\vec{v}_2|}\left(e^{i\omega_s} - \sqrt{qp}e^{2is} - \sqrt{(1-q)(1-p)}\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= U \begin{pmatrix} e^{i\omega_s t/2} & 0 \\ 0 & e^{-i\omega_s t/2} \end{pmatrix} \begin{pmatrix} \frac{1}{|\vec{v}_1|}\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right) \\ \frac{1}{|\vec{v}_2|}\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right) \end{pmatrix} \end{aligned}$$

$$= U \begin{pmatrix} \frac{e^{i\omega_s t/2}}{|\vec{v}_1|}\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right) \\ \\ \frac{e^{-i\omega_s t/2}}{|\vec{v}_2|}\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^{i\omega_s t/2}}{|\vec{v}_1|^2}\left(\sqrt{q(1-p)}e^{-2is} - \sqrt{(1-q)p}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right) \\ +\frac{e^{-i\omega_s t/2}}{|\vec{v}_2|^2}\left(\sqrt{q(1-p)}e^{-2is} - \sqrt{(1-q)p}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right) \\ \\ \frac{e^{i\omega_s t/2}}{|\vec{v}_1|^2}\left(e^{i\omega_s} - \sqrt{qp}e^{-2is} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right) \\ +\frac{e^{-i\omega t/2}}{|\vec{v}_2|^2}\left(e^{-i\omega_s} - \sqrt{qp}e^{-2is} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right) \end{pmatrix}$$

Now we have expressions for $\hat{a}_t(s)$ and $\hat{b}_t(s)$.

$$\hat{a}_t(s) = \left(\frac{e^{i\omega_s t/2}}{|\vec{v}_1|^2} + \frac{e^{-i\omega_s t/2}}{|\vec{v}_2|^2}\right)\left(\sqrt{q(1-p)}e^{-2is} - \sqrt{(1-q)p}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right)$$

$$\begin{aligned}\hat{b}_t(s) &= \frac{e^{i\omega_s t/2}}{|\vec{v}_1|^2}\left(e^{i\omega_s} - \sqrt{qp}e^{-2is} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right) \\ &\quad + \frac{e^{-i\omega t/2}}{|\vec{v}_2|^2}\left(e^{-i\omega_s} - \sqrt{qp}e^{-2is} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right)\end{aligned}$$

4.3. **Inverse Fourier Transform.** Using the inverse Fourier transformation described in (14), we can derive final results for $a_j(t)$ and $b_j(t)$. We also use the substitution $s^* = s + \pi$ from before. This also implies $ds^* = ds$ and $s = s^* - \pi$. We can also derive that

$$\begin{aligned}\omega_s = \omega_{s^*-\pi} &= \sqrt{qp}\cos\left(2\left(s^* - \pi\right)\right) + \sqrt{(1-q)(1-p)} \\ &= \sqrt{qp}\cos\left(2s^* - 2\pi\right) + \sqrt{(1-q)(1-p)} \\ &= \sqrt{qp}\cos\left(2s^*\right) + \sqrt{(1-q)(1-p)} = \omega_{s^*}.\end{aligned}$$

Using these substitutions, perform the following inverse Fourier transforms.

$$\begin{aligned}a_j(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{a}_t(s)e^{ijs}ds \\ &= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\omega_s t/2}}{|\vec{v}_1|^2}\left(\sqrt{q(1-p)}e^{-2is} - \sqrt{(1-q)p}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right)e^{ijs}ds}_{A} \\ &\quad + \frac{1}{2\pi}\int_{-2\pi}^{0} \frac{e^{-i\omega_s t/2}}{|\vec{v}_2|^2}\left(\sqrt{q(1-p)}e^{-2is} - \sqrt{(1-q)p}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right)e^{ijs}ds \\ &= A + \frac{1}{2\pi|\vec{v}_2|^2}\int_{-\pi}^{\pi}\left(\sqrt{q(1-p)}e^{-2i(s^*-\pi)} - \sqrt{(1-q)p}\right)\left(\sqrt{q(1-p)}e^{2i(s^*-\pi)} - \sqrt{(1-q)p}\right)e^{-i\omega_{s^*-\pi}t/2}e^{ij(s^*-\pi)}ds^*\end{aligned}$$

$$= A + \frac{1}{2\pi|\vec{v}_2|^2}\int_{-\pi}^{\pi}\left(\sqrt{q(1-p)}e^{-2is^*} - \sqrt{(1-q)p}\right)\left(\sqrt{q(1-p)}e^{2is^*} - \sqrt{(1-q)p}\right)e^{-i\omega_{s^*}t/2}e^{ijs^*}e^{-i\pi j}ds^*$$
$$= A + \frac{(-1)^j}{2\pi|\vec{v}_2|^2}\int_{-\pi}^{\pi}\left(\sqrt{q(1-p)}e^{-2is^*} - \sqrt{(1-q)p}\right)\left(\sqrt{q(1-p)}e^{2is^*} - \sqrt{(1-q)p}\right)e^{i(js^*-\omega_{s^*}t/2)}ds^*$$
$$= \frac{1}{2\pi|\vec{v}_1|^2}\int_{-\pi}^{\pi}\left(\sqrt{q(1-p)}e^{-2is} - \sqrt{(1-q)p}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right)e^{i(\omega_s t/2+js)}ds$$
$$+ \frac{(-1)^j}{2\pi|\vec{v}_2|^2}\int_{-\pi}^{\pi}\left(\sqrt{q(1-p)}e^{-2is} - \sqrt{(1-q)p}\right)\left(\sqrt{q(1-p)}e^{2is} - \sqrt{(1-q)p}\right)e^{i(js-\omega_s t/2)}ds$$

And on to $b_j(t)$.

$$b_j(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\omega_s t/2}}{|\vec{v}_1|^2}\left(e^{i\omega_s} - \sqrt{qp}\ e^{-2is} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}\ e^{2is} - \sqrt{(1-q)p}\right)e^{ijs}ds$$
$$+ \frac{1}{2\pi}\int_{-2\pi}^{0}\frac{e^{-i\omega t/2}}{|\vec{v}_2|^2}\left(e^{-i\omega_s} - \sqrt{qp}e^{-2is} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}\ e^{2is} - \sqrt{(1-q)p}\right)e^{ijs}ds$$

$$= \underbrace{\frac{1}{2\pi|\vec{v}_1|^2}\int_{-\pi}^{\pi}\left(e^{i\omega_s} - \sqrt{qp}\ e^{-2is} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}\ e^{2is} - \sqrt{(1-q)p}\right)e^{i(\omega_s t/2+js)}ds}_{B}$$
$$+ \frac{1}{2\pi|\vec{v}_2|^2}\int_{-\pi}^{\pi}\left(e^{-i\omega_s} - \sqrt{qp}\ e^{-2is} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}\ e^{2is} - \sqrt{(1-q)p}\right)e^{-i\omega_s t/2}e^{ijs}ds$$

$$= B + \frac{1}{2\pi|\vec{v}_2|^2}\int_{-\pi}^{\pi}\left(e^{i\omega_{s^*}} - \sqrt{qp}\ e^{-2is^*} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}\ e^{2is^*} - \sqrt{(1-q)p}\right)e^{i\omega_{s^*}t/2}e^{ij(s^*-\pi)}ds^*$$

$$= B + \frac{(-1)^j}{2\pi|\vec{v}_2|^2}\int_{-\pi}^{\pi}\left(e^{i\omega_{s^*}} - \sqrt{qp}\ e^{-2is^*} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}\ e^{2is^*} - \sqrt{(1-q)p}\right)e^{i(\omega_{s^*}t/2+js^*)}ds^*$$

$$= \frac{1}{2\pi|\vec{v}_1|^2}\int_{-\pi}^{\pi}\left(e^{i\omega_s} - \sqrt{qp}\ e^{-2is} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}\ e^{2is} - \sqrt{(1-q)p}\right)e^{i(\omega_s t/2+js)}ds$$
$$+ \frac{(-1)^j}{2\pi|\vec{v}_2|^2}\int_{-\pi}^{\pi}\left(e^{-i\omega_s} - \sqrt{qp}\ e^{-2is} - \sqrt{(1-q)(1-p)}\right)\left(\sqrt{q(1-p)}\ e^{2is} - \sqrt{(1-q)p}\right)e^{i(js-\omega_s t/2)}ds$$

5. Rotations Walk Analysis

Recall the alternating Rotations quantum walk with coin operators as defined in (23) and (24).

5.1. **Fourier Transformation.** Now we would like to find the Fourier transform of the recursion formulas for the C_{even} operator by substituting the formulas for $a_j(t+1)$ and $b_j(t+1)$ into (6).

$$\begin{aligned}\hat{a}_{t+1}(s) &= \sum_j a_j(t)e^{-ijs} \\ &= \sum_j \cos(\theta)a_{j-1}(t)e^{-ijs} + \sum_j \sin(\theta)b_{j-1}(t)e^{-ijs} \\ &= \sum_k \cos(\theta)a_k(t)e^{-i(k+1)s} + \sum_k \sin(\theta)b_k(t)e^{-i(k+1)s} && \text{Change of variable: } k = j-1 \\ &= \cos(\theta)e^{-is}\sum_k a_k(t)e^{-iks} + \sin(\theta)e^{-is}\sum_k b_k(t)e^{-iks} \\ &= \cos(\theta)e^{-is}\hat{a}_t(s) + \sin(\theta)e^{-is}\hat{b}_t(s) && \text{Defn. of Fourier transform} \end{aligned} \tag{38}$$

$$\begin{aligned}\hat{b}_{t+1}(s) &= \sum_j b_j(t)e^{-ijs} \\ &= -\sum_j \sin(\theta)a_{j-1}(t)e^{-ijs} + \sum_j \cos(\theta)b_{j-1}(t)e^{-ijs} \\ &= -\sum_k \sin(\theta)a_k(t)e^{-i(k-1)s}a_k(t) + \sum_k \cos(\theta)b_k(t)e^{-i(k-1)s} && \text{Change of variable: } k = j+1 \\ &= -\sin(\theta)e^{is}\sum_k a_k(t)e^{-iks} + \cos(\theta)e^{is}\sum_k b_k(t)e^{-iks} \\ &= -\sin(\theta)e^{is}\hat{a}_t(s) + \cos(\theta)e^{is}\hat{b}_t(s) && \text{Defn. of Fourier transform} \end{aligned} \tag{39}$$

There the recurrence relations for the walk at even times may be written as a product of matrices in the following manner:

$$\begin{pmatrix}\hat{a}_{t+1}(s) \\ \hat{b}_{t+1}(s)\end{pmatrix} = \begin{pmatrix}\cos(\theta)e^{-is} & \sin(\theta)e^{-is} \\ -\sin(\theta)e^{is} & \cos(\theta)e^{is}\end{pmatrix}\begin{pmatrix}\hat{a}_t(s) \\ \hat{b}_t(s)\end{pmatrix}.$$

We define

$$M_e = \begin{pmatrix}\cos(\theta)e^{-is} & \sin(\theta)e^{-is} \\ -\sin(\theta)e^{is} & \cos(\theta)e^{is}\end{pmatrix} \tag{40}$$

to represent the evolution of the walk at even positions.

In a similar manner, using the recurrences obtained for C_{odd}, we define the operator to represent the evolution of the quantum walk at odd positions as:

$$M_o = \begin{pmatrix}\cos(\theta)e^{-is} & -\sin(\theta)e^{-is} \\ \sin(\theta)e^{is} & \cos(\theta)e^{is}\end{pmatrix}. \tag{41}$$

Although we apply the operators in alternating order[1], we may still write them as:

[1]Note that the operators M_e and M_o are noncommutative.

$$M^t = (M_o M_e)^{t/2} \qquad \text{(t even)}$$
$$M^t = M_e (M_o M_e)^{(t-1)/2} \qquad \text{(t odd)}$$

where

$$M_o M_e = \begin{pmatrix} e^{-2is}\cos^2(\theta) + \sin^2(\theta) & \cos(\theta)\sin(\theta)\left(e^{-2is} - 1\right) \\ -\cos(\theta)\sin(\theta)\left(e^{2is} - 1\right) & e^{2is}\cos^2(\theta) + \sin^2(\theta) \end{pmatrix}. \tag{42}$$

Given initial states $\hat{a}_0(s)$ and $\hat{b}_0(s)$ and the matrix $M_o M_e$ as defined above, we define

$$\begin{pmatrix} \hat{a}_t(s) \\ \hat{b}_t(s) \end{pmatrix} = (M_o M_e)^{t/2} \begin{pmatrix} \hat{a}_0(s) \\ \hat{b}_0(s) \end{pmatrix} \tag{43}$$

for even time steps. For convenience let us rename $M := M_o M_e$, so that we can diagonalize M as in (10).

5.2. **Diagonalization.** Once again diagonalizing M as defined in (42) involves finding its eigenvalues and a pair of orthonormal eigenvectors. The full details of finding these formulas may be found in Appendix D.

For the eigenvalues we find

$$\lambda_{1,2} = 1 - 2\sin^2(s)\cos^2(\theta) \pm 2i\sin(s)\cos(\theta)\sqrt{1 - \sin^2(s)\cos^2(\theta)}. \tag{44}$$

Both eigenvalues lie on the unit circle, and are conjugates of each other, so we can take the argument of each eigenvalue. Let $\lambda_1 = e^{i\omega_s}$ and $\lambda_2 = e^{-i\omega_s}$ where ω_s is the angle in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ where $\omega_s = 2\arcsin(\sin(s)\cos(\theta))$ is uniquely defined. Additionally, we have $\cos(\omega_s) = 1 - 2\sin^2(s)\cos^2(\theta)$. Let $A := \sin^2(s)\cos^2(\theta)$, so we have $\cos(\omega_s) = 1 - 2A$. Note the definition of A from now on.

Now, we would like to find the corresponding orthonormal eigenvectors, $\vec{u_1}$ and $\vec{u_2}$, for each eigenvalue. This is accomplished by solving $(M - \lambda I)\vec{v} = 0$ for $\lambda = \lambda_1$ and $\lambda = \lambda_2$ where $v = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$.

First, we find an orthonormal eigenvector for $\lambda_1 = e^{i\omega_s}$. We have

$$\left(\cos^2(\theta)e^{-2is} + \sin^2(\theta) - e^{i\omega_s}\right)\eta_1 + \cos(\theta)\sin(\theta)\left(e^{-2is} - 1\right)\eta_2 = 0. \tag{45}$$

The details of solving (45) to obtain an eigenvector and then normalizing this eigenvector are left to Appendix D. After doing so, we have the following normalized eigenvector for $\lambda_1 = e^{i\omega_s}$.

$$\vec{u_1} := \frac{1}{\sqrt{N_1(s)}} \begin{pmatrix} \cos(\theta)\sin(\theta)\left(e^{-2is} - 1\right) \\ e^{i\omega_s} - \cos^2(\theta)e^{-2is} - \sin^2(\theta) \end{pmatrix} \tag{46}$$

where $N_1(s) := 8A\left(1 - A + \cos(s)\cos(\theta)\sqrt{1 - A}\right)$.

Now find an orthonormal eigenvector for $\lambda_2 = e^{-i\omega_s}$. We obtain the following equation

$$\left(\cos^2(\theta)e^{-2is} + \sin^2(\theta) - e^{-i\omega_s}\right)\eta_1 + \cos(\theta)\sin(\theta)\left(e^{-2is} - 1\right)\eta_2 = 0. \tag{47}$$

Again, it remains to solve (47) for η_2 to find an eigenvector and then normalize it for which we leave the details in Appendix D. To sum up, we get

$$\vec{u_2} := \frac{1}{\sqrt{N_2(s)}} \begin{pmatrix} \cos(\theta)\sin(\theta)\left(e^{-2is}-1\right) \\ e^{-i\omega_s} - \cos^2(\theta)e^{-2is} - \sin^2(\theta) \end{pmatrix}, \tag{48}$$

where $N_2(s) := 8A\left(1 - A - \cos(s)\cos(\theta)\sqrt{1-A}\right)$, as an orthonormal eigenvector for $\lambda_2 = e^{-i\omega_s}$.

5.3. **Finding Formulas for $\hat{a}_t(s)$ and $\hat{b}_t(s)$.** For now, we assume that the initial condition is $|{\uparrow}\rangle \otimes |0\rangle$, which is purely up. From this we know that $a_0(t) = \delta_{0,j}$, so

$$\hat{a}_0(s) = \sum_j \delta_{0,j} e^{-ijs} = \delta_{0,0}e^0 = 1(1) = 1.$$

Likewise, $b_0(t) = 0$ which implies that $\hat{b}_0(s) = 0$.

Therefore, we have that

$$\begin{pmatrix} \hat{a}_0(s) \\ \hat{b}_0(s) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{49}$$

Hence, if we let $U = (\vec{u_1}, \vec{u_2})$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ we have the diagonalization of M. Then, if we substitute (49) into the general formula in (37) and perform the matrix multiplications, then we obtain the expressions presented below.

$$\hat{a}_t(s) = \frac{\left(\cos^2(\theta)\sin^2(\theta)\left(e^{-2is}-1\right)\left(e^{2is}-1\right)\right)}{N_1(s)} e^{i\omega_s t/2} + \frac{\left(\cos^2(\theta)\sin^2(\theta)\left(e^{-2is}-1\right)\left(e^{2is}-1\right)\right)}{N_2(s)} e^{-i\omega_s t/2} \tag{50}$$

$$\hat{b}_t(s) = \frac{\cos(\theta)\sin(\theta)\left(e^{2is}-1\right)\left(e^{i\omega_s} - \cos^2(\theta)e^{-2is} - \sin^2(\theta)\right)}{N_1(s)} e^{i\omega_s t/2} + \frac{\cos(\theta)\sin(\theta)\left(e^{2is}-1\right)\left(e^{-i\omega_s} - \cos^2(\theta)e^{-2is} - \sin^2(\theta)\right)}{N_2(s)} e^{-i\omega_s t/2} \tag{51}$$

The formulas in (50) and (51) can be simplified further simplified to a form that is easier to work with. Since these are longer calculations, we have placed them in Appendix D and only present the final forms of $\hat{a}_t(s)$ and $\hat{b}_t(s)$ here.

$$\hat{a}_t(s) = \frac{1}{2}\left[\left(1 - \frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right) e^{i\omega_s t/2} + \left(1 + \frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right) e^{-i\omega_s t/2}\right] \tag{52}$$

$$\hat{b}_t(s) = \frac{e^{is}\sin(\theta)}{2\sqrt{1-A}}\left(e^{i\omega_s t/2} - e^{-i\omega_s t/2}\right) \tag{53}$$

5.4. **Inverse Fourier Transform.** Our main objective is to find formulas for $a_j(t)$ and $b_j(t)$. This is accomplished by taking inverse Fourier transforms of (52) and (53). We substitute both (52) and (53) into (14) to obtain the following coefficients for the quantum walk at even positions.

$$\begin{aligned} a_j(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{a}_t(s) e^{ijs} ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left[\left(1 - \frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right) e^{i\omega_s t/2} + \left(1 + \frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right) e^{-i\omega_s t/2}\right] e^{ijs} ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(1 - \frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right) e^{i\omega_s t/2}\cdot e^{ijs} ds + \frac{1}{2\pi}\int_{-2\pi}^{0} \frac{1}{2}\left(1 + \frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right) e^{-i\omega_s t/2}\cdot e^{ijs} ds \end{aligned}$$

Now we perform a change of variable. Let $s^* := s + \pi$. Then, we have that $s = s^* - \pi$, $ds^* = ds$, $s^*(-2\pi) = -\pi$, and $s^*(0) = \pi$. Furthermore, it can be shown that $\omega_{s^*-\pi} = -\omega_{s^*}$ and that $\cos(s^* - \pi) = -\cos(s^*)$ and $\sin^2(s^* - \pi) = \sin^2(s^*)$. Using this information the second integral above may be rewritten as:

$$\begin{aligned} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(1 + \frac{\cos(s^*-\pi)\cos(\theta)}{\sqrt{1-\sin^2(s^*-\pi)\cos^2(\theta)}}\right) e^{-i\omega_{s^*-\pi} t/2}\cdot e^{ij(s^*-\pi)} ds^* \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(1 - \frac{\cos(s^*)\cos(\theta)}{\sqrt{1-\sin^2(s^*)\cos^2(\theta)}}\right) e^{i\omega_{s^*} t/2}\cdot e^{ijs^*}\cdot e^{-i\pi j} ds^* \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\cdot(-1)^j \left(1 - \frac{\cos(s^*)\cos(\theta)}{\sqrt{1-\sin^2(s^*)\cos^2(\theta)}}\right) e^{i(js^* + \omega_{s^*} t/2)} ds^* \\ &= \frac{(-1)^j}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(1 - \frac{\cos(s^*)\cos(\theta)}{\sqrt{1-\sin^2(s^*)\cos^2(\theta)}}\right) e^{i(js^* + \omega_{s^*} t/2)} ds^* \end{aligned}$$

At this stage, rename $s := s^*$ in the derivation above. Combine and simplify the two halves of the integral to get the final form of $a_j(t)$.

$$a_j(t) = \frac{1 + (-1)^j}{2}\int_{-\pi}^{\pi} \frac{ds}{2\pi}\left(1 - \frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right) e^{i(js + \omega_s t/2)} \tag{54}$$

Continuing with our calculations, we derive the formula for $b_j(t)$ in a similar manner.

$$\begin{aligned} b_j(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{b}_t(s) e^{ijs} ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{is}\sin(\theta)}{2\sqrt{1-A}}\left(e^{i\omega_s t/2} - e^{-i\omega_s t/2}\right) e^{ijs} ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{is}\sin(\theta)}{2\sqrt{1-A}}\left(e^{i\omega_s t/2}\cdot e^{ijs}\right) ds - \frac{1}{2\pi}\int_{-2\pi}^{0} \frac{e^{is}\sin(\theta)}{2\sqrt{1-A}}\left(e^{-i\omega_s t/2}\cdot e^{ijs}\right) ds \end{aligned}$$

As before let $s^* := s + \pi$ within the second integral from above and reduce the integral in the following way:

$$\begin{aligned}
&= \frac{-1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i(s^*-\pi)}\sin(\theta)}{2\sqrt{1-\sin^2(s^*-\pi)\cos^2(\theta)}} e^{-i\omega_{s^*-\pi}t/2}\cdot e^{ij(s^*-\pi)}ds^* \\
&= \frac{-1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{is^*}\cdot e^{-i\pi}\sin(\theta)}{2\sqrt{1-\sin^2(s^*)\cos^2(\theta)}} e^{i\omega_{s^*}t/2}e^{ijs^*}\cdot e^{-i\pi j}ds^* \\
&= \frac{\cancel{-}1}{2\pi}\int_{-\pi}^{\pi} \frac{\cancel{(-1)}(-1)^j e^{is^*}\sin(\theta)}{2\sqrt{1-\sin^2(s^*)\cos^2(\theta)}} e^{i(js^*+\omega_{s^*}t/2)}ds^* \\
&= \frac{(-1)^j}{2\pi}\int_{-\pi}^{\pi} \frac{e^{is^*}\sin(\theta)}{2\sqrt{1-\sin^2(s^*)\cos^2(\theta)}} e^{i(js^*+\omega_{s^*}t/2)}ds^*
\end{aligned}$$

Once again rename $s := s^*$ in the derivation above and combine the two halves of the integral to get the final formula for $b_j(t)$.

$$b_j(t) = \frac{1+(-1)^j}{2}\int_{-\pi}^{\pi}\frac{ds}{2\pi}\frac{e^{is}\sin(\theta)}{\sqrt{1-A}}e^{i(js+\omega_s t/2)} \tag{55}$$

We observe that for *odd* positions of j, the amplitudes of both $a_j(t)$ and $b_j(t)$ cancel out, as desired.

Thus, we have obtained two integrals that describe the coefficients of the wave function at a position j and time t. Therefore, we calculate the probability of being at j at time t as $P_j(t) = |a_j(t)|^2 + |b_j(t)|^2$.

5.5. **Asymptotic Expansion of $a_j(t)$ and $b_j(t)$.** Using the method of stationary phase, described in Appendix C, we would like to asymptotically expand the integrals (54) and (55) in order to analyze the behavior of the wave function as t tends to $+\infty$. To do this, we consider an integral of the form:

$$I(\alpha, t) = \int_{-\pi}^{\pi}\frac{ds}{2\pi}g(s)e^{i\phi(s,\alpha)t} \tag{56}$$

If we substitute $j = \alpha t$ for $\alpha \in [-1, 1]$ into the expressions in (54) and (55), we can obtain integrals in the form of (56). After doing so we can define[2]

$$\begin{aligned}
I_1(\alpha, t) &:= \frac{1+(-1)^{\alpha t}}{2}\int_{-\pi}^{\pi}\frac{ds}{2\pi}\left(1-\frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right)e^{it(\alpha s+\omega_s/2)} && \text{for } a_j(t) \\
I_2(\alpha, t) &:= \frac{1+(-1)^{\alpha t}}{2}\int_{-\pi}^{\pi}\frac{ds}{2\pi}\frac{e^{is}\sin(\theta)}{\sqrt{1-A}}e^{it(\alpha s+\omega_s/2)} && \text{for } b_j(t)
\end{aligned}$$

Furthermore, we can let

[2]Note that t is defined to be the *total* number of steps taken during the walk.

$$g_1(s) := \left(1 - \frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right),$$
$$g_2(s) := \frac{e^{is}\sin(\theta)}{\sqrt{1-A}}$$

and since both I_1 and I_2 have the same phase term, define

$$\phi(s,\alpha) := \alpha s + \frac{1}{2}\omega_s$$

Initially, we would like to calculate some derivatives which will be useful in our analysis later.

$$\frac{\partial \phi}{\partial s} = \alpha + \frac{1}{2}\omega_s = \alpha + \frac{\cos(\theta)\cos(s)}{\sqrt{1-\sin^2(s)\cos^2(\theta)}}$$

$$\frac{\partial^2 \phi}{\partial s^2} = \frac{1}{2}\omega_s'' = \frac{-\sin(s)\cos(\theta)\sin^2(\theta)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{3/2}}$$

$$\frac{\partial^3 \phi}{\partial s^3} = \frac{1}{2}\omega_s''' = \frac{-\sin^2(\theta)\cos(\theta)\cos(s)\left(1+2\sin^2(s)\cos^2(\theta)\right)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{5/2}}$$

After calculating these derivatives, we need to find an appropriate region in terms of α for which to analyze the asymptotic behavior of the wave function Ψ as $t \to +\infty$. We would like to choose α, so that $I_1(\alpha,t)$ and $I_2(\alpha,t)$ decay faster than any inverse polynomial in t in the region $|\alpha| + \epsilon$, go as $t^{-1/3}$ in the regions around $|\alpha|$, and as $t^{-1/2}$ in the third interval [4].

In order to look for stationary points of order 2 around $|\alpha|$, we would like to have $\frac{\partial \phi}{\partial s} = \frac{\partial^2 \phi}{\partial s^2} = 0$, but $\frac{\partial^3 \phi}{\partial s^3} \neq 0$.

First off, if

$$\frac{\partial^2 \phi}{\partial s^2} = \frac{-\sin(s)\cos(\theta)\sin^2(\theta)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{3/2}} = 0,$$

we must have $\sin(s) = 0$, so $s = 0, \pi$ are the stationary points.

For

$$\begin{aligned}\frac{\partial \phi}{\partial s} &= \alpha + \frac{\cos(\theta)\cos(s)}{\sqrt{1-\sin^2(s)\cos^2(\theta)}} = 0\\ &= \alpha + \frac{\pm\cos(\theta)\sqrt{1-\sin^2(s)}}{\sqrt{1-\sin^2(s)\cos^2(\theta)}} = 0\\ \alpha &= \frac{\mp\cos(\theta)\sqrt{1-\sin^2(s)}}{\sqrt{1-\sin^2(s)\cos^2(\theta)}}\end{aligned}$$

If $s = 0, \pi$, then $\alpha = \pm\cos(\theta)$. However, we must be careful and observe that for ϕ, $s = 0 \Rightarrow \alpha = -\cos(\theta)$ and $s = \pi \Rightarrow \alpha = \cos(\theta)$.

Now we would like to analyze the behaviors of I_1 and I_2 for $|\alpha| = \cos(\theta)$ and $|\alpha| < \cos(\theta) - \epsilon$ for any constant $\epsilon > 0$.

We begin with the points $\alpha = \cos(\theta), -\cos(\theta)$ where ϕ has stationary points of order 2 at $s = 0, \pi$. Therefore we apply the method of stationary phase with $p = 3$ to find the leading terms for I_1 and I_2.

First, we evaluate $\phi^{(3)}(s, \alpha)$ at $s = 0, \pi$ which will be used in the formula for the integral.

$$\phi^{(3)}(0, \alpha) = -\sin^2(\theta)\cos(\theta)$$
$$\phi^{(3)}(\pi, \alpha) = \sin^2(\theta)\cos(\theta).$$

Thus, using the above formulas we get

$$\begin{aligned} I_1(\alpha, t) &\sim \frac{1 + (-1)^{\alpha t}}{2} \frac{g_1(s)}{2\pi} e^{i(\phi(s,\alpha)t \pm \pi/6)} \frac{\Gamma\left(\frac{1}{3}\right)}{3} \left[\frac{3!}{t\left|\phi^{(3)}(s, \alpha)\right|}\right]^{1/3} \\ &\sim \frac{1 + (-1)^{\alpha t}}{2} \frac{g_1(s)}{6\pi} e^{i(\phi(s,\alpha)t \pm \pi/6)} \Gamma\left(\frac{1}{3}\right) \left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3} \end{aligned}$$

and

$$I_2(\alpha, t) \sim \frac{1 + (-1)^{\alpha t}}{2} \frac{g_2(s)}{6\pi} e^{i(\phi(s,\alpha)t \pm \pi/6)} \Gamma\left(\frac{1}{3}\right) \left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3}$$

We also need that for I_1, $g_1(0) = 1 - \cos(\theta)$ and $g_1(\pi) = 1 + \cos(\theta)$. Similarly, for I_2, $g_2(0) = \sin(\theta)$ and $g_2(\pi) = -\sin(\theta)$. Furthermore, we can find that $\phi(0, -\cos(\theta)) = 0$ and $\phi(\pi, \cos(\theta)) = \pi\cos(\theta)$.

Using all of the above information we can finally write more specific for the leading terms in the integrals I_1 and I_2.

$$\begin{aligned} I_1\left(-\cos(\theta), t\right) &\sim \frac{1 + (-1)^{\alpha t}}{2} \frac{g_1(0)}{6\pi} e^{i(\phi(0,-\cos(\theta))t - \pi/6)} \Gamma\left(\frac{1}{3}\right) \left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3} \\ &\sim \frac{1 + (-1)^{\alpha t}}{2} \frac{(1 - \cos(\theta))}{6\pi} e^{-i\pi/6} \Gamma\left(\frac{1}{3}\right) \left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3} \\ &\sim \frac{1 + (-1)^{\alpha t}}{2} \frac{(1 - \cos(\theta))}{6\pi} \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \Gamma\left(\frac{1}{3}\right) \left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3} \end{aligned} \tag{57}$$

$$\begin{aligned} I_1\left(\cos(\theta), t\right) &\sim \frac{1 + (-1)^{\alpha t}}{2} \frac{g_1(\pi)}{6\pi} e^{i(\phi(\pi,\cos(\theta))t + \pi/6)} \Gamma\left(\frac{1}{3}\right) \left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3} \\ &\sim \frac{1 + (-1)^{\alpha t}}{2} \frac{(1 + \cos(\theta))}{6\pi} e^{i(\pi\cos(\theta)t + \pi/6)} \Gamma\left(\frac{1}{3}\right) \left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3} \end{aligned}$$

$$\sim \frac{1+(-1)^{\alpha t}}{2}\frac{(1+\cos(\theta))}{6\pi}\left(\cos\left(\pi\cos(\theta)t+\frac{\pi}{6}\right)+i\sin\left(\pi\cos(\theta)t+\frac{\pi}{6}\right)\right)$$

$$\times\Gamma\left(\frac{1}{3}\right)\left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3} \tag{58}$$

$$I_2\left(-\cos(\theta),t\right)\sim\frac{1+(-1)^{\alpha t}}{2}\frac{g_2(0)}{6\pi}e^{i(\phi(0,-\cos(\theta))t-\pi/6)}\Gamma\left(\frac{1}{3}\right)\left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3}$$

$$\sim\frac{1+(-1)^{\alpha t}}{2}\frac{\sin(\theta)}{6\pi}e^{-i\pi/6}\Gamma\left(\frac{1}{3}\right)\left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3}$$

$$\sim\frac{1+(-1)^{\alpha t}}{2}\frac{\sin(\theta)}{6\pi}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\Gamma\left(\frac{1}{3}\right)\left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3} \tag{59}$$

$$I_2\left(\cos(\theta),t\right)\sim\frac{1+(-1)^{\alpha t}}{2}\frac{g_2(\pi)}{6\pi}e^{i(\phi(\pi,\cos(\theta))t+\pi/6)}\Gamma\left(\frac{1}{3}\right)\left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3}$$

$$\sim\frac{1+(-1)^{\alpha t}}{2}\frac{-\sin(\theta)}{6\pi}e^{i(\pi\cos(\theta)t+\pi/6)}\Gamma\left(\frac{1}{3}\right)\left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3}$$

$$\sim\frac{1+(-1)^{\alpha t}}{2}\frac{-\sin(\theta)}{6\pi}\left(\cos\left(\pi\cos(\theta)t+\frac{\pi}{6}\right)+i\sin\left(\pi\cos(\theta)t+\frac{\pi}{6}\right)\right)$$

$$\times\Gamma\left(\frac{1}{3}\right)\left[\frac{6}{\left|t\sin^2(\theta)\cos(\theta)\right|}\right]^{1/3} \tag{60}$$

Now we turn to the interval of most importance to us, $\left[-\cos(\theta)+\epsilon,\cos(\theta)-\epsilon\right]$. When α lies in this region we would like to have $\frac{\partial\phi}{\partial s}=0$, but $\frac{\partial^2\phi}{\partial s^2}\neq 0$, so each of ϕ has two stationary points in this region.

Recall that

$$\frac{\partial\phi}{\partial s}=\alpha+\frac{\cos(\theta)\cos(s)}{\sqrt{1-\sin^2(s)\cos^2(\theta)}}=0$$

Solving for s gives

$$s_\alpha=\pm\arccos\left(-\sqrt{\frac{\alpha^2\tan^2(\theta)}{1-\alpha^2}}\right)$$

$$s_\alpha=\pm\arccos\left(-\frac{\alpha\tan(\theta)}{\sqrt{1-\alpha^2}}\right). \tag{61}$$

Note that we get two stationary points for each value of alpha, namely s_α and $-s_\alpha$.

Once again, we can employ the method of stationary phase but this time for $p = 2$. In the most general form, we have [3]

$$\begin{aligned} I_{1,2}(\alpha,t) &\sim \cancel{2} \times \frac{1+(-1)^{\alpha t}}{2}\frac{g_{1,2}(s_\alpha)}{2\pi}e^{i\left(\phi(s_\alpha),\alpha)t\pm\pi/4\right)}\left[\frac{2!}{t\left|\phi^{(2)}(s_\alpha,\alpha)\right|}\right]^{1/2}\frac{\Gamma(\frac{1}{2})}{\cancel{2}} \\ &\sim \frac{1+(-1)^{\alpha t}}{2}\frac{\sqrt{2\pi}}{2\pi}\cdot\frac{g_{1,2}(s_\alpha)}{\sqrt{t\left|\omega''_{s_\alpha}\right|}}e^{i\left(\phi(s_\alpha,\alpha)t\pm\pi/4\right)} \\ &\sim \frac{1+(-1)^{\alpha t}}{2}\frac{g_{1,2}(s_\alpha)}{\sqrt{2\pi t\left|\omega''_{s_\alpha}\right|}}\left(\cos\left(\phi(s_\alpha,\alpha)t+\frac{\pi}{4}\right)\pm\sin\left(\phi(s_\alpha,\alpha)t+\frac{\pi}{4}\right)\right) \end{aligned}$$

which can be simplified to

$$I_{1,2}(\alpha,t) \sim \frac{g_{1,2}(s_\alpha)}{\sqrt{2\pi t\left|\omega''_{s_{\alpha_{1,2}}}\right|}} \times \begin{cases} 2\cos\left(\phi(s_\alpha,\alpha)t+\pi/4\right) & \text{if } g \text{ is even} \\ 2i\sin\left(\phi(s_\alpha\alpha)t+\pi/4\right) & \text{if } g \text{ is odd} \end{cases} \tag{62}$$

The phase is

$$\phi(\pm s_\alpha,\alpha) = \pm\alpha s_\alpha + \frac{1}{2}\omega_{s_\alpha}$$

where

$$\begin{aligned} \omega(\pm s_\alpha) &= 2\arcsin\left(\sin(\pm s_\alpha)\cos(\theta)\right) \\ &= 2\arcsin\left(\sin\left(\pm\arccos\left(\frac{-\alpha\tan(\theta)}{\sqrt{1-\alpha^2}}\right)\right)\cos(\theta)\right) \\ &= 2\arcsin\left(\pm\cos(\theta)\sqrt{\frac{1-\alpha^2\sec^2(\theta)}{1-\alpha^2}}\right) \\ &= \pm 2\arcsin\left(\sqrt{\frac{\cos^2(\theta)-\alpha^2}{1-\alpha^2}}\right) \end{aligned} \tag{63}$$

and the second derivative is [4]

$$\begin{aligned} \frac{\partial^2}{\partial s^2}(\pm s_\alpha,\alpha) &= \frac{-\sin(\pm s_\alpha)\sin^2(\theta)\cos(\theta)}{\left(1-\sin^2(\pm s_\alpha)\cos^2(\theta)\right)^{3/2}} \\ &= \mp\sin^2(\theta)\cos(\theta)\sqrt{\frac{1-\alpha^2\sec^2(\theta)}{1-\alpha^2}}\left(\frac{\sin^2(\theta)}{1-\alpha^2}\right)^{-3/2} \end{aligned}$$

[3]Note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

[4]See Appendix D for the expansion of $\sin(s_\alpha)$ and $1-\sin^2(s_\alpha)\cos^2(\theta)$.

$$= \mp \sin^2(\theta)\cos(\theta)\sqrt{\frac{1-\alpha^2\sec^2(\theta)}{1-\alpha^2}}\left(\frac{1-\alpha^2}{\sin^2(\theta)}\right)^{3/2}$$
$$= \mp \sin^2(\theta)\cdot\left(\sin^2(\theta)\right)^{-3/2}\left(1-\alpha^2\right)^{-1/2}\cdot\left(1-\alpha^2\right)^{3/2}\cos(\theta)\sqrt{1-\alpha^2\sec^2(\theta)}$$
$$= \mp\left(\sin(\theta)^2\right)^{-1}\left(1-\alpha^2\right)\cos(\theta)\sqrt{1-\alpha^2\sec^2(\theta)}$$
$$= \frac{\mp(1-\alpha^2)\sqrt{\cos^2(\theta)-\alpha^2}}{\sin(\theta)}$$

Therefore, for $\left|\omega''_{s_\alpha}\right|$ we can just use

$$\left|\omega''_{s_\alpha}\right| = \frac{(1-\alpha^2)\sqrt{\cos^2(\theta)-\alpha^2}}{\left|\sin(\theta)\right|}. \tag{64}$$

Next, we would also like to find $g_1(\pm s_\alpha)$ and $g_2(\pm s_\alpha)$:

$$g_1(\pm s_\alpha) = \left(1 - \frac{\cos(\pm s_\alpha)\cos(\theta)}{\sqrt{1-\sin^2(\pm s_\alpha)\cos^2(\theta)}}\right)$$
$$= \left(1 - \frac{-\alpha\tan(\theta)\cos(\theta)}{\cancel{\sqrt{1-\alpha^2}}}\cdot\frac{\cancel{\sqrt{1-\alpha^2}}}{\sqrt{\sin^2(\theta)}}\right)$$
$$= \left(1 + \frac{\alpha\tan(\theta)\cos(\theta)}{\sin(\theta)}\right)$$
$$= \left(1+\alpha\arctan(\theta)\cdot\tan(\theta)\right)$$
$$= (1+\alpha)$$

Also, we get

$$g_2(\pm s_\alpha) = \frac{e^{\pm i s_\alpha}\sin(\theta)}{\sqrt{1-\sin^2(\pm s_\alpha)\cos^2(\theta)}}$$
$$= e^{\pm i s_{\alpha_1}}\sin(\theta)\left(\frac{\sin^2(\theta)}{1-\alpha^2}\right)^{-1/2}$$
$$= e^{\pm i s_{\alpha_1}}\cancel{\sin(\theta)}\left(\frac{1-\alpha^2}{\cancel{\sin^2(\theta)}}\right)^{1/2}$$
$$= \sqrt{1-\alpha^2}e^{\pm i s_{\alpha_1}}$$
$$= \sqrt{1-\alpha^2}\left(\cos(s_{\alpha_1}) \pm i\sin(s_{\alpha_1})\right)$$
$$= \cancel{\sqrt{1-\alpha^2}}\left(\frac{-\alpha\tan(\theta)}{\cancel{\sqrt{1-\alpha^2}}} \pm i\frac{\sqrt{1-\alpha^2\sec^2(\theta)}}{\cancel{\sqrt{1-\alpha^2}}}\right)$$
$$= -\alpha\tan(\theta) \pm i\sqrt{1-\alpha^2\sec^2(\theta)}$$

Combining the aforementioned calculations, we can write asymptotic expression for $a_j(\alpha t, t)$ and $b_j(\alpha t, t)$, where $a_j(\alpha t, t) := I_1$ and $b_j(\alpha t, t) := I_2$.

$$\left.\begin{array}{c} a_j(\alpha t, t) \\ \\ b_j(\alpha t, t) \end{array}\right\} \sim \frac{1 + (-1)^{\alpha t}}{\sqrt{2\pi t |\omega''_{s_\alpha}|}} \times \left\{\begin{array}{l} (1+\alpha)\cos\left(\phi(s_\alpha, \alpha)\, t + \pi/4\right) \\ \\ -\alpha \tan(\theta)\cos\left(\phi(s_\alpha, \alpha)\, t + \pi/4\right) \\ -\sqrt{1 - \alpha^2 \sec^2(\theta)}\sin\left(\phi(s_\alpha, \alpha)\, t + \pi/4\right) \end{array}\right. \tag{65}$$

Therefore, we can calculate the probability of observing the particle at any point $j = \alpha t$. The asymptotic distribution for points $\alpha = j/t$ between $-\cos(\theta) + \epsilon$ and $\cos(\theta) - \epsilon$, for any small constant $\epsilon > 0$ is given by (31).

6. Future Work

Many aspects of our project could be further generalized. We worked with the same initial state, $|\uparrow\rangle \otimes |0\rangle$, for both our walks, but there are infinitely many initial conditions that could be tested. Similarly, there are many ways to make a coin space-inhomogeneous. For example, we could define a coin that is different at every point on the line. We could also study a walk that is strictly time-inhomogeneous, independent of position, and compare results with similar space-inhomogeneous walks.

However, there is still much to learn about the two specific walks we covered. Time constraints prevented us from finding asymptotic approximations for the PQ-walk. Having accurate approximations could let us better examine key features of the probability distributions for any set of parameters. We found some key features of the Rotation walk, but more could be discovered and justified using our asymptotics. Learning to predict and control the behavior of quantum walks using these variables could make them even more powerful tools in computing and other applications.

The discovery that the Rotations walk could replicate the Hadamard walk raises several poignant questions. Are there other walks we can model using the Rotations walk or the PQ-walk? Furthermore, can *all* homogeneous walks be modeled by inhomogeneous walks? Methods to control walks by adjusting their parameters could lead to insights for answering these questions.

Appendix A. Shift Operator

The shift operator is defined as

$$S = \sum_{j\in\mathbb{Z}} |\uparrow\rangle\langle\uparrow| \otimes |j+1\rangle\langle j| + \sum_{j\in\mathbb{Z}} |\downarrow\rangle\langle\downarrow| \otimes |j-1\rangle\langle j|.$$

This looks rather intimidating at first, so consider an example. Assume that after applying the coin operator we have an expression $|\uparrow\rangle \otimes |1\rangle$. Now apply the shift operator.

$$\begin{aligned} S(H\otimes I)|\psi(t)\rangle &= S(|\uparrow\rangle \otimes |1\rangle) \\ &= \left(\sum_{j\in\mathbb{Z}} |\uparrow\rangle\langle\uparrow| \otimes |j+1\rangle\langle j| + \sum_{j\in\mathbb{Z}} |\downarrow\rangle\langle\downarrow| \otimes |j-1\rangle\langle j|\right)(|\uparrow\rangle \otimes |1\rangle) \end{aligned}$$

$$= \left(\sum_{j\in\mathbb{Z}} |\uparrow\rangle\langle\uparrow| \otimes |j+1\rangle\langle j|\right)(|\uparrow\rangle \otimes |1\rangle) + \left(\sum_{j\in\mathbb{Z}} |\downarrow\rangle\langle\downarrow| \otimes |j-1\rangle\langle j|\right)(|\uparrow\rangle \otimes |1\rangle)$$

One useful property of the tensor product is that for any matrices A, B, C, and D,

$$(A \otimes B)(C \otimes D) = (AC \otimes BD).$$

This identity leads to a useful result:

$$S(H \otimes I)|\psi(t)\rangle = \sum_{j\in\mathbb{Z}} |\uparrow\rangle\langle\uparrow|\uparrow\rangle \otimes |j+1\rangle\langle j|1\rangle + \sum_{j\in\mathbb{Z}} |\downarrow\rangle\langle\downarrow|\uparrow\rangle \otimes |j-1\rangle\langle j|1\rangle.$$

Recall that a bra multiplied by a ket is an inner product. Since $\mathcal{H}_c$ has an orthonormal basis of $\{|\uparrow\rangle, |\downarrow\rangle\}$, we know $\langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 0$ and $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$. The elements of $\mathcal{H}_p$ follow a similar convention. For positions $|u\rangle, |v\rangle \in \mathcal{H}_p$,

$$\langle u|v\rangle = \delta_{u,v} = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v. \end{cases}$$

This means the sum containing the inner product of up and down is zero, canceling out that term completely. Only the first sum containing the inner product of up with itself remains.

$$S(H \otimes I)|\psi(t)\rangle = \sum_{j\in\mathbb{Z}} |\uparrow\rangle \otimes |j+1\rangle\langle j|1\rangle$$

The inner product of j and 1 will always equal 0 if $j \neq 1$, so the only term that doesn't cancel out will be the $j = 1$ term. After substituting $j = 1$, we get

$$\begin{aligned} S(H \otimes I)|\psi(t)\rangle &= |\uparrow\rangle \otimes |2\rangle\langle 1|1\rangle \\ &= |\uparrow\rangle \otimes |2\rangle. \end{aligned}$$

Appendix B. PQ-walk Eigenvalues and Eigenvectors

Find the eigenvalues using the characteristic equation.

$$\begin{aligned} \det(M - \lambda I) = {} & (\sqrt{qp}\, e^{-2is} + \sqrt{(1-q)(1-p)} - \lambda)(\sqrt{(1-q)(1-p)} + \sqrt{qp}\, e^{2is} - \lambda) \\ & - (\sqrt{(1-q)p} - \sqrt{q(1-p)}\, e^{2is})(\sqrt{q(1-p)}\, e^{-2is} - \sqrt{(1-q)p}) \end{aligned}$$

$$\begin{aligned} = {} & \cancel{\sqrt{qp(1-q)(1-p)}\, e^{-2is}} + qp - \sqrt{qp}\, e^{-2is}\lambda + (1-q)(1-p) \\ & + \cancel{\sqrt{qp(1-q)(1-p)}\, e^{2is}} - 2\sqrt{(1-q)(1-p)}\lambda - \sqrt{qp}\, e^{2is}\lambda + \lambda^2 \end{aligned}$$

$$- \cancel{\sqrt{qp(1-q)(1-p)}\ e^{-2is}} + p(1-q) + q(1-p) - \cancel{\sqrt{qp(1-q)(1-p)}\ e^{2is}}$$

$$\begin{aligned} &= \lambda^2 - \left(\sqrt{qp}\ e^{2is} + \sqrt{qp}\ e^{-2is} + 2\sqrt{(1-q)(1-p)}\right)\lambda \\ &\quad + \big(qp + (1-q)(1-p) + p(1-q) + q(1-p)\big) \end{aligned}$$

$$\begin{aligned} &= \lambda^2 - \left(\sqrt{qp}\left(\cos(2s) + \cancel{i\sin(2s)} + \cos(2s) - \cancel{i\sin(2s)}\right) + 2\sqrt{(1-q)(1-p)}\right)\lambda + 1 \\ &= \lambda^2 - 2\left(\sqrt{qp}\cos(2s) + \sqrt{(1-q)(1-p)}\right)\lambda + 1 \end{aligned}$$

Solve for the roots of the characteristic equation.

$$\lambda_{1,2} = \frac{2\left(\sqrt{qp}\cos(2s) + \sqrt{(1-q)(1-p)}\right) \pm \sqrt{4\left(\sqrt{qp}\cos(2s) + \sqrt{(1-q)(1-p)}\right)^2 - 4}}{2}$$

$$\begin{aligned} &= \sqrt{qp}\cos(2s) + \sqrt{(1-q)(1-p)} \pm \sqrt{\left(\sqrt{qp}\cos(2s) + \sqrt{(1-q)(1-p)}\right)^2 - 1} \\ &= \underbrace{\sqrt{qp}\cos(2s) + \sqrt{(1-q)(1-p)}}_{Re(\lambda)} \pm i \underbrace{\sqrt{1 - \left(\sqrt{qp}\cos(2s) + \sqrt{(1-q)(1-p)}\right)^2}}_{Im(\lambda)} \end{aligned}$$

Using these eigenvalues, we can solve for the corresponding eigenvectors $\vec{v_1}, \vec{v_2}$.

$$\left(\sqrt{qp}\ e^{-2is} + \sqrt{(1-q)(1-p)} - e^{\pm i\omega_s}\right)x_1 + \underbrace{\left(\sqrt{q(1-p)}\ e^{-2is} - \sqrt{(1-q)p}\right)}_{\text{Call this } x_1} x_2 = 0$$

Choose x_1 to be the second coefficient, then we can factor and divide x_1 out. The derivation becomes much simpler this way.

$$\begin{aligned} x_1\left(\sqrt{qp}\ e^{-2is} + \sqrt{(1-q)(1-p)} - e^{\pm i\omega_s} + x_2\right) &= 0 \\ \left(\sqrt{qp}\ e^{-2is} + \sqrt{(1-q)(1-p)} - e^{\pm i\omega_s}\right) + x_2 &= 0 \\ x_2 &= e^{\pm i\omega_s} - \sqrt{qp}\ e^{-2is} - \sqrt{(1-q)(1-p)} \end{aligned}$$

This would result in these non-normalized eigenvectors.

$$\vec{v}_{1,2} = \begin{pmatrix} \sqrt{q(1-p)}\ e^{-2is} - \sqrt{(1-q)p} \\ e^{\pm i\omega_s} - \sqrt{qp}\ e^{-2is} - \sqrt{(1-q)(1-p)} \end{pmatrix}$$

In rectangular complex form, that can be written as

$$\vec{v}_{1,2} = \begin{pmatrix} \overbrace{\left(\sqrt{q(1-p)}\ \cos(2s) - \sqrt{(1-q)p}\right)}^{a} - i\overbrace{\left(\sqrt{q(1-p)}\ \sin(2s)\right)}^{b} \\ \underbrace{\left(\cos(\omega_s) - \sqrt{qp}\ \cos(2s) - \sqrt{(1-q)(1-p)}\right)}_{c} + i\underbrace{\left(\sqrt{qp}\ \sin(2s) \pm \sin(\omega_s)\right)}_{d} \end{pmatrix}.$$

So the magnitude squared of the eigenvector is $|\vec{v}|^2 = a^2 + b^2 + c^2 + d^2$.

$$\begin{aligned} \left|\vec{v}_{1,2}\right|^2 &= \left(\sqrt{q(1-p)}\cos(2s) - \sqrt{(1-q)p}\right)^2 + \left(\sqrt{q(1-p)}\sin(2s)\right)^2 \\ &\quad + \left(\cos(\omega_s) - \sqrt{qp}\ \cos(2s) - \sqrt{(1-q)(1-p)}\right)^2 + \left(\sqrt{qp}\sin(2s) \pm \sin(\omega_s)\right)^2 \end{aligned}$$

$$\begin{aligned} &= \overbrace{q(1-p)\cos^2(2s)} - 2\sqrt{qp(1-q)(1-p)}\cos(2s) + p(1-q) + \overbrace{q(1-p)\sin^2(2s)} + \underbrace{\cos^2(\omega_s)} \\ &\quad - 2\sqrt{qp}\cos(2s)\cos(\omega_s) - 2\sqrt{(1-q)(1-p)}\cos(\omega_s) + \overbrace{qp\cos^2(2s)} \\ &\quad + 2\sqrt{qp(1-q)(1-p)}\cos(2s) + (1-q)(1-p) + \overbrace{qp\sin^2(2s)} \\ &\quad \pm 2\sqrt{qp}\sin(2s)\sin(\omega_s) + \underbrace{\sin^2(\omega_s)} \end{aligned}$$

$$\begin{aligned} &= q(1-p) - \cancel{2\sqrt{qp(1-q)(1-p)}\cos(2s)} + p(1-q) + 1 - 2\sqrt{qp}\cos(2s)\cos(\omega_s) \\ &\quad - 2\sqrt{(1-q)(1-p)}\cos(\omega_s) + qp + \cancel{2\sqrt{qp(1-q)(1-p)}\cos(2s)} \\ &\quad + (1-q)(1-p) \pm 2\sqrt{qp}\sin(2s)\sin(\omega_s) \end{aligned}$$

$$= 2 - 2\sqrt{qp}\cos(2s)\cos(\omega_s) - 2\sqrt{(1-q)(1-p)}\cos(\omega_s) \pm 2\sqrt{qp}\sin(2s)\sin(\omega_s)$$

Now plug in our identities for $\cos(\omega_s)$ and $\sin(\omega_s)$.

$$\begin{aligned} \left|\vec{v}_{1,2}\right|^2 &= 2 - 2\sqrt{qp}\cos(2s)\left(\sqrt{pq}\cos(2s) + \sqrt{(1-q)(1-p)}\right) \\ &\quad - 2\sqrt{(1-q)(1-p)}\left(\sqrt{pq}\cos(2s) + \sqrt{(1-q)(1-p)}\right) \pm 2\sqrt{qp}\sin(2s)\sin(\omega_s) \end{aligned}$$

$$\begin{aligned} &= 2 - 2qp\cos^2(2s) + \cancel{2\sqrt{qp(1-q)(1-p)}}\cos(2s) - \cancel{2\sqrt{qp(1-q)(1-p)}} \\ &\quad - 2(1-q)(1-p) \pm 2\sqrt{qp}\sin(2s)\sin(\omega_s) \end{aligned}$$

$$= 2q + 2p - 2qp - 2qp\cos^2(2s) \pm 2\sqrt{qp}\sin(2s)\sqrt{1 - \left(\sqrt{pq}\cos(2s) + \sqrt{(1-q)(1-p)}\right)^2}$$

Appendix C. The Method of Stationary Phase

The integral expressions for our coefficients $a_j(t)$ and $b_j(t)$ clearly cannot be solved by ordinary integration techniques, so we must consider a well-known method which allows for the asymptotic expansion of integrals called the *method of stationary phase* [4].

We consider an integral of the form:

$$I(t) = \int_a^b g(s)e^{it\phi(s)}ds \tag{66}$$

as t tends to infinity. We assume that the exponential term in the integral oscillates rapidly when t is large and if ϕ, called the *phase* of the integral, is not constant in any sub-interval. Also, if $g(s)$ is a smooth function of s, then terms from adjacent sub-intervals will almost cancel each other out, meaning that the major contribution to the value of the integral comes from regions where the oscillations are slow. These regions of slow oscillations occur exactly at the *stationary points*, i.e the points where the phase term is stationary. More precisely, the points c where $\phi'(c) = 0$. Thus, the significant terms in the expansion come from a small interval around the stationary points.

Without loss of generality, we assume that ϕ has exactly one stationary point occurring at the left endpoint of the interval, a. Furthermore, we make the assumption that g is smooth and non-vanishing at a. The *order* of a stationary point, c, corresponds to the last derivative of ϕ in the Taylor expansion of $\phi(s)$ at c which is nonzero at c. Suppose that the order of a is $p-1$, then $\phi'(a) = \phi^{(2)}(a) = \cdots = \phi^{(p-1)}(a) = 0$, but $\phi^{(p)}(a) \neq 0$. Then, the dominant behavior of I is given by

$$I(t) \sim g(s)e^{it\phi(s)\pm i\pi/2p}\left[\frac{p!}{t\left|\phi^{(p)}(a)\right|}\right]^{1/p}\frac{\Gamma(1/p)}{p}, \qquad t \to +\infty, \tag{67}$$

where we use $e^{i\pi/2p}$ $(e^{-i\pi/2p})$ if $\phi^{(p)}(a) > 0$ $(\phi^{(p)}(a) < 0)$.

Appendix D. Alternating Rotation Walk Calculations

D.1. **Eigenvalues and Eigenvectors.** In order to diagonalize the matrix M, we first solve for the eigenvalues of (42). The determinant of the matrix M is 1 and its trace is

$$\begin{aligned}
trace(M) &= e^{-2is}\cos^2(\theta) + \sin^2(\theta) + e^{2is}\cos^2(\theta) + \sin^2(\theta) \\
&= \left(e^{-2is} + e^{2is}\right)\cos^2(\theta) + 2\sin^2(\theta) \\
&= 2\left(\underbrace{\cos(2s)}\cos^2(\theta) + \sin^2(\theta)\right) \\
&= 2\left((-2\sin^2(s) + 1)\cos^2(\theta) + \sin^2(\theta)\right) \\
&= 2\left(-2\sin^2(s)\cos^2(\theta) + \underbrace{\cos^2(\theta) + \sin^2(\theta)}\right) \\
&= 2\left(1 - 2\sin^2(s)\cos^2(\theta)\right).
\end{aligned}$$

Therefore, the characteristic equation is

$$\lambda^2 - 2\left(1 - 2\sin^2(s)\cos^2(\theta)\right)\lambda + 1 = 0.$$

Solving for the roots of the characteristic equation, we get

$$\begin{aligned}
\lambda_{1,2} &= \frac{2\left(1-2\sin^2(s)\cos^2(\theta)\right) \pm \sqrt{\left(-2\left(1-2\sin^2(s)\cos^2(\theta)\right)\right)^2 - 4}}{2} \\
&= \frac{2\left(1-2\sin^2(s)\cos^2(\theta)\right) \pm \sqrt{4\left(\left(1-2\sin^2(s)\cos^2(\theta)\right)^2 - 1\right)}}{2} \\
&= 1 - 2\sin^2(s)\cos^2(\theta) \pm \sqrt{\left(1-2\sin^2(s)\cos^2(\theta)\right)^2 - 1} \\
&= 1 - 2\sin^2(s)\cos^2(\theta) \pm \sqrt{4\sin^4(s)\cos^4(\theta) - 4\sin^2(s)\cos^2(\theta) + \not{1} - \not{1}} \\
&= 1 - 2\sin^2(s)\cos^2(\theta) \pm \sqrt{4\sin^2(s)\cos^2(\theta)\left(\sin^2(s)\cos^2(\theta) - 1\right)} \\
&= 1 - 2\sin^2(s)\cos^2(\theta) \pm 2\sin(s)\cos(\theta)\sqrt{-1\left(1 - \sin^2(s)\cos^2(\theta)\right)} \\
&= 1 - 2\sin^2(s)\cos^2(\theta) \pm 2i\sin(s)\cos(\theta)\sqrt{1 - \sin^2(s)\cos^2(\theta)}.
\end{aligned}$$

If we let $\eta_1 = \cos(\theta)\sin(\theta)\left(e^{-2is} - 1\right)$, solving (45) for η_2 gives $\eta_2 = e^{i\omega_s} - \cos^2(\theta)e^{-2is} - \sin^2(\theta)$. Therefore, we can define

$$\vec{v_1} := \begin{pmatrix} \cos(\theta)\sin(\theta)\left(e^{-2is} - 1\right) \\ e^{i\omega_s} - \cos^2(\theta)e^{-2is} - \sin^2(\theta) \end{pmatrix}.$$

It remains to normalize $\vec{v_1}$.

$$\begin{aligned}
|\vec{v_1}|^2 &= \left|\cos(\theta)\sin(\theta)\left(e^{-2is} - 1\right)\right|^2 + \left|e^{i\omega_s} - \cos^2(\theta)e^{-2is} - \sin^2(\theta)\right|^2 \\
&= 4A\sin^2(\theta) + 4A\left(1 + \cos(2s)\cos^2(\theta) + 2\cos(s)\cos(\theta)\sqrt{1-A}\right) \qquad \text{See Notes 1 and 2 for details.} \\
&= 4A\left(1 + \sin^2(\theta) + \underbrace{\cos(2s)}\cos^2(\theta) + 2\cos(s)\cos(\theta)\sqrt{1-A}\right) \\
&= 4A\left(1 + \sin^2(\theta) + (1 - 2\sin^2(s))\cos^2(\theta) + 2\cos(s)\cos(\theta)\sqrt{1-A}\right) \\
&= 4A\left(1 + \underbrace{\sin^2(\theta) + \cos^2(\theta)} - 2\sin^2(s)\cos^2(\theta) + 2\cos(s)\cos(\theta)\sqrt{1-A}\right) \\
&= 4A\left(2 - 2A + 2\cos(s)\cos(\theta)\sqrt{1-A}\right) \\
&= 8A\left(1 - A + \cos(s)\cos(\theta)\sqrt{1-A}\right)
\end{aligned}$$

Denote $N_1(s) := 8A\left(1 - A + \cos(s)\cos(\theta)\sqrt{1-A}\right)$ as the squared-norm of $\vec{v_1}$.

Once again, let $\eta_1 = \cos(\theta)\sin(\theta)\left(e^{-2is} - 1\right)$ and solve (47) for η_2 to get $\eta_2 = e^{-i\omega_s} - \cos^2(\theta)e^{-2is} - \sin^2(\theta)$. Define:

$$\vec{v_2} := \begin{pmatrix} \cos(\theta)\sin(\theta)\left(e^{-2is} - 1\right) \\ e^{-i\omega_s} - \cos^2(\theta)e^{-2is} - \sin^2(\theta) \end{pmatrix}.$$

Again we need to normalize $\vec{v_2}$.

$$\begin{aligned}
|\vec{v_2}|^2 &= \left|\cos(\theta)\sin(\theta)\left(e^{-2is} - 1\right)\right|^2 + \left|e^{-i\omega_s} - \cos^2(\theta)e^{-2is} - \sin^2(\theta)\right|^2 \\
&= 4A\sin^2(\theta) + 4A\left(1 + \cos(2s)\cos^2(\theta) - 2\cos(s)\cos(\theta)\sqrt{1-A}\right) \qquad \text{See Notes 1 and 2 for details.} \\
&= 4A\left(1 + \sin^2(\theta) + \underbrace{\cos(2s)}\cos^2(\theta) - 2\cos(s)\cos(\theta)\sqrt{1-A}\right) \\
&= 4A\left(1 + \sin^2(\theta) + (1 - 2\sin^2(s))\cos^2(\theta) - 2\cos(s)\cos(\theta)\sqrt{1-A}\right) \\
&= 4A\left(1 + \underbrace{\sin^2(\theta) + \cos^2(\theta)} - 2\sin^2(s)\cos^2(\theta) - 2\cos(s)\cos(\theta)\sqrt{1-A}\right) \\
&= 4A\left(2 - 2A - 2\cos(s)\cos(\theta)\sqrt{1-A}\right) \\
&= 8A\left(1 - A - \cos(s)\cos(\theta)\sqrt{1-A}\right)
\end{aligned}$$

Denote $N_2(s) := 8A\left(1 - A - \cos(s)\cos(\theta)\sqrt{1-A}\right)$.

Note 1: Expansion of $\left|\cos(\theta)\sin(\theta)\left(e^{-2is} - 1\right)\right|^2$**:**

$$\begin{aligned}
\left|\cos(\theta)\sin(\theta)\left(e^{-2is} - 1\right)\right|^2 &= \left|\cos(\theta)\sin(\theta)\left(\left(\cos(2s) - i\sin(2s)\right) - 1\right)\right|^2 \\
&= \left|\left(\cos(\theta)\sin(\theta)\cos(2s) - \cos(\theta)\sin(\theta)\right) - i\left(\cos(\theta)\sin(\theta)\sin(2s)\right)\right|^2 \\
&= \left(\cos(\theta)\sin(\theta)\left(\cos(2s) - 1\right)\right)^2 + \left(\cos(\theta)\sin(\theta)\sin(2s)\right)^2 \\
&= \cos^2(\theta)\sin^2(\theta)\left(\underbrace{\cos^2(2s)} - 2\cos(2s) + 1\right) + \cos^2(\theta)\sin^2(\theta)\underbrace{\sin^2(2s)} \\
&= \cos^2(\theta)\sin^2(\theta)\left(\underbrace{\cos^2(2s) + \sin^2(2s)} - 2\cos(2s) + 1\right) \\
&= \cos^2(\theta)\sin^2(\theta)\left(2 - 2\cos(2s)\right) \\
&= 2\cos^2(\theta)\sin^2(\theta)\left(1 - \underbrace{\cos(2s)}\right) \\
&= 2\cos^2(\theta)\sin^2(\theta)\left(\cancel{1} - \left(\cancel{1} - 2\sin^2(s)\right)\right) \\
&= 4A\sin^2(\theta)
\end{aligned}$$

Note 2: Expansion of $\left|e^{i\omega_s} - \cos^2(\theta)e^{-2is} - \sin^2(\theta)\right|^2$**:**

$$\begin{aligned}
&\left|e^{i\omega_s} - \cos^2(\theta)e^{-2is} - \sin^2(\theta)\right|^2 \\
&\quad= \left|\cos(\omega_s) + i\sin(\omega_s) - \cos^2(\theta)\left(\cos(2s) - i\sin(2s)\right) - \sin^2(\theta)\right|^2 \\
&\quad= \left|\left(\cos(\omega_s) - \cos^2(\theta)\cos(2s) - \sin^2(\theta)\right) + i\left(\sin(\omega_s) + \cos^2(\theta)\sin(2s)\right)\right|^2 \\
&\quad= \left(\underbrace{\cos(\omega_s)} - \cos^2(\theta)\underbrace{\cos(2s)} - \sin^2(\theta)\right)^2 + \left(\sin(\omega_s) + \cos^2(\theta)\sin(2s)\right)^2 \\
&\quad= \left(1 - 2\sin^2(s)\cos^2(\theta) - \cos^2(\theta)\left(1 - 2\sin^2(s)\right) - \sin^2(\theta)\right)^2 + \left(\sin(\omega_s) + \cos^2(\theta)\sin(2s)\right)^2 \\
&\quad= \left(1 - \cancel{2\sin^2(s)\cos^2(\theta)} + \cancel{2\sin^2(s)\cos^2(\theta)} - \left(\underbrace{\cos^2(\theta) + \sin^2(\theta)}\right)\right)^2 \\
&\qquad + \left(\sin(\omega_s) + \cos^2(\theta)\sin(2s)\right)^2 \\
&\quad= (1-1)^2 + \left(\sin(\omega_s) + \cos^2(\theta)\sin(2s)\right)^2 \\
&\quad= \left(\sin(\omega_s) + \cos^2(\theta)\sin(2s)\right)^2 \\
&\quad= \underbrace{\sin^2(\omega_s)} + 2\cos^2(\theta)\sin(2s)\sin(\omega_s) + \cos^4(\theta)\underbrace{\sin^2(2s)} \\
&\quad= \left(2\sin(s)\cos(\theta)\sqrt{1-A}\right)^2 + 2\cos^2(\theta)\sin(2s)\cdot 2\sin(s)\cos(\theta)\sqrt{1-A} + \cos^4(\theta)\sin^2(2s) \\
&\quad= 4\sin^2(s)\cos^2(\theta)(1-A) + 4\cos^3(\theta)\sin(s)\underbrace{\sin(2s)}\sqrt{1-A} + \cos^4(\theta)\underbrace{\sin^2(2s)} \\
&\quad= 4\sin^2(s)\cos^2(\theta)(1-A) + 4\cos^3(\theta)\sin(s)\cdot 2\sin(s)\cos(s)\sqrt{1-A} + 4\cos^4(\theta)\sin^2(s)\cos^2(s) \\
&\quad= 4\sin^2(s)\cos^2(\theta)(1-A) + 8\cos^3(\theta)\sin^2(s)\cos(s)\sqrt{1-A} + 4\cos^4(\theta)\sin^2(s)\cos^2(s) \\
&\quad= 4A\left(1 + \cos^2(\theta)\left(\underbrace{\cos^2(s) - \sin^2(s)}\right) + 2\cos(s)\cos(\theta)\sqrt{1-A}\right) \\
&\quad= 4A\left(1 + \cos(2s)\cos^2(\theta) + 2\cos(s)\cos(\theta)\sqrt{1-A}\right)
\end{aligned}$$

D.2. **Formula Simplification of $\hat{a}_t(s)$ and $\hat{b}_t(s)$.** We begin by reducing the formula for $\hat{a}_t(s)$. First, reduce the first coefficient in the formula.

$$\begin{aligned}
\frac{\left(\cos^2(\theta)\sin^2(\theta)\left(e^{-2is}-1\right)\left(e^{2is}-1\right)\right)}{N_1(s)} &= \frac{1}{N_1(s)}\left(\cos^2(\theta)\sin^2(\theta)\left(e^{-2is}\cdot e^{2is} - e^{2is} - e^{-2is} - 1\right)\right) \\
&= \frac{1}{N_1(s)}\left(\cos^2(\theta)\sin^2(\theta)\left(1 - \left(\underbrace{e^{2is} + e^{-2is}}\right) - 1\right)\right) \\
&= \frac{1}{N_1(s)}\left(\cos^2(\theta)\sin^2(\theta)\left(-2\underbrace{\cos(2s)} + 2\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N_1(s)}\left(\cos^2(\theta)\sin^2(\theta)\left(-2\left(1-2\sin^2(s)\right)+2\right)\right)\\
&= \frac{1}{N_1(s)}\left(\cos^2(\theta)\sin^2(\theta)\left(\cancel{-2}+4\sin^2(s)\cancel{+2}\right)\right)\\
&= \frac{4\sin^2(s)\cos^2(\theta)\sin^2(\theta)}{N_1(s)}\\
&= \frac{4A\sin^2(\theta)}{N_1(s)}\\
&= \frac{4A\sin^2(\theta)}{8A\left(1-A+\cos(s)\cos(\theta)\sqrt{1-A}\right)}\\
&= \frac{\sin^2(\theta)}{2\left(1-A+\cos(s)\cos(\theta)\sqrt{1-A}\right)}\times\frac{\left(1-A-\cos(s)\cos(\theta)\sqrt{1-A}\right)}{\left(1-A-\cos(s)\cos(\theta)\sqrt{1-A}\right)}\\
&= \frac{\cancel{\sin^2(\theta)(1-A)}\left(1-\cos(s)\cos(\theta)(1-A)^{-1/2}\right)}{2\cancel{\sin^2(\theta)(1-A)}}\\
&= \frac{1}{2}\left(1-\frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right)
\end{aligned}
$$

Second, we need to simplify the second coefficient in $\hat{a}_t(s)$. However, since the numerator is the same as in the first coefficient, we can skip directly to the step where:

$$
\begin{aligned}
\frac{\cos^2(\theta)\sin^2(\theta)\left(e^{-2is}-1\right)\left(e^{2is}-1\right)}{N_2(s)} &= \frac{4A\sin^2(\theta)}{N_2(s)}\\
&= \frac{4A\sin^2(\theta)}{8A\left(1-A-\cos(s)\cos(\theta)\sqrt{1-A}\right)}\\
&= \frac{\sin^2(\theta)}{2\left(1-A-\cos(s)\cos(\theta)\sqrt{1-A}\right)}\times\frac{\left(1-A+\cos(s)\cos(\theta)\sqrt{1-A}\right)}{\left(1-A+\cos(s)\cos(\theta)\sqrt{1-A}\right)}\\
&= \frac{\cancel{\sin^2(\theta)(1-A)}\left(1+\cos(s)\cos(\theta)(1-A)^{-1/2}\right)}{2\cancel{\sin^2(\theta)(1-A)}}\\
&= \frac{1}{2}\left(1+\frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right)
\end{aligned}
$$

Therefore, $\hat{a}_t(s)$ is reduced to the more manageable form:

$$
\hat{a}_t(s) = \frac{1}{2}\left[\left(1-\frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right)e^{i\omega_s t/2}+\left(1+\frac{\cos(s)\cos(\theta)}{\sqrt{1-A}}\right)e^{-i\omega_s t/2}\right].
$$

Similarly, we would like to simplify the complicated formula for $\hat{b}_t(s)$ beginning with the first coefficient in $\hat{b}_t(s)$.

$$\frac{\cos(\theta)\sin(\theta)\left(e^{2is}-1\right)\left(e^{i\omega_s}-\cos^2(\theta)e^{-2is}-\sin^2(\theta)\right)}{N_1(s)}$$

$$\begin{aligned}
&= \frac{1}{N_1(s)}\cos(\theta)\sin(\theta)\left(-2ie^{is}\left(\frac{-1}{2}i\left(\underbrace{e^{is}-e^{-is}}\right)\right)\right)\left(e^{i\omega_s}-\cos^2(\theta)e^{-2is}-\sin^2(\theta)\right)\\
&= \frac{-2ie^{is}}{N_1(s)}\sin(s)\cos(\theta)\sin(\theta)\left(e^{i\omega_s}-\cos^2(\theta)e^{-2is}-\sin^2(\theta)\right)\\
&= \frac{-2ie^{is}}{N_1(s)}\sin(s)\cos(\theta)\sin(\theta)\left(\cos(\omega_s)+i\sin(\omega_s)-\cos^2(\theta)\left(\cos(2s)-i\sin(2s)\right)-\sin^2(\theta)\right)\\
&= \frac{-2ie^{is}}{N_1(s)}\sin(s)\cos(\theta)\sin(\theta)\left(\left(\underbrace{\cos(\omega_s)}-\cos^2(\theta)\underbrace{\cos(2s)}-\sin^2(\theta)\right)+i\left(\sin(\omega_s)+\cos^2(\theta)\sin(2s)\right)\right)\\
&= \frac{-2ie^{is}}{N_1(s)}\sin(s)\cos(\theta)\sin(\theta)\left(\left(1-2\sin^2(s)\cos^2(\theta)-\cos^2(\theta)(1-2\sin^2(s))-\sin^2(\theta)\right)\right.\\
&\qquad\qquad \left.+\,i\left(\sin(\omega_s)+\cos^2(\theta)\sin(2s)\right)\right)\\
&= \frac{-2ie^{is}}{N_1(s)}\sin(s)\cos(\theta)\sin(\theta)\left(\left(\cancel{1}-\cancel{2\sin^2(s)\cos^2(\theta)}-\cancel{(\cos^2(\theta)+\sin^2(\theta))}+\cancel{2\sin^2(s)\cos^2(\theta)}\right)\right.\\
&\qquad\qquad \left.+\,i\left(\sin(\omega_s)+\cos^2(\theta)\sin(2s)\right)\right)\\
&= \frac{-2i^2e^{is}}{N_1(s)}\sin(s)\cos(\theta)\sin(\theta)\left(\sin(\omega_s)+\cos^2(\theta)\sin(2s)\right)\\
&= \frac{2e^{is}}{N_1(s)}\sin(s)\cos(\theta)\sin(\theta)\left(\underbrace{\sin(\omega_s)}+\cos^2(\theta)\sin(2s)\right)\\
&= \frac{2e^{is}}{N_1(s)}\sin(s)\cos(\theta)\sin(\theta)\left(2\sin(s)\cos(\theta)\sqrt{1-A}+\cos^2(\theta)\underbrace{\sin(2s)}\right)\\
&= \frac{2e^{is}}{N_1(s)}\sin(s)\cos(\theta)\sin(\theta)\left(2\sin(s)\cos(\theta)\sqrt{1-A}+2\sin(s)\cos(s)\cos^2(\theta)\right)\\
&= \frac{4e^{is}}{N_1(s)}\sin^2(s)\cos^2(\theta)\sin(\theta)\left(\cos(s)\cos(\theta)+\sqrt{1-A}\right)\\
&= \frac{4Ae^{is}}{N_1(s)}\sin(\theta)\left(\cos(s)\cos(\theta)+\sqrt{1-A}\right)\\
&= \frac{4Ae^{is}\sin(\theta)\left(\cos(s)\cos(\theta)+\sqrt{1-A}\right)}{8A\left(1-A+\cos(s)\cos(\theta)\sqrt{1-A}\right)}\\
&= \frac{e^{is}\sin(\theta)\left(\cos(s)\cos(\theta)+\sqrt{1-A}\right)}{2\left(1-A+\cos(s)\cos(\theta)\sqrt{1-A}\right)}\times\frac{\left(1-A-\cos(s)\cos(\theta)\sqrt{1-A}\right)}{\left(1-A-\cos(s)\cos(\theta)\sqrt{1-A}\right)}\\
&= \frac{e^{is}\sin(\theta)\left(\cancel{\cos(s)\cos(\theta)}-\cancel{\cos(s)\cos(\theta)}-\cancel{A\cos(s)}\cos(\theta)+\cancel{A\cos(s)\cos(\theta)}-\sqrt{1-A}\left(1-A-\cos^2(s)\cos^2(\theta)\right)\right)}{2\sin^2(\theta)\left(1-A\right)}
\end{aligned}$$

$$= \frac{e^{is}\sqrt{1-A}\left(1-A-\cos^2(s)\cos^2(\theta)\right)}{2\sin(\theta)\,(1-A)}$$
$$= \frac{e^{is}\sqrt{1-A}\left(1-\sin^2(s)\cos^2(\theta)-\cos^2(s)\cos^2(\theta)\right)}{2\sin(\theta)\,(1-A)}$$
$$= \frac{e^{is}\left(1-\cos^2(\theta)\left(\underbrace{\sin^2(s)+\cos^2(s)}\right)\right)}{2\sin(\theta)\sqrt{1-A}}$$
$$= \frac{e^{is}\left(1-\cos^2(\theta)\right)}{2\sin(\theta)\sqrt{1-A}}$$
$$= \frac{e^{is}\sin^2(\theta)}{2\sin(\theta)\sqrt{1-A}}$$
$$= \frac{e^{is}\sin(\theta)}{2\sqrt{1-A}}$$

Next, it remains to reduce the second coefficient in the formula for $\hat{b}_t(s)$. Given that the difference in the numerator from the first coefficient is $e^{-i\omega_s}$ instead of $e^{i\omega_s}$, the first few steps may be simplified by exchanging $\sin(\omega_s)$ with $-\sin(\omega_s)$. Thus, we arrive at:

$$\frac{\cos(\theta)\sin(\theta)\left(e^{2is}-1\right)\left(e^{-i\omega_s}-\cos^2(\theta)e^{-2is}-\sin^2(\theta)\right)}{N_2(s)}$$

$$= \frac{2e^{is}\sin(s)\cos(\theta)\sin(\theta)\left(-\sin(\omega_s)+\cos^2(\theta)\sin(2s)\right)}{N_2(s)}$$
$$= \frac{2e^{is}\sin(s)\cos(\theta)\sin(\theta)\left(-2\sin(s)\cos(\theta)\sqrt{1-A}+\cos^2(\theta)\underbrace{\sin(2s)}\right)}{N_2(s)}$$
$$= \frac{2e^{is}\sin(s)\cos(\theta)\sin(\theta)\left(-2\sin(s)\cos(\theta)\sqrt{1-A}+2\cos^2(\theta)\sin(s)\cos(s)\right)}{N_2(s)}$$
$$= \frac{4e^{is}\sin^2(s)\cos^2(\theta)\sin(\theta)\left(\cos(s)\cos(\theta)-\sqrt{1-A}\right)}{N_2(s)}$$
$$= \frac{4Ae^{is}\sin(\theta)\left(\cos(s)\cos(\theta)-\sqrt{1-A}\right)}{N_2(s)}$$
$$= \frac{4Ae^{is}\sin(\theta)\left(\cos(s)\cos(\theta)-\sqrt{1-A}\right)}{8A\left(1-A-\cos(s)\cos(\theta)\sqrt{1-A}\right)}$$
$$= \frac{e^{is}\sin(\theta)\left(\cos(s)\cos(\theta)-\sqrt{1-A}\right)}{2\left(1-A-\cos(s)\cos(\theta)\sqrt{1-A}\right)} \times \frac{\left(1-A+\cos(s)\cos(\theta)\sqrt{1-A}\right)}{\left(1-A+\cos(s)\cos(\theta)\sqrt{1-A}\right)}$$
$$= \frac{e^{is}\sin(\theta)\left(\cancel{\cos(s)\cos(\theta)}-\cancel{\cos(s)\cos(\theta)}-\cancel{A\cos(s)}\cos(\theta)+\cancel{A\cos(s)\cos(\theta)}-\sqrt{1-A}\left(1-A-\cos^2(s)\cos^2(\theta)\right)\right)}{2\sin^2(\theta)\,(1-A)}$$

$$= \frac{e^{is}\sqrt{1-A}\left(-1+A+\cos^2(s)\cos^2(\theta)\right)}{2\sin(\theta)\left(1-A\right)}$$
$$= \frac{e^{is}\sqrt{1-A}\left(-1+\sin^2(s)\cos^2(\theta)+\cos^2(s)\cos^2(\theta)\right)}{2\sin(\theta)\left(1-A\right)}$$
$$= \frac{e^{is}\left(-1+\cos^2(\theta)\left(\underbrace{\sin^2(s)+\cos^2(s)}\right)\right)}{2\sin(\theta)\sqrt{1-A}}$$
$$= \frac{e^{is}\left(-1+\cos^2(\theta)\right)}{2\sin(\theta)\sqrt{1-A}}$$
$$= \frac{e^{is}(-\sin^2(\theta))}{2\sin(\theta)\sqrt{1-A}}$$
$$= \frac{-e^{is}\sin(\theta)}{2\sqrt{1-A}}$$

Hence, from the above results, $\hat{b}_t(s)$ is simplified to:

$$\hat{b}_t(s) = \frac{e^{is}\sin(\theta)}{2\sqrt{1-A}}\left(e^{i\omega_s t/2} - e^{-i\omega_s t/2}\right)$$

Note 3: Multiplication of $\left(1-A-\cos(s)\cos(\theta)\sqrt{1-A}\right)\left(1-A+\cos(s)\cos(\theta)\sqrt{1-A}\right)$ used in the rationalization of the denominator.

$$\left(1-A-\cos(s)\cos(\theta)\sqrt{1-A}\right)\left(1-A+\cos(s)\cos(\theta)\sqrt{1-A}\right)$$

$$= 1-A-A+A^2-\cos^2(s)\cos^2(\theta)(1-A)$$
$$= \left((1-A)-A(1-A)-\cos^2(s)\cos^2(\theta)(1-A)\right)$$
$$= (1-A)(1-A-\cos^2(s)\cos^2(\theta))$$
$$= (1-A)(1-\sin^2(s)\cos^2(\theta)-\cos^2(s)\cos^2(\theta)) \qquad \text{Re-substitute A}=\sin^2(s)\cos^2(\theta)$$
$$= (1-A)\left(1-\cos^2(s)\cancel{(\sin^2(\theta)+\cos^2(\theta))}\right)$$
$$= (1-A)(1-\cos^2(\theta))$$
$$= \sin^2(\theta)(1-A)$$

D.3. **Derivatives of ϕ.**

$$\frac{\partial\phi}{\partial s} = \alpha + \frac{1}{2}\omega_s$$
$$= \alpha + \frac{1}{\cancel{2}}\cdot\frac{\partial}{\partial s}\left(\arcsin\left(\cancel{2}\sin(s)\cos(\theta)\right)\right)$$
$$= \alpha + \frac{\partial}{\partial s}\left(\arcsin\left(\sin(s)\cos(\theta)\right)\right)$$

$$
\begin{aligned}
&= \alpha + \frac{\cos(\theta)\cos(s)}{\sqrt{1-\big(\sin(s)\cos(\theta)\big)^2}} \\
&= \alpha + \frac{\cos(\theta)\cos(s)}{\sqrt{1-\sin^2(s)\cos^2(\theta)}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\phi}{\partial s^2} &= \frac{1}{2}\omega_s'' \\
&= \frac{\partial}{\partial s}\left(\frac{\cos(\theta)\cos(s)}{\sqrt{1-\sin^2(s)\cos^2(\theta)}}\right) \\
&= \frac{-\cos(\theta)\sin(s)\left(1-\sin^2(s)\cos^2(\theta)\right)^{1/2} - \frac{1}{2}\cos^3(\theta)\cos^2(s)\left(1-\sin^2(s)\cos^2(\theta)\right)^{-1/2}\left(-2\sin(s)\right)}{1-\sin^2(s)\cos^2(\theta)} \\
&= \frac{-\cos(\theta)\sin(s)\left(1-\sin^2(s)\cos^2(\theta)\right)^{1/2} + \cos^3(\theta)\cos^2(s)\sin^2(s)\left(1-\sin^2(s)\cos^2(\theta)\right)^{-1/2}}{1-\sin^2(s)\cos^2(\theta)} \\
&= \frac{\cos^3(\theta)\cos^2(s)\sin(s)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{3/2}} - \frac{\cos(\theta)\sin(s)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{1/2}} \\
&= \frac{\cos^3(\theta)\cos^2(s)\sin(s) - \cos(\theta)\sin(s)\left(1-\sin^2(s)\cos^2(\theta)\right)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{3/2}} \\
&= \frac{\cos^3(\theta)\cos^2(s)\sin(s) - \cos(\theta)\sin(s) + \cos^3(\theta)\sin^3(s)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{3/2}} \\
&= \frac{\sin(s)\cos(\theta)\left(\cos^2(\theta)\left(\underbrace{\cos^2(s)+\sin^2(s)}\right)-1\right)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{3/2}} \\
&= \frac{\sin(s)\cos(\theta)\left(\cos^2(\theta)-1\right)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{3/2}} \\
&= \frac{-\sin(s)\cos(\theta)\sin^2(\theta)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{3/2}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3\phi}{\partial s^3} &= \frac{1}{2}\omega_s''' \\
&= \frac{\partial}{\partial s}\left(\frac{-\sin(s)\cos(\theta)\sin^2(\theta)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{3/2}}\right) \\
&= \frac{-\sin^2(\theta)\cos(\theta)\cos(s)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{3/2}} - \frac{3\sin^2(\theta)\cos^3(\theta)\sin^2(s)\cos(s)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{5/2}} \\
&= \frac{-\sin^2(\theta)\cos(\theta)\cos(s)\left(1-\sin^2(s)\cos^2(\theta)\right) - 3\sin^2(\theta)\cos^3(\theta)\sin^2(s)\cos(s)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{5/2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-\sin^2(\theta)\cos(\theta)\cos(s) + \sin^2(\theta)\cos^3(\theta)\sin^2(s)\cos(s) - 3\sin^2(\theta)\cos^3(\theta)\sin^2(s)\cos(s)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{5/2}} \\
&= \frac{-\sin^2(\theta)\cos(\theta)\cos(s) - 2\sin^2(\theta)\cos^3(\theta)\sin^2(s)\cos(s)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{5/2}} \\
&= \frac{\sin^2(\theta)\cos(\theta)\cos(s)\left(-1 - 2\sin^2(s)\cos^2(\theta)\right)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{5/2}} \\
&= \frac{-\sin^2(\theta)\cos(\theta)\cos(s)\left(1 + 2\sin^2(s)\cos^2(\theta)\right)}{\left(1-\sin^2(s)\cos^2(\theta)\right)^{5/2}}
\end{aligned}
$$

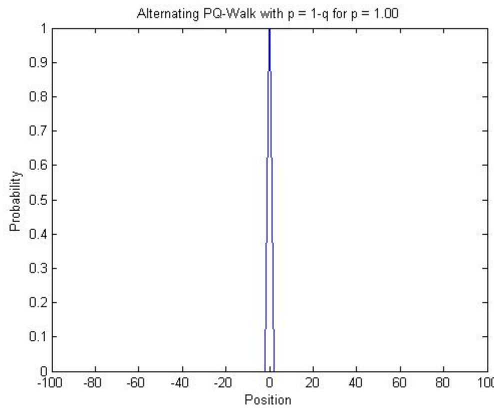
D.4. **Expansion of** $\sin(s_\alpha)$ **and** $1-\sin^2(s_\alpha)\cos^2(\theta)$**:**

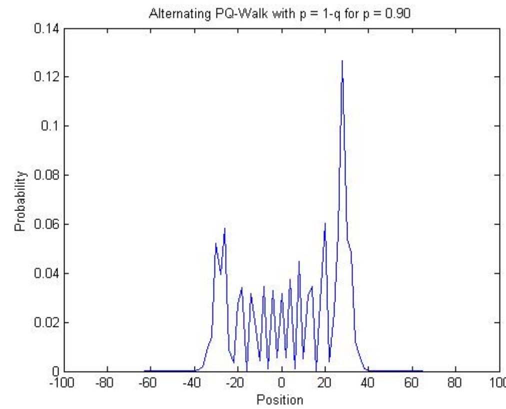
$$
\begin{aligned}
\sin(s_\alpha) &= \sin\left(\pm \arccos\left(\frac{\pm\alpha\tan(\theta)}{\sqrt{1-\alpha^2}}\right)\right) \\
&= \pm\sqrt{1 - \frac{\alpha^2\tan^2(\theta)}{1-\alpha^2}} \\
&= \pm\sqrt{\frac{1-\alpha^2-\alpha^2\tan^2(\theta)}{1-\alpha^2}} \\
&= \pm\sqrt{\frac{-\alpha^2\underbrace{(1+\tan^2(\theta))}+1}{1-\alpha^2}} \\
&= \pm\sqrt{\frac{1-\alpha^2\sec^2(\theta)}{1-\alpha^2}}
\end{aligned}
$$

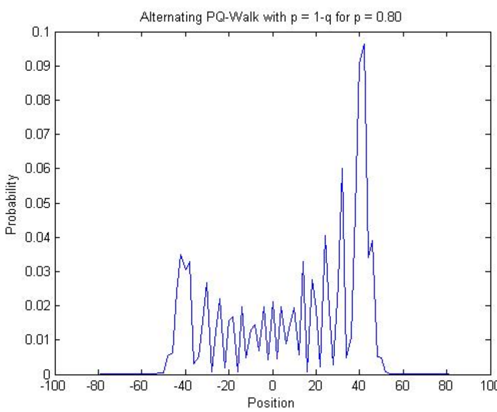
$$
\begin{aligned}
1-\sin^2(s_\alpha)\cos^2(\theta) &= 1-\cos^2(\theta)\left(\pm\sqrt{\frac{1-\alpha^2\sec^2(\theta)}{1-\alpha^2}}\right) \\
&= 1-\cos^2(\theta)\left(\frac{1-\alpha^2\sec^2(\theta)}{1-\alpha^2}\right) \\
&= \frac{1-\cancel{\alpha^2}-\cos^2(\theta)+\cancel{\alpha^2}}{1-\alpha^2} \\
&= \frac{1-\cos^2(\theta)}{1-\alpha^2} \\
&= \frac{\sin^2(\theta)}{1-\alpha^2}
\end{aligned}
$$

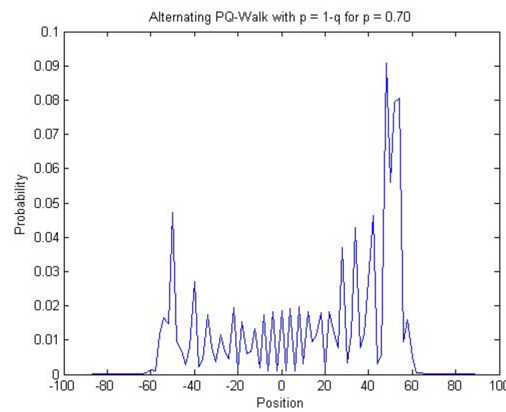
Appendix E. Numerical Results and Graphs

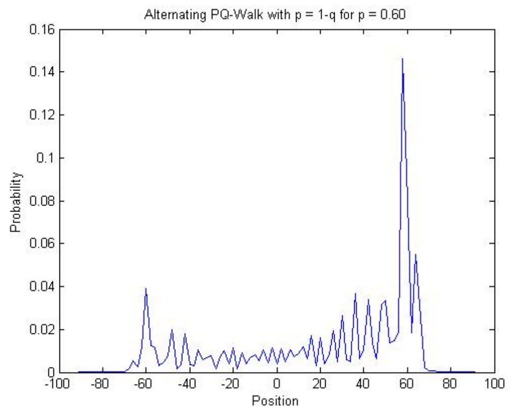
E.1. ***PQ*-Walk Numerical Simulations.** Below we present numerical simulations for 100 time steps for the alternating PQ-walk with initial condition $|\uparrow\rangle \otimes |0\rangle$ with the restriction $p = 1 - q$ with $0 \leq p,\ q \leq 1$. Note that only the points for even positions are graphed, since the probabilities at odd positions are zero.

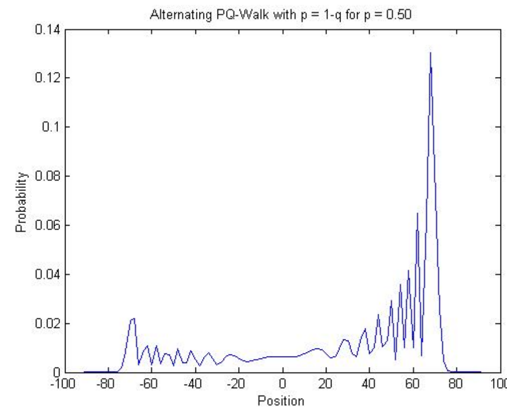


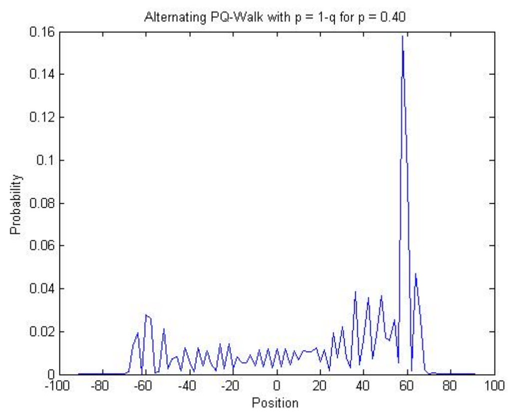


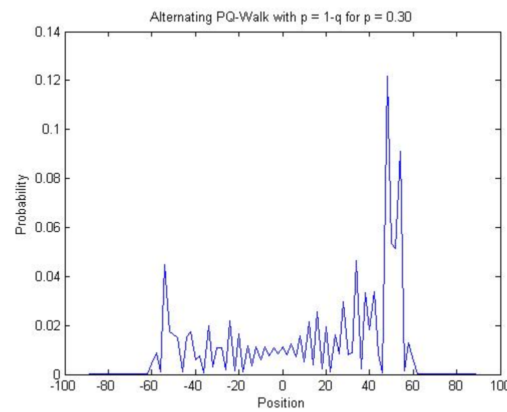


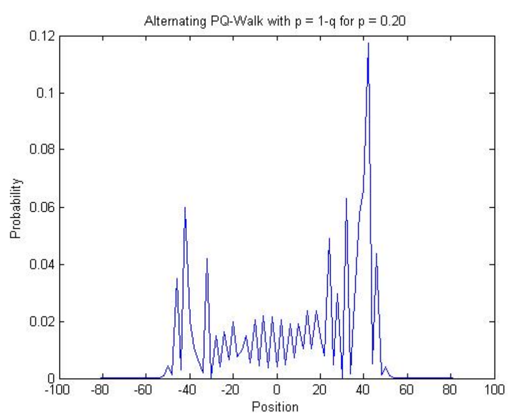


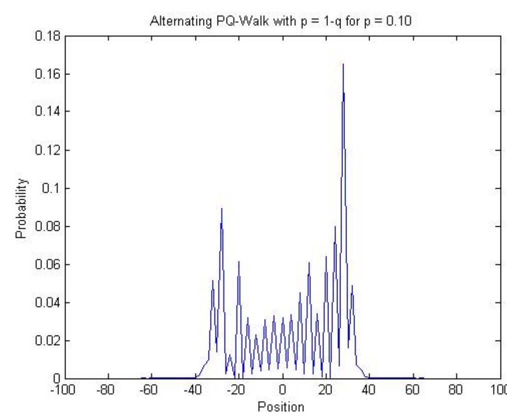






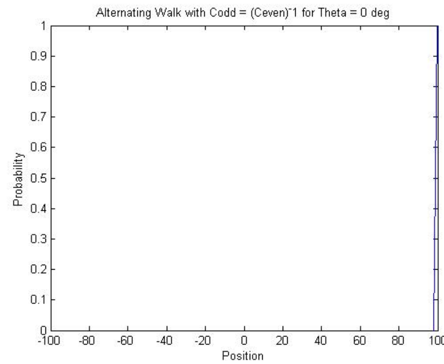


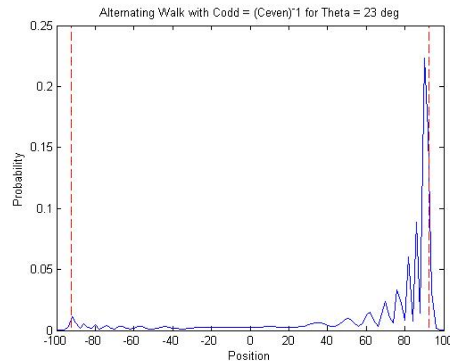


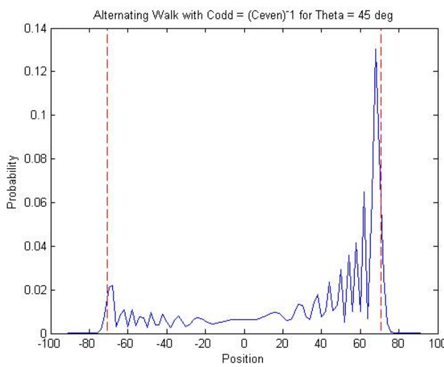


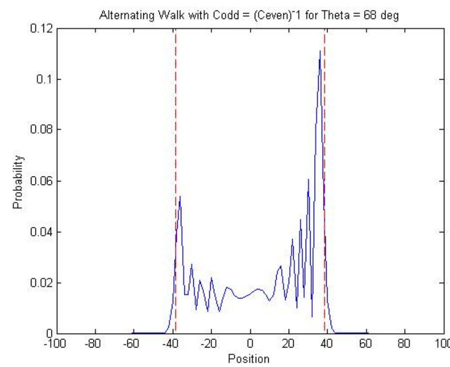
E.2. **Rotations Walk Numerical Simulations.** Below we present numerical simulations for 100 time steps for the alternating Rotations quantum walk with initial condition $|\uparrow\rangle \otimes |0\rangle$. Note that only the points for even positions are graphed as the probabilities at odd positions are zero.

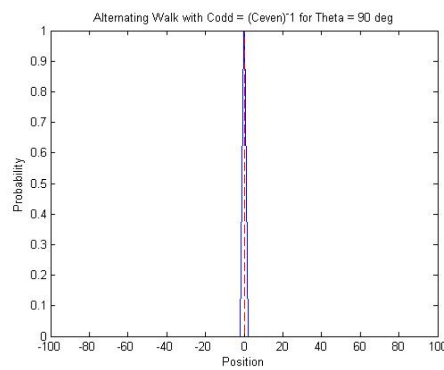
The vertical dashed lines mark the stationary points of order 2 which occur at $\pm\cos(\theta)\,t$.

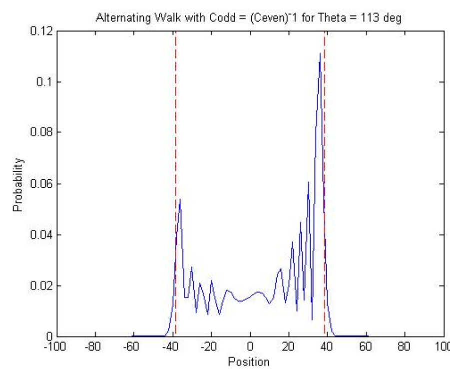

Alternating Walk with Codd = (Ceven)^-1 for Theta = 0 deg
Probability
Position

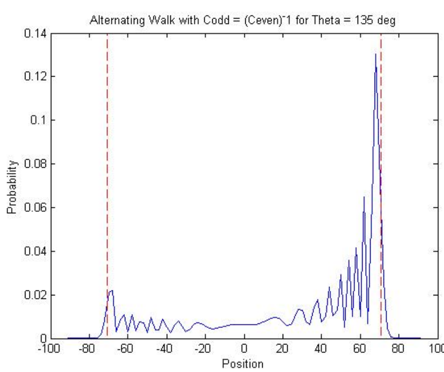

Alternating Walk with Codd = (Ceven)^-1 for Theta = 23 deg
Probability
Position

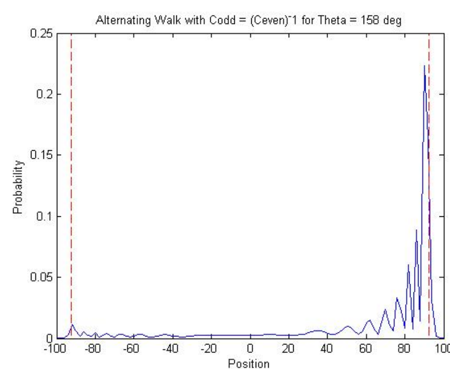

Alternating Walk with Codd = (Ceven)^-1 for Theta = 45 deg
Probability
Position

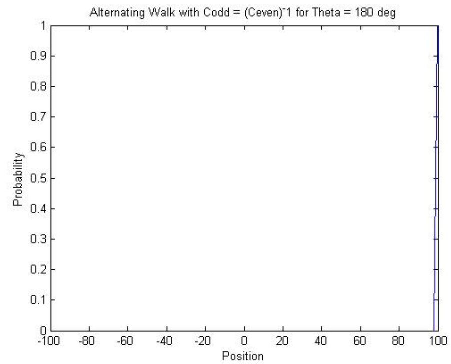

Alternating Walk with Codd = (Ceven)^-1 for Theta = 68 deg
Probability
Position

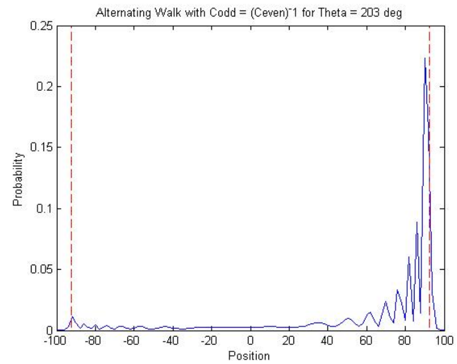

Alternating Walk with Codd = (Ceven)^-1 for Theta = 90 deg
Probability
Position

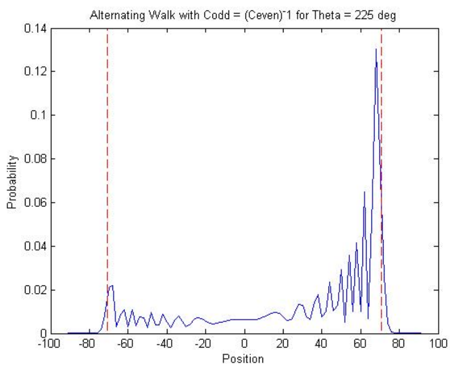

Alternating Walk with Codd = (Ceven)^-1 for Theta = 113 deg
Probability
Position

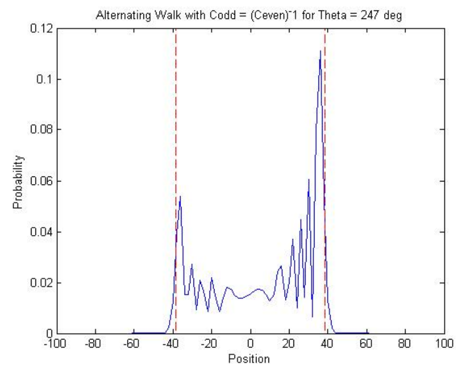

Alternating Walk with Codd = (Ceven)^-1 for Theta = 135 deg
Probability
Position

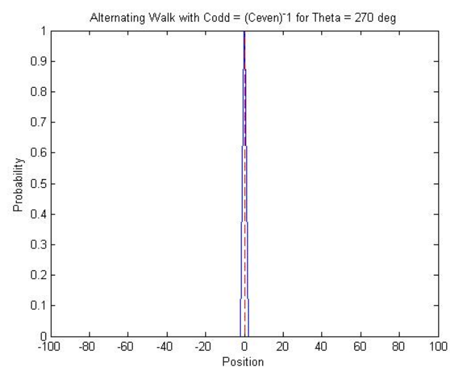

Alternating Walk with Codd = (Ceven)^-1 for Theta = 158 deg
Probability
Position

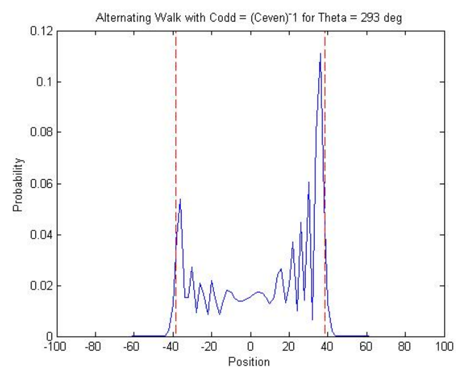

Alternating Walk with Codd = (Ceven)^-1 for Theta = 180 deg
Probability
Position

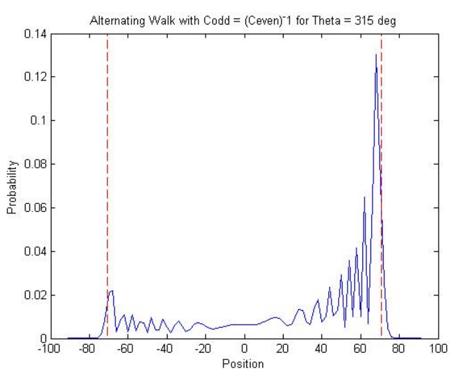

Alternating Walk with Codd = (Ceven)^-1 for Theta = 203 deg
Probability
Position

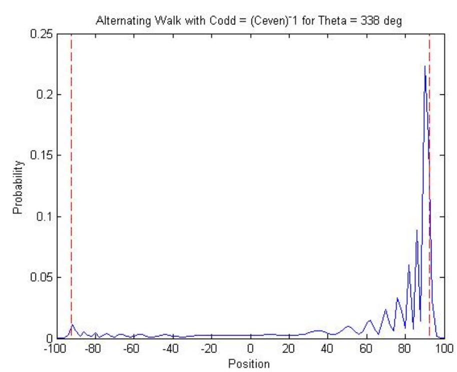

Alternating Walk with Codd = (Ceven)^-1 for Theta = 225 deg
Probability
Position


Alternating Walk with Codd = (Ceven)^-1 for Theta = 247 deg
Probability
Position


Alternating Walk with Codd = (Ceven)^-1 for Theta = 270 deg
Probability
Position


Alternating Walk with Codd = (Ceven)^-1 for Theta = 293 deg
Probability
Position


Alternating Walk with Codd = (Ceven)^-1 for Theta = 315 deg
Probability
Position


Alternating Walk with Codd = (Ceven)^-1 for Theta = 338 deg
Probability
Position

E.3. **MatLab Code.**

E.3.1. *Generating a General Alternating Quantum Walk.* The following code provides a generic algorithm for simulating any alternating quantum walk with initial condition $|\uparrow\rangle \otimes |0\rangle$.

```
%This program simulates an alternating quantum walk.

%N is the total number is steps, C_even and C_odd are 2x2 unitary matrices

function returnVal = AlternateWalkSim(N, C_even, C_odd)

%These are the starting chirality amplitudes of the particle whose
%sum of squares must be equal to 1.
tup = 1;
tdown = 0;

%vertical vectors to store the ampitudes
a = zeros(2 * N + 1, 1);
b = zeros(2 * N + 1, 1);

a(N + 1) = tup;
b(N + 1) = tdown;

dstep = floor(N/2); %number of combined steps

for t = 1:dstep

    %First, do the even step

    % chirality update (action of the coin operator C_even)
    e = zeros(size(a));
    f = zeros(size(b));

    e(1:end) = a(1:end) * C_even(1,1) + b(1:end) * C_even(1,2);
    f(1:end) = a(1:end) * C_even(2,1) + b(1:end) * C_even(2,2);

    % particle movement (action of the shift operator)
    a(2:end) = e(1:end-1);
    b(1:end-1) = f(2:end);

    %Now for the odd step.

    % chirality update (action of the coin operator C_odd)
    e = zeros(size(a));
    f = zeros(size(b));

    e(1:end) = a(1:end) * C_odd(1,1) + b(1:end) * C_odd(1,2);
    f(1:end) = a(1:end) * C_odd(2,1) + b(1:end) * C_odd(2,2);

    % particle movement (action of the shift operator)
    a(2:end) = e(1:end-1);
```

```
    b(1:end-1) = f(2:end);

end

% Square and add the amplitudes to get the probabilities of the
% particle's location regardless of chirality.
amp = zeros(size(a));
amp(1:end) = a(1:end).^2 + b(1:end).^2;

returnVal = amp; % returns a 1-D vector of the probabilities
```

E.3.2. *Generating the PQ-walk.* The program below was used to generate the simulations for the PQ-walk in Appendix E.1.

```
%This program simulates the PQ-Walk in the case where p = 1-q for various
%values of q.

function PQWalkSim(N) % N is the number of time steps

inc = 0.1; % This value may be modified.

for q = 0:inc:1

   p = 1-q; %Definition of p which may be changed.

   C_even = [sqrt(p), sqrt(1-p); sqrt(1-p), -sqrt(p)];
   C_odd = [sqrt(q), sqrt(1-q); sqrt(1-q), -sqrt(q)];

   prob = AlternateWalkSim(N, C_even, C_odd);

   figure
   plot(-N:2:N, prob(1:2:end))% Plot only even points b/c odd points
                              % have probability zero.
   str = sprintf('Alternating PQ-Walk with p = 1-q for p = %.2f', p);
   title(str)
   xlabel('Position')
   ylabel('Probability')

end
```

E.3.3. *Generating the Rotations Walk.* The program below was used to generate the simulations for the Rotations walk in Appendix E.2.

```
%This program simulates the alternating walk where C_odd = (C_even)^-1
%for various values of theta.

function InversesWalkSim(N) % N is the number of time steps

inc = pi/8; % This value may be modified.
```

```
for theta = 0: inc: 2* pi % increments of theta to 2*pi

   C_even = [cos(theta), sin(theta); -sin(theta), cos(theta)];
   C_odd = [cos(theta), -sin(theta); sin(theta), cos(theta)];

   prob = AlternateWalkSim(N, C_even, C_odd);

   figure
   plot(-N:2:N, prob(1:2:end))% Plot only even points b/c odd points
                              % have probability zero.
   x = cos(theta) * N;
   line([-x, -x], ylim, 'Color', 'r', 'LineStyle', '--')
   line([x, x], ylim, 'Color', 'r', 'LineStyle', '--')
   str = sprintf('Alternating Walk with Codd = (Ceven)^-1 for Theta = %d deg',round(radtodeg(theta)));
   title(str) %Note degrees are rounded in the title
   xlabel('Position')
   ylabel('Probability')

end
```

References

[1] Kevin Schwenkler Angelica Deibel and Laura Veith. Inhomogeneous quantum walks. Technical report, Oregon State University, Kidder Hall 368, Oregon State University, Corvallis, OR 97331-4605, 8 2012.

[2] Zlatko Dimcovic. *Discrete-time Quantum Walks via Interchange Framework and Memory in Quantum Evolution*. PhD thesis, Oregon State University, Kidder Hall 368, Oregon State University, Corvallis, OR 97331-4605, 6 2012.

[3] J Kempe. Quantum random walks - an introductory overview. *Contemporary Physics*, 44(4):307–327, 2003.

[4] Ashwin Nayak and Ashvin Vishwanath. Quantum walk on the line. *arXiv preprint quant-ph/0010117*, 2000.

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