Irreducible Plane Curves

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Abstract

Progress in the classification of plane curves in the last five years has centered largely around the work of Arnold, Vassiliev, and Aicardi. The classical index theorem of H. Whitney (1937) classifies curves by index, up to isotopy. Arnold has recently proposed new curve invariants (J^+, J^-, and St) with the aim of finding classifications of curves with the same index, up to ambient diffeomorphisms of the plane and reparametrizations of the curve.

Since these new invariants still do not uniquely determine curves, new ways of classifying curves up to diffeomorphisms have been sought. A certain class of curves (so called "reducible" curves) appears to be classifiable by direct consideration via relatively simple combinatorial methods involving combinations of irreducible curves. This shifts the focus to the classification of the irreducible curves, whose characterization appears to be simplified by association with certain types of planar graphs.

1 Reducibility of Curves

Throughout this article, "ambient diffeomorphisms of the plane and reparametrizations of the curve" may be shortened to "diffeomorphisms." "Distinct" is short for "distinct up to diffeomorphisms."

Definition 1.1. An *immersion of a circle into the plane* is a smooth mapping of a circle into the plane $\gamma : S^1 \to \mathbb{R}^2$ whose derivative never vanishes.

Definition 1.2. A *double point* (respectively, *n-point*) on an immersion of a circle into the plane is a point that is the image of exactly two (respectively, n) points on S^1 under γ. The term *crossing* will be used interchangeably with double point.

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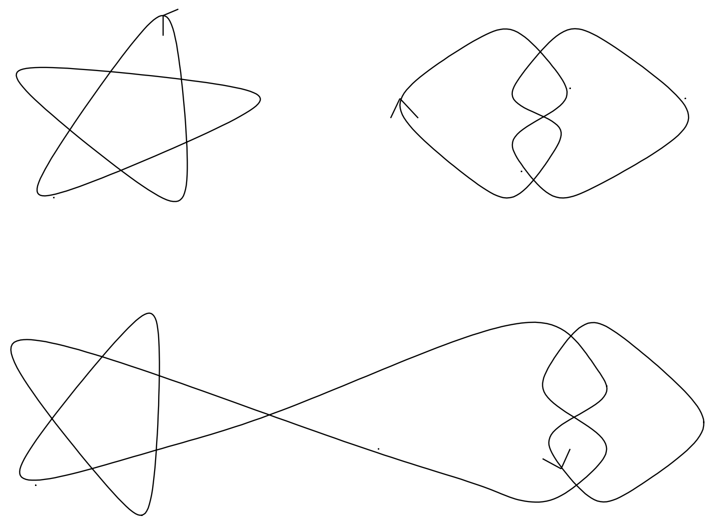


Figure 1: An inverted sum of two curves.

Definition 1.3. A *curve* shall be defined as an immersion of a circle into the plane that contains no self-tangencies and where every n-point is a double point.

Definition 1.4. A *strand* of a curve is a segment of the curve that runs from one crossing to another with no crossings in between. For consistency we also consider the whole image of S^1 to be a strand even though it has no crossings.

Note 1. Let Str_γ denote the number of strands on a curve γ, and R_γ denote the number of regions into which the curve divides the plane. For every n-crossing curve besides S^1, $Str_\gamma = 2n$. Recall that Str_{S^1} is defined to be 1.

Definition 1.5. The *inverted sum* between two strands of two curves is the new curve shown in Figure 1.

Definition 1.6. A *reduction cut* at a crossing on a curve is a surgery and subsequent smoothing that separates the image of the curve into two disjoint, well-defined curves (see Figure 2).

Note that the reduction cut has the opposite effect of the inverted sum procedure (see Figure 2).

Definition 1.7. A *reduction point* is a double point on a curve at which a reduction cut can be performed.

Definition 1.8. A curve is said to be *reducible* if it has a reduction point. That is, there exists some crossing on the curve that divides the image of the curve into two curves with no other points in common. A curve is *completely reducible*

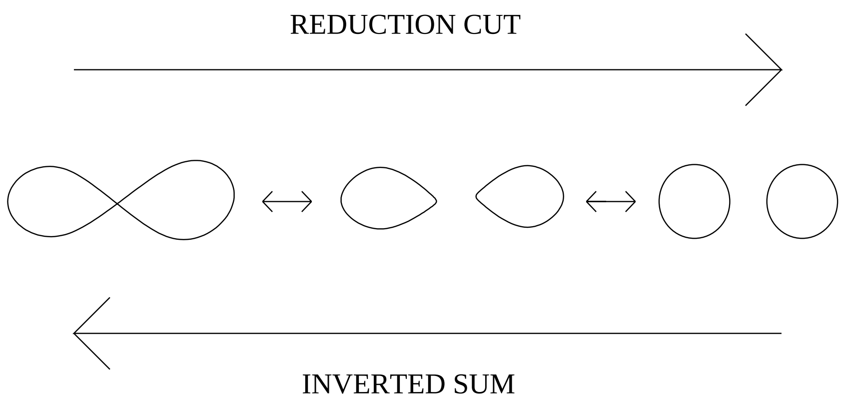


Figure 2: Opposite procedures.

if every crossing on the curve is a reduction point. A curve is *partially reducible* if it has a reduction point and also has a crossing that is not a reduction point. A curve is *irreducible* if it has no reduction points.

Figure 3 shows a list of the irreducible curves with five or fewer crossings. The numbers associated with the curves will be used later to aid in referring to them.

Proposition 1.1. *The inverted sum of two immersions always has a reduction point — the double point created by the inverted sum procedure.*

Proof. Evident, by doing the reduction cut procedure corresponding to whatever inverted sum procedure yielded the inverted sum. □

Note 2. Let γ_1 and γ_2 be two arbitrary curves. After taking any inverted sum of γ_1 and γ_2, any crossings that had been reduction points on γ_1 or γ_2 will be reduction points on the curve resulting from the inverted sum, and any crossings on γ_1 and γ_2 that had not been reduction points will not be reduction points on the resulting curve. The new crossing formed by the inverted sum procedure will by Proposition 1.1 be a reduction point. Similarly, performing a reduction cut procedure at a crossing on a curve cannot change the status of any of the other double points with respect to reducibility.

Proposition 1.2. *Any completely reducible curve can be expressed as a sequence of inverted sums of standard circles. Conversely, every sequence of inverted sums of standard circles yields a completely reducible curve.*

Proof. Consider an arbitrary completely reducible curve with n crossings (hence n reduction points). Perform a reduction cut at every crossing; this yields a set of $n + 1$ circles. Now it is clear that performing the inverted sums that undo the reduction cuts will produce the original curve.

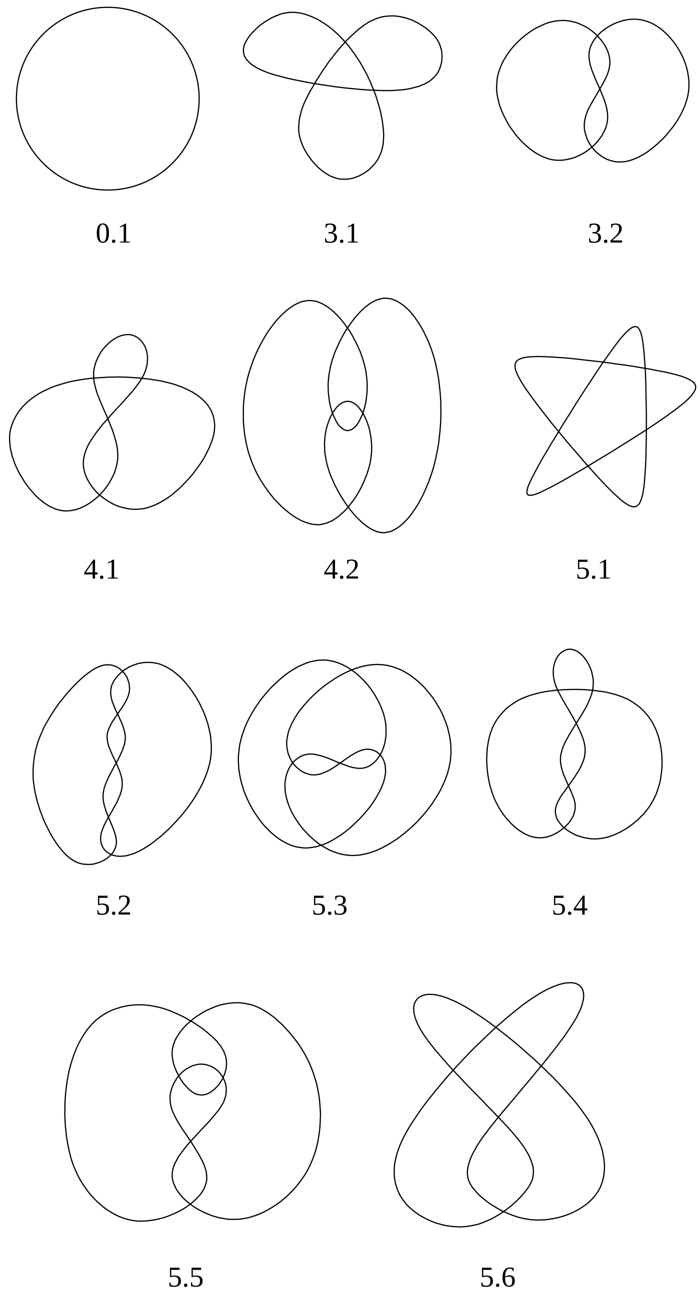


Figure 3: The irreducible curves with $n \leq 5$ double points. The number to the left of the decimal indicates the number of crossings. The number to the right is just an arbitrary indexing number.

For the converse, we proceed by induction: taking one inverted sum between two standard circles can only result in two possible curves, both of which have one crossing and are completely reducible by Proposition 1.1 (see Figure 5). It follows directly from Note 2 that an inverted sum of any two completely reducible curves must yield a completely reducible curve. By induction, every sequence of inverted sums of standard circles must produce a completely reducible curve. □

Corollary 1.3. *Any sequence of inverted sums of completely reducible curves must result in a completely reducible curve.*

Proof. Evident, via the theorem. □

Proposition 1.4. *Every reducible curve can be expressed as a sequence of inverted sums of irreducible curves; there is no sequence of inverted sums of curves that yields an irreducible curve.*

Proof. Consider an n-crossing reducible curve with m reduction points. Perform the reduction cut at every reduction point; this must yield $m+1$ curves, and by Note 2, no crossings on any of these $m+1$ curves can be reduction points, so each of the curves must be irreducible. Now it is clear that performing the inverted sums that undo the reduction cuts we made before will produce the original curve.

Consider an arbitrary sequence of inverted sums of curves. By Proposition 1.1 and Note 2, the resulting curve must have a reduction point, hence it is reducible. □

Proposition 1.5. *Any partially reducible curve can be expressed as a sequence of inverted sums of curves where at least one of the factor curves is not completely reducible. Conversely, every sequence of inverted sums of curves where at least one of the factor curves is not completely reducible yields a partially reducible curve.*

Proof. Consider an arbitrary partially reducible curve. Since it has a reduction point it can be reduced into two curves, so it is clear that it is the result of some inverted sum of curves. The question is whether it is necessary that one of the factor curves not be completely reducible. Corollary 1.3 shows that it is.

For the converse, consider an arbitrary sequence of inverted sums of curves where at least one of the factor curves is not completely reducible. By Proposition 1.4, this sequence of sums cannot yield an irreducible curve. Note 2 shows additionally that the resulting curve cannot be completely reducible. The resulting curve is therefore partially reducible. □

2 Irreducible Curves as Building Blocks

We would like to be able to write every reducible curve as a set of n irreducible curves together with $n-1$ pairs of strands from those curves that indicate

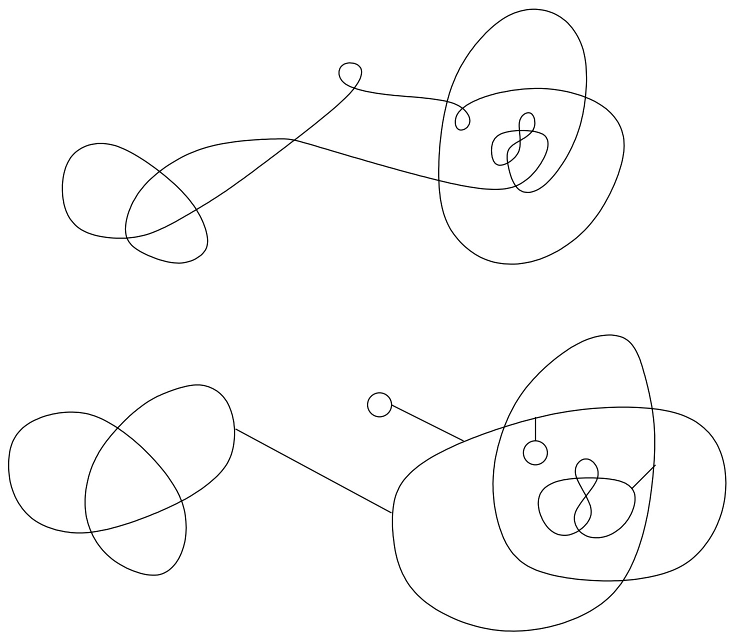


Figure 4: The decomposition of a reducible curve.

which inverted sums would need to be taken to reconstruct the reducible curve in question. Figure 4 shows a complicated reducible curve reduced down to its constituent irreducible curves and recipe of connected sums. But is this decomposition unique? That is, is there any other set of irreducible curves or recipe of inverted sums that would yield the same reducible curve? The following theorem shows there is not.

Theorem 2.1. *Upon decomposition, every reducible curve gives rise to a unique set of irreducible curves together with a unique set of inverted sums among those curves that yields the curve in question.*

Proof. Consider an arbitrary reducible curve with m reduction points. Recall from Note 2 that performing the reduction cut at these reduction points doesn't change the status of any of the other double points with respect to reducibility. Thus after performing the m possible reduction cuts, we will have $m + 1$ irreducible curves, as well as a set of m reduction cuts to repair with m specific inverted sums. Since these reduction cuts can only be made in one way, the irreducible factor curves are uniquely determined; the inverted sums are uniquely determined as the reverse procedures of the reduction cuts. This proves the theorem. $\square$

Assuming the above theorems, if one were to know all the irreducible curves with $n < k$ crossings, then the problem of constructing all of the curves with

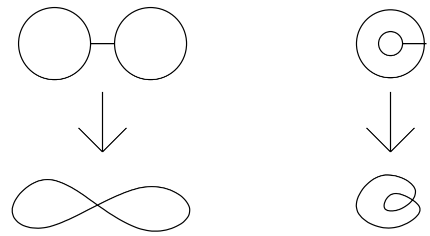


Figure 5: All the distinct inverted sums of two copies of S^1 yields all the distinct curves with one double point.

$n \leq k$ crossings comes down to more basic combinatorics. One would try to find the distinct combinations of inverted sums involving irreducible curves with $n < k$ crossings; this should produce all the reducible curves with $n \leq k$ crossings. As a simple example, we can derive all the 1-crossing curves by taking all the possible inverted sums of two copies of S^1. Since each circle has only one strand, there are only two distinct ways of taking their inverted sum (see Figure 5).

Definition 2.1. A reducible curve is *n-reducible* if it has n reduction points. Note that such a curve has exactly $n + 1$ irreducible factor curves.

Definition 2.2. The *inverted sum set (ISS)* of two irreducible curves (denoted $ISS(\gamma_1, \gamma_2)$) is the set of all distinct curves that can result from any inverted sum between γ_1 and γ_2. For example, Figure 5 shows $ISS(S^1, S^1)$.

Proposition 2.2. *We can put an upper bound on the cardinality of $ISS(\gamma_1, \gamma_2)$, where γ_1 is an m-crossing curve and γ_2 is an n-crossing curve, by*

$$Card(ISS(\gamma_1, \gamma_2)) \leq Str_{\gamma_1} \cdot Str_{\gamma_2} \cdot (m + n + 3)$$

and, provided that neither γ_1 nor γ_2 is S^1,

$$Card(ISS(\gamma_1, \gamma_2)) \leq 4mn(m + n + 3)$$

Proof. The formula derives easily by considering the possible cases: first consider the case where the two curves are unnested (i.e., one in the left half-plane and the other in the right half-plane). The worst case scenario would be for neither curve to have any kind of symmetry, forcing every strand to be considered individually. Then every unique pair of strands, one from γ_1 and the other from γ_2, could have an associated inverted sum between the two curves and every one of those inverted sums could result in a distinct curve. So far this gives $Str_{\gamma_1} Str_{\gamma_2}$ as an upper bound. Now consider the cases where one curve is nested inside one of the bounded regions of the plane delineated by the other. The worst case again gives $Str_{\gamma_1} Str_{\gamma_2}$ as the upper bound. There are an additional $n + 1$ cases for γ_1 nested in γ_2, and $m + 1$ cases for γ_2 nested in γ_1. The total cases are $(n + 1) + (m + 1) + 1 = m + n + 3$, which is exhaustive. $\square$

Remark 2.1. This upper bound is very generous and there are probably very simple considerations that could be taken into account to make it tighter. For example, $Card(ISS(S^1, 3.1))$, where 3.1 is the three-crossing trefoil curve, is given an upper bound of 36 by the formula, when the actual number is only 5. It is nice, however, to be able to confidently give an upper bound on the number of 1-reducible 6-crossing curves. The only inverted sum sets that yield such curves are sets where one curve has one crossing and the other has five, or where both curves have three crossings (there are no 2-crossing irreducible curves to pair with any of the 4-crossing irreducibles). The above formula gives 1452 as the maximum possible number of such curves. Similar methods could be applied to find upper bounds for 2-reducible curves and beyond, which, together with an upper bound on the number of irreducibles, would yield upper bounds on the total number of curves with a given number of crossings.

Note how, in Figure 5, the symmetry of the two curves (in this case, the "symmetry" is the fact that the two factor curves are identical) has made one of the possible three cases equivalent — the inverted sum with circle A inside of circle B and the inverted sum with circle B inside of circle A are equivalent cases. In general, any symmetry within either of the curves involved or between the two curves seems to reduce the number of distinct inverted sums. The fact that many curves have several symmetries is another reason the upper bound given above is so generous.

When we can identify how many symmetrically distinct regions irreducible curves have and how many symmetrically distinct strands border each region (this seems very easy, at least for curves with low numbers of crossings), we can give an exact number for the cardinality of any ISS involving these curves.

Notation. For any irreducible curve γ, let U be the unbounded region of the plane delineated by γ and label the rest of the symmetrically distinct regions $A, B, C, \cdots$. Let u be the number of symmetrically distinct strands bordering U, and $a, b, c \cdots$ be the number of symmetrically distinct strands bordering $A, B, C, \cdots$, respectively.

Proposition 2.3. *The cardinality of $ISS(\gamma_1, \gamma_2)$ is given by*

$$Card(ISS(\gamma_1, \gamma_2)) = u_1 u_2 + u_1(a_2 + b_2 + \cdots + j_2) + u_2(a_1 + b_1 + \cdots + k_1)$$

if γ_1 and γ_2 are different curves, and

$$Card(ISS(\gamma_1, \gamma_2)) = ({u_1}^2 + u_1)/2 + u_1(a_2 + b_2 + \cdots + j_2)$$

if the two curves are identical.

Proof. This proof follows the proof of Proposition 2.2, reducing the number of regions and strands to the symmetrically distinct ones. The arguments are analogous except where noted. The $u_1 u_2$ term derives from the case where the two curves are unnested. The $u_1(a_2 + b_2 + \cdots + j_2)$ term accounts for the cases where γ_1 is nested in γ_2, and the $u_2(a_1 + b_1 + \cdots + k_1)$ term accounts for the cases where γ_2 is nested in γ_1. When γ_1 and γ_2 are identical curves, the cases where γ_1 is nested in γ_2 are not distinct from the cases where γ_2 is nested in γ_1, so we drop the $u_2(a_1 + b_1 + \cdots + k_1)$ term. When they are unnested the redundant combinations of distinct strands end up eliminating half of the cases where two different strands are involved, giving the $({u_1}^2 + u_1)/2$ term. □

For example, to verify that $Card(ISS(S^1, S^1)) = 2$, label the region inside the first circle A_1, the region inside the second circle A_2. Then $u_1 = u_2 = a_1 = a_2 = 1$, so we have

$$Card(ISS(S^1, S^1)) = 1 + 1 = 2$$

It is helpful to have a listing of the values of $u, a, b, c, \cdots$ for the irreducible curves. These are the values for the irreducible curves up to five crossings.

curve	u	a	b	c	d	e
0.1	1	1	-	-	-	-
3.1	1	2	1	-	-	-
3.2	1	2	1	-	-	-
4.1	2	3	2	2	1	-
4.2	1	2	2	1	-	-
5.1	1	2	1	-	-	-
5.2	1	3	1	1	-	-
5.3	1	2	2	1	-	-
5.4	2	4	2	2	1	1
5.5	1	4	2	2	2	1
5.6	2	3	2	2	1	-

With this table we can easily compute, for example,

$$Card(ISS(5.5, 5.6)) = 2 + 1(3 + 2 + 2 + 1) + 2(4 + 2 + 2 + 2 + 1) = 32$$

As noted above, the upper bound formula for the ISS gives 1452 as an upper bound on the number of 1-reducible 6-crossing curves. Now we can compute the exact number:

$$Card(ISS(3.1, 3.1)) = 4$$

$$Card(ISS(3.1, 3.2)) = 7$$

$$Card(ISS(3.2, 3.2)) = 4$$

$$Card(ISS(5.1, 0.1)) = 5$$

$$Card(ISS(5.2, 0.1)) = 7$$

$$Card(ISS(5.3, 0.1)) = 7$$
$$Card(ISS(5.4, 0.1)) = 14$$
$$Card(ISS(5.5, 0.1)) = 13$$
$$Card(ISS(5.6, 0.1)) = 12$$
$$(Total = 73)$$

For the determination of the set of distinct inverted sums of curves with low numbers of crossings using few inverted sums, the combinatorics involved in constructing the reducible curves is simple, but for higher numbers of crossings or more inverted sums the combinatorics problem is difficult, as it is at least as hard as the problem of finding the distinct trees with n vertices. In [**2**], F. Aicardi discusses combinatorial structures for completely reducible curves in terms of trees; it appears that the combinatorics for the completely reducible curves can be easily adapted for partially reducible curves. Such a modification of Aicardi's theory seems a promising direction for future research.

3 Finding the Irreducibles

So we see that in some sense the irreducible curves are the building blocks for the space of plane curves, for if we know them we can construct the other curves by combinatorial methods based on the nesting of the factor irreducible curves and the choice of which of their strands to join with the inverted sum. From here the most imperative problem seems to be to find, characterize, and classify the irreducible curves. The most promising approach so far has been to associate a certain type of planar graph with irreducible curves and then to find all the distinct graphs of that type.

Definition 3.1. A *closed planar graph* is a graph in which none of the edges intersect and where removing any edge would decrease the number of regions formed by the graph by one.

It will become clear later that every irreducible curve has a corresponding closed planar graph that is unique to that curve. The nature of that correspondence is well known; the following explanation and claims are based on information from pp. 51-55 of [**1**]. (The validity of the following method seems intuitively correct and has so far been very successful in deriving irreducible curves, but proving the legitimacy of the method rigorously would have taken more time than was available — perhaps future work could be done to fill in the details, assuming similar work has not already been carried out.)

Every n-crossing curve divides the plane into $n + 2$ regions that are 2-colorable. That is, the regions can be colored so that no like colored regions are adjacent (see Figure 6). (To readers of [**3**] a new proof of this fact is evident using Arnold's perestroikas on the K_i curves, by showing that the moves J^+, J^-, and St do not change the colorability.) We call a region bounded by a curve

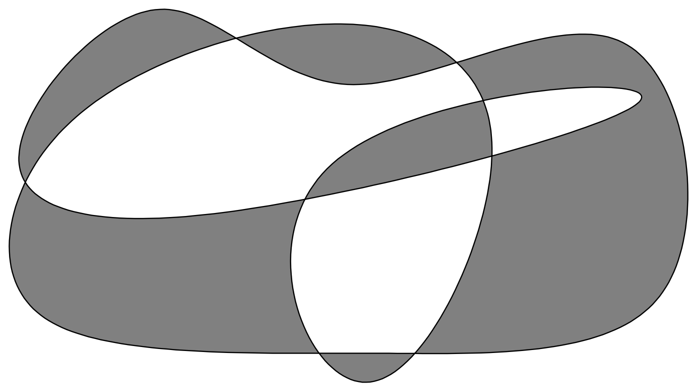


Figure 6: A 2-coloration of an irreducible curve.

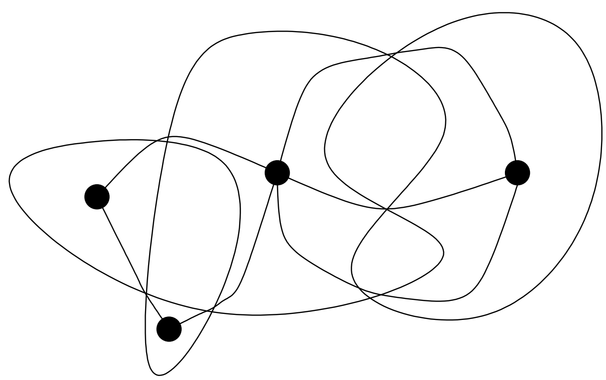


Figure 7: The closed planar graph associated with an 6-crossing irreducible curve.

an *exterior* region if it must receive the same color as the unbounded region in a 2-coloration; we call a region *interior* if it is not exterior.

A closed planar graph associated with an irreducible curve is constructed as follows. Place a vertex inside each interior region and then place edges that connect the vertices through the crossings without letting the edges cross, as in Figure 7. Notice that every crossing is traversed by exactly one edge, and that every region of the graph encloses exactly one exterior region of the curve (with the exception of the unbounded region). It is not difficult to see that an irreducible a curve can always be constructed uniquely from its graph, and that every irreducible curve has a unique associated closed planar graph.

This representation of irreducible curves as closed planar graphs reduces the problem of finding all the irreducible curves to an apparently simpler combinatorics problem: find all the distinct closed planar graphs with certain restrictions. These restrictions fall out easily from basic properties of irreducible curves. In the following, V shall denote the number of vertices of a graph, E the number of edges, R the number of (bounded) regions enclosed by the graph, n the number of crossings on the curve, r_e the number of exterior regions of the curve (excluding the unbounded region), and r_i the number of interior regions of the curve.

From the Euler equation we have

$$V + R - 1 = E = n$$

and it is clear from the construction (of a curve from its graph) that

$$V = r_i \geq 1$$

and

$$R = r_e$$

for every irreducible curve. Also note that every curve that is not completely reducible must clearly have at least one exterior region.

Remark 3.1. It would be helpful to know, before attempting to find all the closed planar graphs, whether every graph with these restrictions corresponds to an irreducible curve. Unfortunately, there are more of such graphs than there are irreducible curves, since many of the graphs correspond to immersions of more than one circle (call these immersions of m circles, $m > 1$, m-curves). So far a way of determining whether a given closed planar graph will correspond to an m-curve has not been found, other than actually constructing the curve. This just means we will have to test more cases, but provided we can find all the distinct planar graphs with n edges with the necessary restrictions, we can simply construct all the corresponding curves to find all the irreducible n-crossing curves (and as a bonus we will have found all the irreducible n-crossing m-curves).

In [**3**], Arnold gives a complete listing of all the curves with $n \leq 5$ crossings. A sound goal for this project would be to find a method of deriving all the curves with $n \leq 7$ crossings (which I estimate number in the tens of thousands). As shown above, if we could find all the irreducible curves with $n = 6$ crossings, the rest would be a simpler combinatorical issue that could possibly even be handled by a computer. To that end, we now attempt to derive the 6-crossing irreducible curves.

An irreducible 6-crossing curve may have from two to six interior regions, so we proceed by cases of the numbers of interior regions of the curve. To obtain all the 6-crossing irreducible curves with two interior regions, we look at closed planar graphs with two vertices and six edges, which from the Euler equation we know will create five bounded regions. For the curves with three interior regions, we look at graphs with three vertices and six edges, and so on (see Figure 9, Figure 10, and Figure 11 — the graphs corresponding to immersions of more than one circle have been omitted). This list is probably not exhaustive, since a combinatorial method of finding all the distinct graphs has not yet been found. Using currently known or unknown combinatorical theory to understand the combinatorics of these closed planar graphs is another possible direction for future work.

4 Irreducible Curves and Alternating Knots

Definition 4.1. The *preimages* on S^1 of a double point a on a curve γ are the two points on S^1 that get sent by the map $\gamma : S^1 \to \mathbb{R}^2$ to the same point a.

Definition 4.2. The *Gauss diagram* of an n-crossing curve is an image of S^1 with a collection of n chords, where each chord connects the two preimages of a double point of the curve (see Figure 8).

Definition 4.3. A *bisection diagram* of a curve is an image of S^1 together with a collection of the chords that are the perpendicular bisectors of the chords in the curve's Gauss diagram (we must add that the preimages on the Gauss diagram must be evenly spaced), with multiplicity noted.

Note that a bisection diagram is just a Gauss diagram with evenly spaced preimages where each chord has been moved normal to its direction so that it crosses the center of S^1, noting multiplicities (see Figure 8).

Proposition 4.1. *A curve is irreducible iff every chord in its Gauss diagram participates in an intersection with another chord (its Gauss diagram is then* totally non-planar*).*

(This holds for all curves listed in [**3**].)

Proof of the proposition. Recall that a reduction point on a curve is a double point that divides the curve into two pieces with no other common points. If a Gauss diagram has a non-intersecting chord, it means (from the construction) that the part of the circle on one side of the chord has no points in common with the part of the circle on the other side of the chord after being mapped by the function γ that defines the curve whose Gauss diagram we are examining. We know, then, that any non-intersecting chord on a Gauss diagram must correspond to a reduction point on the curve. This proves that every irreducible curve must have a totally non-planar Gauss diagram. As for the converse, assume we are given a curve with a totally non-planar Gauss diagram. If this curve has a reduction point, its Gauss diagram must have a corresponding chord. But every chord on the Gauss diagram is intersected by another chord, so the Gauss diagram is such that some point on the part of the circle on one side of any chord in the diagram must get mapped to the same point as some point on the part of the circle that is on the other side of the chord. Hence no points on the curve are reduction points, so the curve is irreducible. ☐

Definition 4.4. An irreducible curve is called *composite* if a new chord can be drawn on the curve's Gauss diagram that 1) divides the diagram's circle into

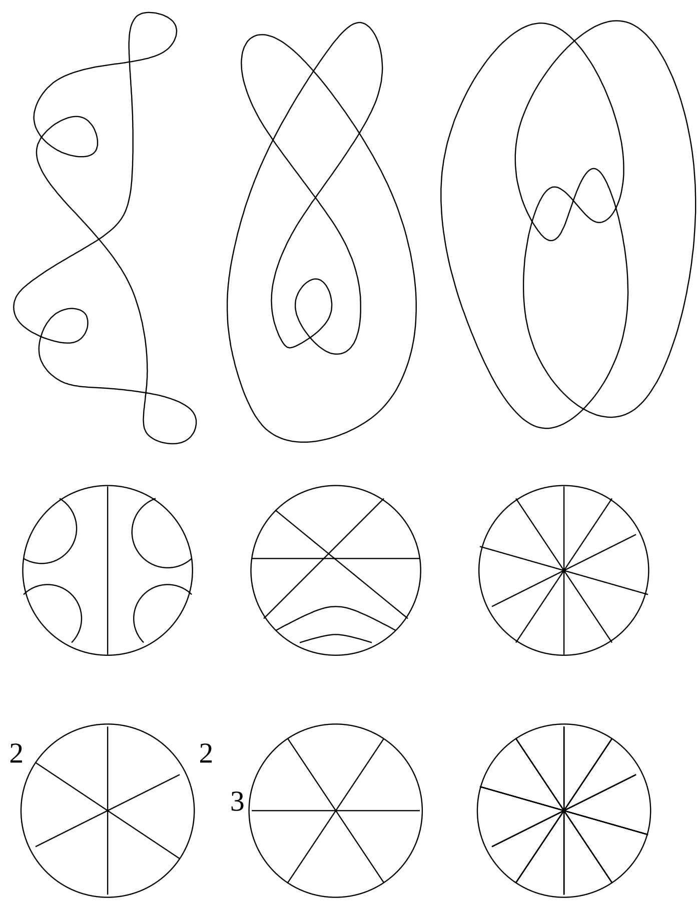


Figure 8: The Gauss and bisection diagrams of a completely reducible curve, a partially reducible curve, and an irreducible curve.

V = 2: NONE

V = 3:

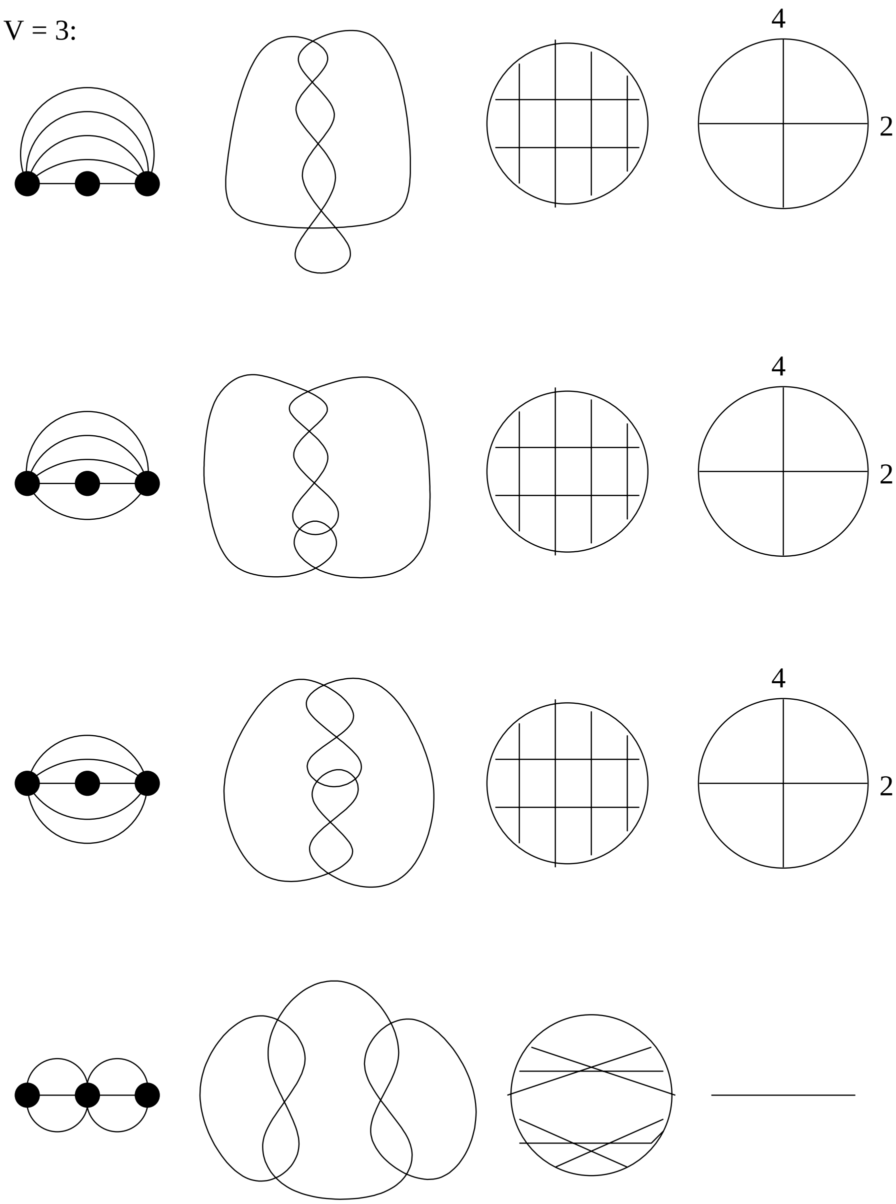


Figure 9: 6-crossing irreducible curves with V vertices as derived from closed planar graphs, listed with their Gauss and bisection diagrams.

V = 4:

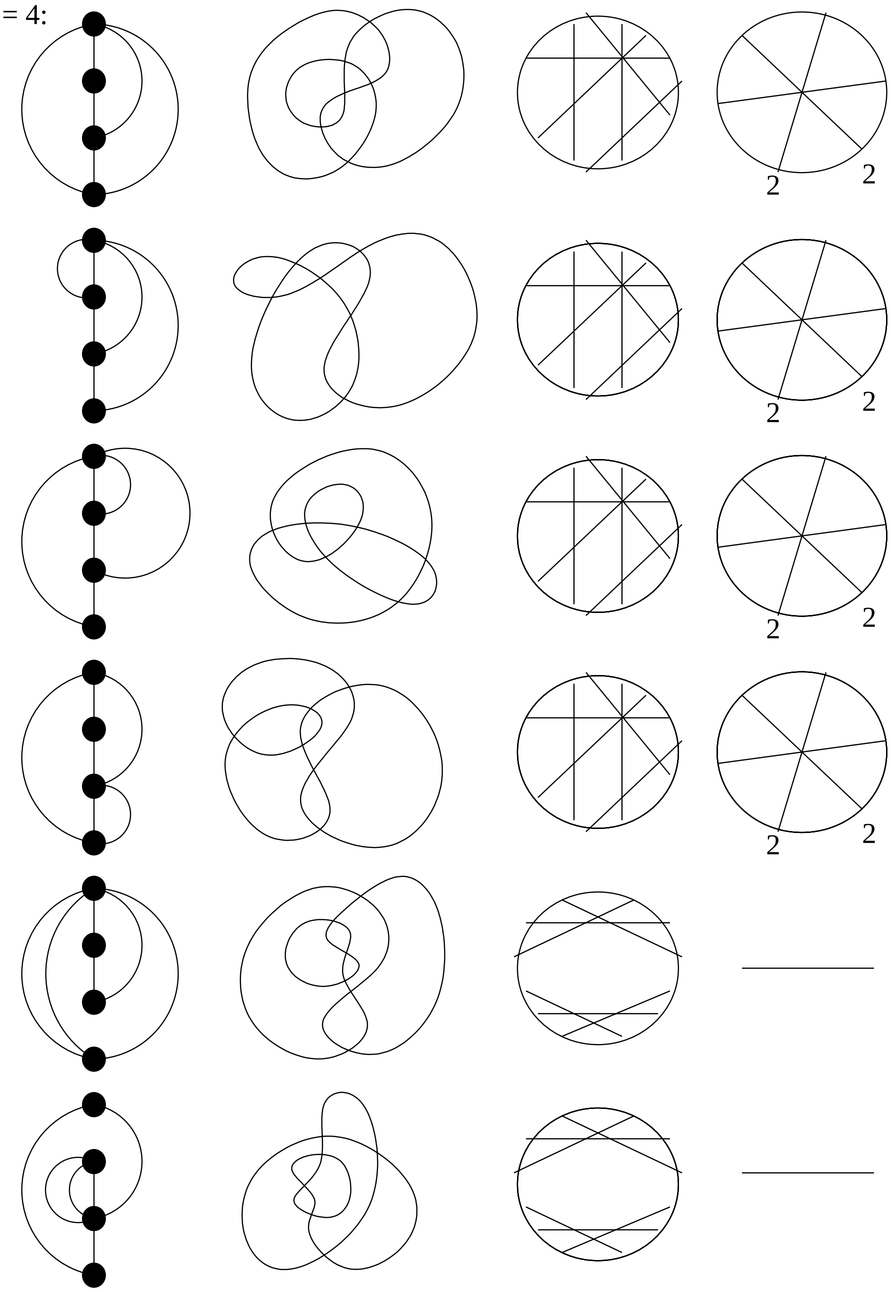
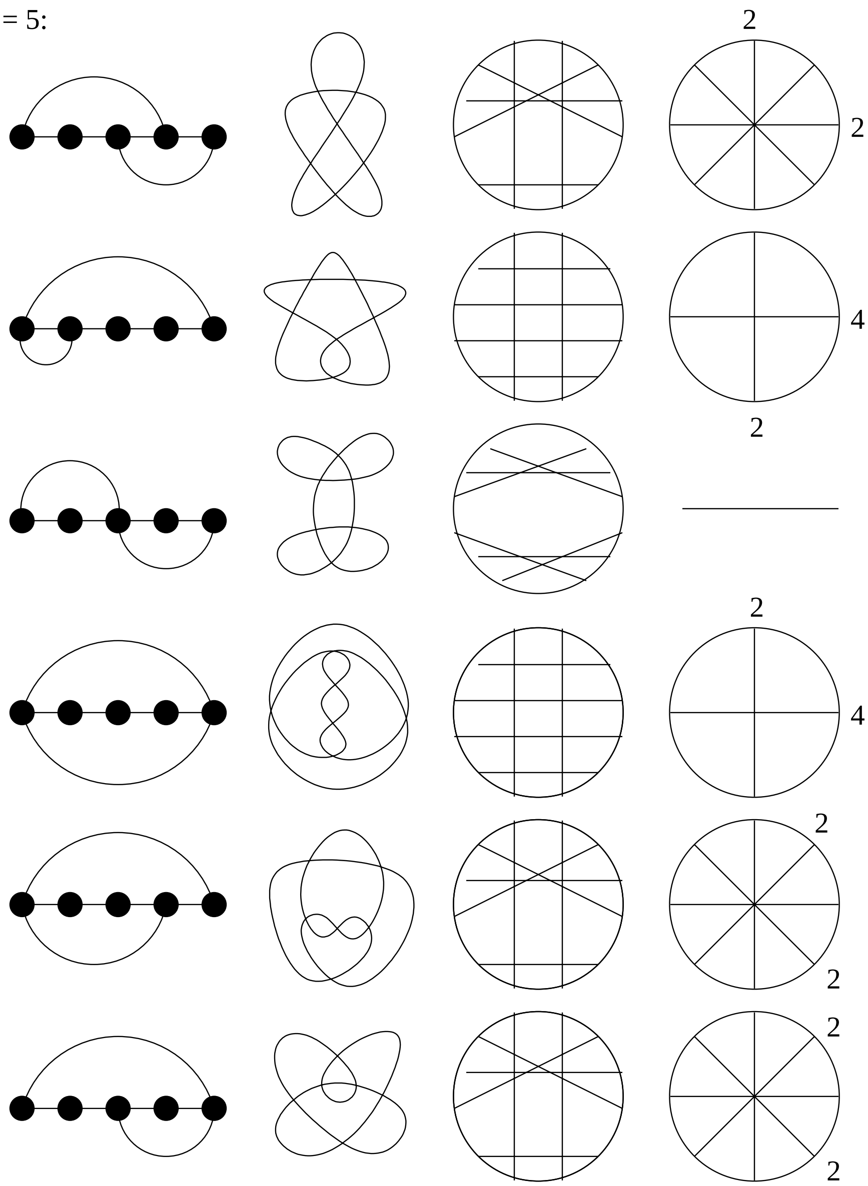


Figure 10: 6-crossing irreducible curves with V vertices as derived from closed planar graphs, listed with their Gauss and bisection diagrams.

V = 5:



V = 6: NONE

Figure 11: 6-crossing irreducible curves with V vertices as derived from closed planar graphs, listed with their Gauss and bisection diagrams.

two regions, both of which contain chords and 2) does not intersect any of the diagram's chords. An irreducible curve is called *prime* if it is not composite.

(This definition is motivated by the projections of prime and composite knots and the Gauss diagrams so far observed as being associated with those projections.)

It is obvious that every irreducible curve can be made into an alternating knot, simply by choosing each crossing to be an under- or over-strand alternately while traversing the curve. It appears that there is a natural bijection between the set of irreducible curves and the set of reduced alternating projections of alternating knots. In this sense, it seems that studying irreducible curves is like studying projections of knots. The question naturally arises, is there anything we have learned about plane curves that can tell us something about knots? Of particular interest has been the question of which characteristics irreducible curves that are different projections of the same alternating knot might share.

The two curves listed as 3.1 and 3.2 in Figure 3 have the same Gauss diagram, and if made into alternating knots they turn into two reduced alternating projections of the alternating knot known as 3_1 (refer to [**1**] for more information). Likewise, the curves 4.1 and 4.2 have identical Gauss diagrams and correspond to reduced alternating projections of the knot 4_1. Similar facts hold for the irreducible curves of five crossings. (Incidentally, it has been observed that composite irreducible curves (as they are defined above) correspond to reduced alternating projections of composite alternating knots.) Could Gauss diagrams be an invariant for alternating knots? Further investigation shows that there are some 7-crossing irreducible curves that has different Gauss diagrams, but correspond to alternating projections of the same alternating knot. Still, for the prime irreducible curves, the Gauss diagrams seem to retain a certain similarity when they correspond to projections of the same knot. This observation led to the formulation of the bisection diagram.

So far the bisection diagram seems like a possible invariant for knots, although it may simply be an invariant that is already understood, but in a different manifestation. It also may be that it is an invariant that is not particularly useful. Future research might reveal whether this is truly an invariant for knots and why, and whether it is of any use.

References

[**1**] C. Adams, *The Knot Book*, W.H. Freeman and Co., New York, 1994.

[**2**] F. Aicardi, Tree-like curves, *Advances in Soviet Mathematics* **21**, Amer. Math. Soc., Providence, 1994, pp. 1-31.

[**3**] V.I. Arnold, Plane curves, their invariants, perestroikas and classifications, *Advances in Soviet Mathematics* **21**, Amer. Math. Soc., Providence, 1994, pp. 33-9.

[**4**] H. Whitney, On regular closed curves in the plane, *Compositio Math* **4**, 1937, pp. 276-284.