Classification of Plane Curves

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August 13, 1999

Abstract

The classification of curves in the plane is a wide-open problem. While this topic has been addressed in the past, new research into the use of topological invariants has shed much light on the subject. We clarify recent research and give new perspectives. We also present a table of curves with extremal values of the invariants.

1 Introduction

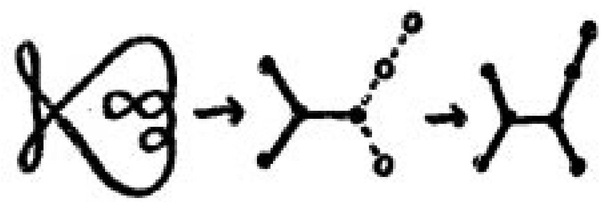
The first systematic attempt to classify plane curves was carried out by H. Whitney in 1937 [3]. He was able to classify isotopy classes of curves by the rotation index. For example, each of the following curves have rotation index 2, so they can be deformed into each other.

All three of these curves have index = 2.

Advances in knot theory make it natural to consider equivalence of curves under ambient diffeomorphisms and reparametrizations of curves. This question has been recently treated by Arnold again in [2]. He approaches the classification question by considering three invariants for such curves, which he calls St, J^+, J^-. While these invariants are well-defined in the equivalence classes of curves, it turns out that they fail to distinguish between different classes of curves. In fact, one can easily find non-equivalent curves of two double points with the same values of the invariants. The classification of all plane curves appears to be a formidable problem. Presently, it is not even known whether the general classification problem can be completely solved by solely considering some sort of local invariants.

Both of these curves have St = 3, J^+ = -6, J^- = -10.

Aicardi, in [1], considers the simpler problem of classifying the so-called tree-like curves, which can be uniquely represented by a planar graph.



A tree-like curve and the associated planar graph.

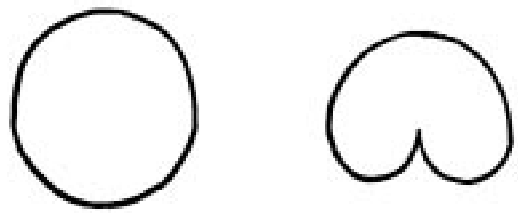
One can see that the correspondence between tree-like curves and planar graphs is not bijective. Consequently, the sought-after classification of all tree-like curves still fails. However, with the help of the planar graphs, Aicardi is able to describe tree-like curves with the maximum or minimum values of various combinations of the three invariants.

The bulk of this paper is devoted to the clarification of the details in the papers by V.I. Arnold [2], and by F. Aicardi [1]. Furthermore, we present a novel method for computing the strangeness invariant of a curve using the Seifert's circles associated with the curve. We also classify all tree-like curves of maximum and minimum values of all of the three invariants, St, J^-, and J^+, up to five crossings. It appears that this classification has not appeared in the literature before.

2 Preliminaries

In this section we present a few basic definitions pertaining to plane curves.

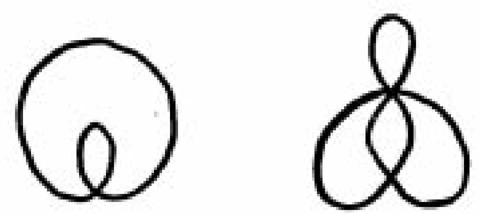
Definition 1 *A curve is a smooth mapping of a circle to a plane whose derivative vanishes nowhere.*



A curve and a non-curve.

Definition 2 *A double point is the image of two distinct points on the circle, with the property that the tangent vectors are distinct.*

Definition 3 *A generic curve intersects itself only in double points.*



A generic curve and a non-generic curve.

Definition 4 *Two curves are equivalent if they are equal up to ambient diffeomorphism.*

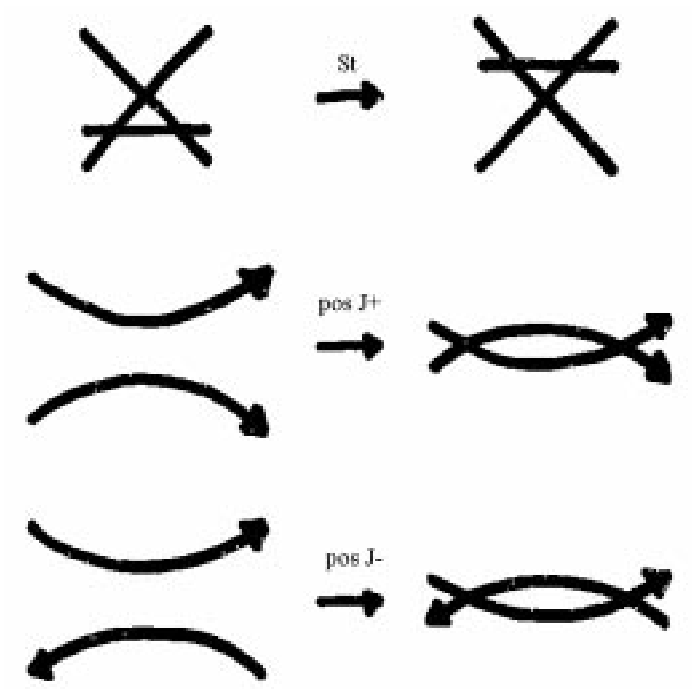
Ths means that two curves are equivalent if there is no double crossings are created or eliminated, and no triple points are created.

3 The Invariants

Arnold classifies the curves by using three invariants: St (strangeness), J^+, and J^-. (For proofs of the existence of these invariants, see [2].) If two curves have the same invariants, they are not necessarily equivalent, but if they have different invariants, they are definitely not equivalent.

3.1 The basic moves

According to Arnold [2], only triple points and self-tangencies are needed to move between generic curves. These special points are passed through by the following moves:



Remark 5 *A $J^{\pm}$ move is positive if the number of double points increases, and negative if the number decreases. A St move is positive if the newborn triangle is positive, and negative if the newborn triangle is negative.*

3.1.1 Triangles

Definition 6 *A vanishing/newborn triangle is a triangle formed by three branches of a curve before/after a St move.*

To find the sign of a triangle, let x be a starting point on the curve which is not on the triangle. As the curves travels from x around the curve, it orients the triangle by the order of its visits to the three sides. The sides of the triangle are oriented in the direction the curve is traveling as it visits each one. Let q = the number of sides whose orientation agrees with that of the triangle.

Definition 7 *The sign of a triangle is $(-1)^q$.*

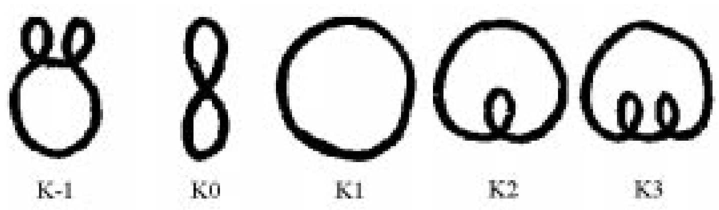
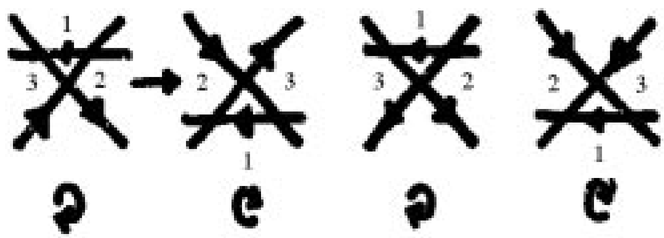


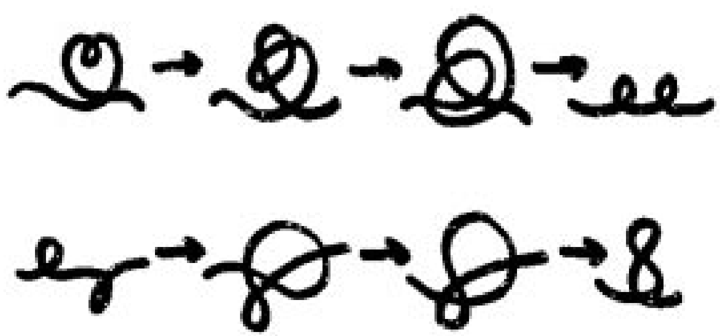
Figure 1:



The signs of various triangles.

3.2 Perestroikas

Definition 8 *A perestroika is a combination of basic moves.*

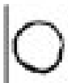
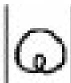
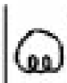


The C (concordant) perestroika and the D (discordant) perestroika.

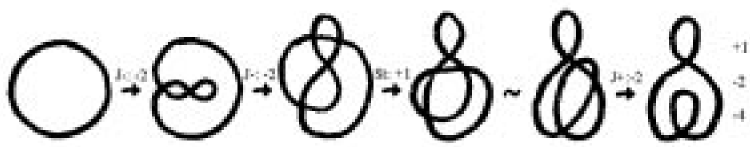
3.3 The basic invariants

The standard curves K_i are representatives of the curves with index i.

Definition 9 *The basic invariants St, J^+ and J^- of plane curves are defined by the values of the standard moves and of the values at the standard curves, as shown below.*

	St move	J+ move	J- move	...						...
St	1	0	0	...	0	0	0	1	2	...
J+	0	2	0	...	0	-0	0	-2	-4	...
J-	0	0	-2	...	-2	-1	0	-3	-6	...

Example 10 *Given a curve C with index i, the invariants are found by susing the basic moves to deform the standard curve K_i to the given curve. The values of the moves are added to the values of K_i to get the invariants.*



Transforming K_1 to a desired curve in order to calculate the invariants.

Remark 11 *The values of J^+ and J^- are chosen such that J^+ = the increase in double points, and J^- = the decrease in double points. Hence $J^+ - J^- = n$.*

4 Index

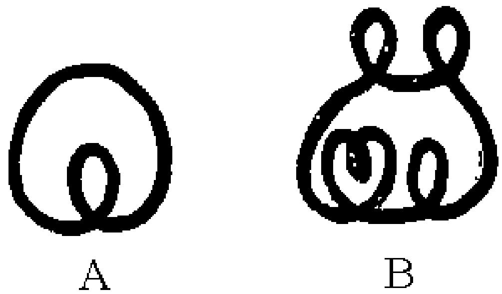
Definition 12 *The index of a curve is the rotation number of the tangent vector.*

The graphs of the tangent vectors of two curves.

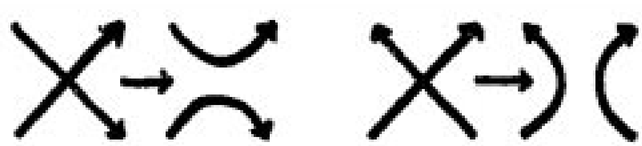
4.1 Whitney's Theorem

Whitney's classical theorem makes possible all subsequent work on plane curves.

Theorem 13 *(Whitney's Theorem) Curve A may be deformed into curve B if and only if A and B have the same index.*



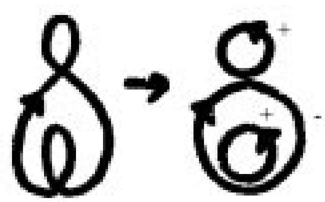
Definition 14 *An oriented surgery is a change on a small neighborhood of each crossing, shown below:*



The orientation of each branch is preserved but the crossing is eliminated.

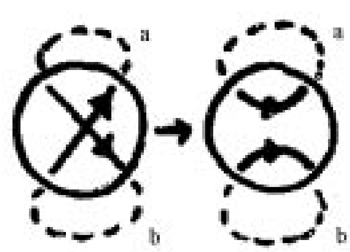
Remark 15 *When all all crossings have been eliminated, the Seifert's circles of the given curve appear.*

Lemma 16 *The index of an oriented curve is equal to the difference between the number of positive and negative Seifert's circles.*



The difference here is 1, giving $ind = 1$.

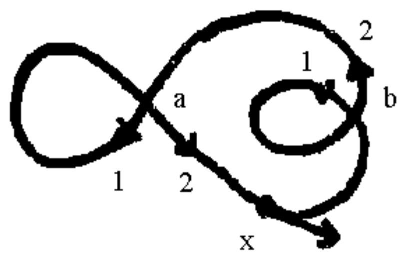
Proof. It is easy to observe that the sum of the indices of the curves after an oriented surgery at one crossing is equal to the index of the original curve:



Then the lemma follows from the fact that the index of a simple closed curve is ± 1 (+1 if oriented counter-clockwise, -1 if clockwise.) ■

Definition 17 *Choose a starting point x on the curve and travel in the direction of the tangent vector. A double point on the curve is positive/negative with respect to x if the first and second branches leaving the double point orient the plane positively/negatively.*

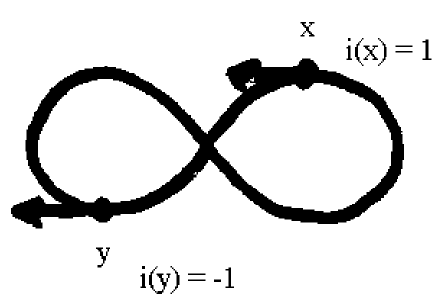
Remark 18 *A simple right hand rule can be used to label the crossings: Let the fingers point in the direction of the first branch leaving the crossing. If the thumb then points in the direction of the second outgoing branch, the crossing is positive; otherwise, it is negative.*



Crossing a is positive, but crossing b is negative.

Definition 19 *The Whitney function defined at ordinary points of a curve is the difference between the numbers $w_+(x)$ and $w_-(x)$ of positive and negative double points with respect to x: $w(x) = w_+(x) - w_-(x)$.*

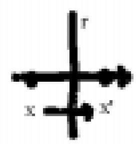
Definition 20 *$i(x)$ is the number of half-turns of the vector connecting x to a point y moving along the curve from x to x.*



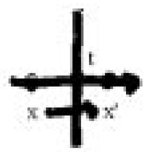
$i(x)$ for two different points on the same curve.

Theorem 21 *The value of the Whitney function at an ordinary (non-double) point of a curve is $w(x) = i(x) - ind$.*

Proof. The jumps (± 2) of $w(x)$ and $i(x)$ as x crosses a double point are equal:



$i(x) = q + r$, and $i(x') = q - r$. The difference is $2r = \pm 2$.



Similarly, $w(x) = s + t$, and $w(x') = s - t$. The difference is $2t = \pm 2$.

If r is positive (with respect to x), then the crossing t is positive. If r is negative, then the crossing t is negative.

Hence $r = t$, the jumps of $w(x)$ and $i(x)$ are equal, and the difference between them is independent of x and is an invariant of the curve.

We now show that $w(x) = i(x) - k$ under J^+. Since the difference between $w(x)$ and $i(x)$ is independent of x, only one case needs to be checked for each move.

The two new crossings are positive, so $\Delta w(x) = +2$.

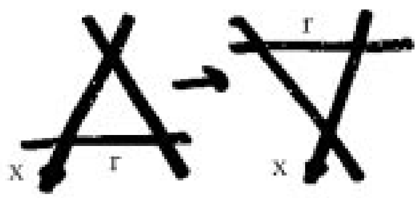
The negative half-curve above x becomes a positive half-curve below x, so $\Delta i(x) = +2$.

Next, we show $w(x) = i(x) - k$ under J^-.

The two new crossings are negative, so $\Delta w(x) = -2$.

The positive half-curve above x becomes a negative half-curve below x, so $\Delta i(x) = -2$.

Finally, $w(x) = i(x) - k$ under St.



$\Delta i(x) = 0$, and since the order and direction of the visits of the crossings are preserved, $\Delta w(x) = 0$ also.

To complete the proof, we show that $w(x) = i(x) - ind$ for the standard curves:

$i = 0$: $ind = +1$, $i(x) = +1$, and $w(x) = 0 = i(x) - ind$.

$i > 0$: $ind = n+1$, $i(x) = +1$, and $w(x) = -n = 1-(n+1) = i(x) - ind$.

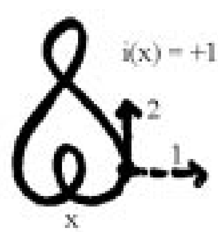
$i < 0$: $ind = 1-n$, $i(x) = +1$, and $w(x) = n = 1-(1-n) = i(x) - ind$.

Since the statement is true for the standard curves, and under the basic moves, it is true for any curve. ∎

Theorem 22 *The index of an immersed circle is* $\sum \varepsilon_i + a$*, where* $\sum \varepsilon_i = -w(x)$ *and* $a = i(x)$*.*

Proof. $w(x) = i(x) - ind \Longrightarrow ind = -w(x) + i(x) = \sum \varepsilon_i + a$. ∎

Remark 23 *A line can be drawn starting at x which is normal to the curve at x, intersects the curve only at x, and extends to infinity. If this line and the tangent vector at x orient the plane positively, then* $i(x) = +1$*. Otherwise,* $i(x) = -1$*.*



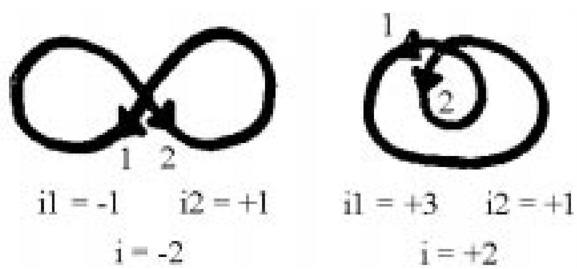
Whitney's original formula for the index of a curve, $(N^+ - N^+) + \mu$, reverses the definitions of positive (N^+) and negative (N^-) crossings. This makes $(N^+ - N^+)$ equal to $\sum \varepsilon_i$. An external point is then chosen such that the curve is entirely inside one of the half planes created by the tangent vector at that point. $\mu = \pm 1$ depending on whether the curve is in the positive or negative half-plane, where the tangent vector is viewed as a positive x-axis. This is equivalent to the method of determining $i(x)$ in the remark above.

5 Half-indices

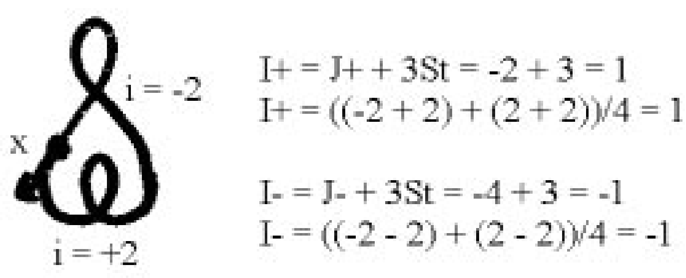
Indices of the double points are a "local" way to calculate combinations of the basic invariants.

Definition 24 *The half index i_j of a double point is the angle of rotation of the vector from that point to a point moving along branch j from the double point to itself, divided by $\pi/2$.*

Definition 25 *The index of a double point is the difference $i = i_1 - i_2$, where branches 1 and 2 are labelled such that they orient the plane positively.*



Theorem 26 *The combinations $I^\pm = J^\pm + 3St$ of the basic invariants of a generic curve are equal to the sums of the indices of all the double points on the curve: $I^\pm = \dfrac{\sum(i \pm 2)}{4}$.*



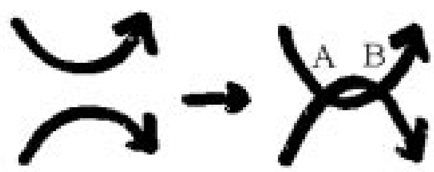
Proof. $I^\pm = \dfrac{\sum(i \pm 2)}{4} \Longrightarrow I^\pm = \frac{1}{4}\sum i \pm \frac{n}{2}$.
True for the standard curves:

$ind \geq 0$: i for each crossing is $i = i_1 - i_2 = 3 - 1 = 2$.
$I^{\pm} = \frac{1}{4}\sum i \pm \frac{n}{2} = \frac{1}{4}(2n) \pm \frac{n}{2} = \frac{n}{2} \pm \frac{n}{2}$.
Then $I^+ = n$, $I^- = 0$.
$St = n$, $J^+ = -2n$ and $J^- = -3n$ by definition.
Then $I^+ = -2n + 3n = n$, and $I^- = -3n + 3n = 0$.

$ind \leq 0$: i for each crossing is $i = i_1 - i_2 = -1 - 1 = -2$.
$I^{\pm} = \frac{1}{4}\sum i \pm \frac{n}{2} = \frac{1}{4}(-2n) \pm \frac{n}{2} = -\frac{n}{2} \pm \frac{n}{2}$.
Then $I^+ = 0$, $I^- = -n$.
$St = 0$, $J^+ = 0$ and $J^- = -n$ by definition.
Then $I^+ = 0 + 3(0) =)$, and $I^- = -n + 3(0) = -n$.

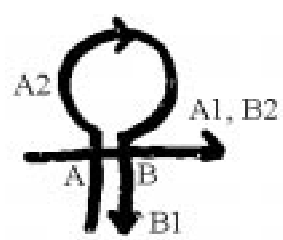
True for the basic moves:

Under a positive J^+ move, $\Delta I^{\pm} = \Delta(J^{\pm} + 3St) = 2$ for I^+, 0 for I^-.

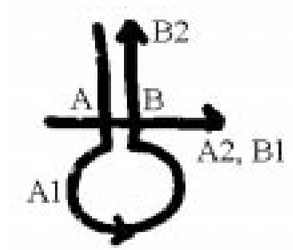


The J^+ move creates two new points A and B.

$\Delta I^{\pm} = \Delta(\frac{1}{4}\sum i \pm \frac{n}{2}) = \frac{1}{4}(i_A + i_B) \pm 1 = \frac{1}{4}((i_{A1} - i_{A2}) + (i_{B1} - i_{B2})) \pm 1 = \frac{1}{4}((i_{A1} - i_{B2}) + (i_{B1} - i_{A2})) \pm 1$.



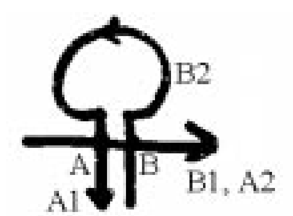
A1 and B2 start out with the same angles of rotation, but after passing A, B2 goes through an extra negative half-turn. This gives $(i_{A1} - i_{B2}) = p - (p-2) = 2$.



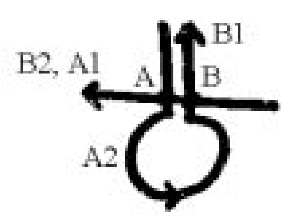
Similarly, B1 and A2 start out with the same angles of rotation, but after passing B, A2 goes through an extra negative half-turn. This gives $(i_{B1} - i_{A2}) = p - (p-2) = 2$. Then $\Delta I^{\pm} = \frac{1}{4}((i_{A1} - i_{B2}) + (i_{B1} - i_{A2})) \pm 1 = 1 \pm 1 = 2$ for I^+, 0 for I^-.

Under a postive J^- move, $\Delta I^{\pm} = \Delta(J^{\pm} + 3St) = 0$ for I^+, 2 for I^-.

The J^- move creates two new points A and B. $\Delta I^{\pm} = \Delta(\frac{1}{4}\sum i \pm \frac{n}{2}) = \frac{1}{4}(i_A + i_B) \pm 1 = \frac{1}{4}((i_{A1} - i_{A2}) + (i_{B1} - i_{B2})) \pm 1 = \frac{1}{4}((i_{A1} - i_{B2}) + (i_{B1} - i_{A2})) \pm 1$.



B1 and A2 start out with the same angles of rotation, but after passing B, A2 goes through an extra positive half-turn. This gives $(i_{B1} - i_{A2}) = p - (p+2) = -2$.



Similarly, A1 and B2 start out with the same angles of rotation, but after passing A, B2 goes through an extra positive quarter-turn. This gives $(i_{A1} - i_{B2}) = p - (p+2) = -2$. Then $\Delta I^{\pm} = \frac{1}{4}((i_{A1} - i_{B2}) + (i_{B1} - i_{A2})) \pm 1 = -1 \pm 1 = 0$ for I^{+}, 2 for I.

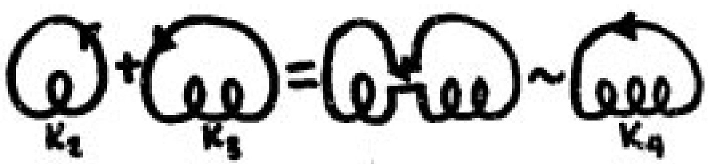
The proof for St moves is checked by example, but this is left as an exercise for the reader.

Since the formula is true for the standard curves and the basic moves, it is true for all generic curves. ■

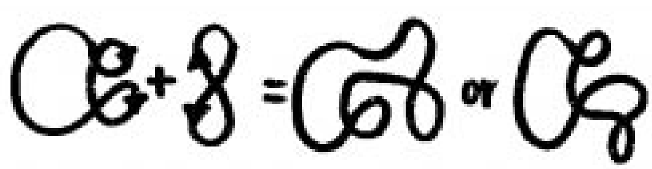
6 Sums

6.1 Connected Sums

Definition 27 *The connected sum of two curves, one in each half of the plane, is defined by adding a bridge between the two curves which intersects the curves only at its endpoints:*

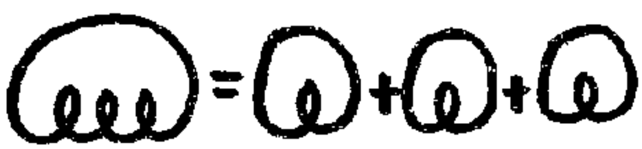


Remark 28 *The connected sum of curves is not an operation, since there may exist many different bridges between two curves.*

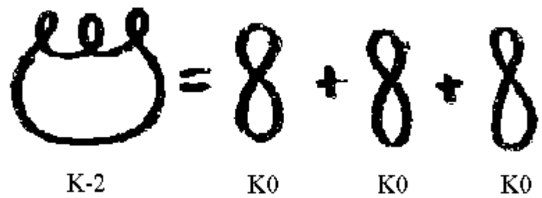


The basic invariants, however, are additive under any choice of the bridge.

The standard curves are connected sums of K_2:

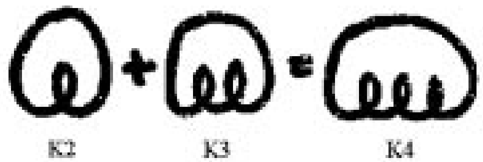


$$K_{i+1} = iK_2 \ (i \geq 0)$$

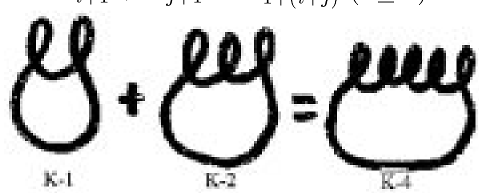


$$K_{i+1} = -iK_0 \ (i \leq 0)$$

There are also formulas for computing sums of the standard curves:



$$K_{i+1} + K_{j+1} = K_{1+(i+j)} \ (i \geq 0)$$

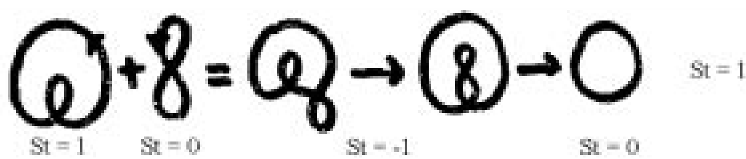


$$K_{i+1} + K_{j+1} = K_{1-(i+j)} \ (i \leq 0)$$

Theorem 29 *All three basic invariants are additive under the connected sum.*

Proof. It is obvious from the defined values of St for the standard curves that St is additive between K_2 curves, and between K_0 curves.

Between the two, the D perestroika shows that St is also additive.



Since any standard curve is the sum of K_2 or K_0, St is additive between any two standard curves.

To show that St is additive between any two curves, consider the case $C_i + C_j = C_p$, where C_a has index a.

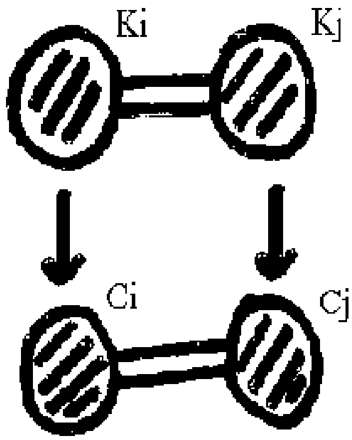
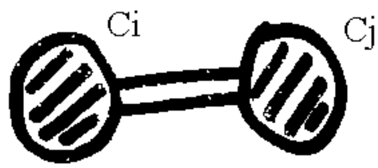


Figure 2:



Deform the curve K_p, the standard curve with the same index as C_p, to C_p:

We now show that $St(C_i) + St(C_j) = St(C_p)$.

From the definition of St, $St(K_i)+\Delta St(K_i) = St(C_i)$ and $St(K_j)+\Delta St(K_j) = St(C_j)$. This gives $St(C_i)+St(C_j) = [St(K_i)+\Delta St(K_i)]+[St(K_j)+\Delta St(K_j)] = [St(K_i) + St(K_j)] + [\Delta St(K_i) + \Delta St(K_j)]$.

From above, $St(K_i) + St(K_j) = St(K_p)$. Since the St moves affect only one side of the curve, $\Delta St(K_i) + \Delta St(K_j) = \Delta St(K_p)$. Hence, $St(C_i) + St(C_j) = St(K_p) + \Delta St(K_p) = St(C_p)$.

For $J^{\pm}$, recall the sum $I^{\pm} = J^{\pm} + 3St = \frac{\sum(i \pm 2)}{4}$.

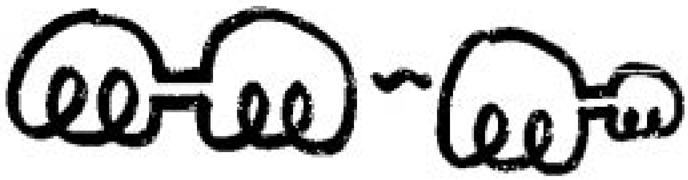
This implies $J^{\pm} = \frac{\sum(i \pm 2)}{4} - 3St = \frac{1}{4}\sum i \pm \frac{n}{2} - 3St$.

St and n are additive, so it only remains to show that the indices of the double points are

additive.

In fact, the curve on the left will not affect the one on the right, and vice versa, since the

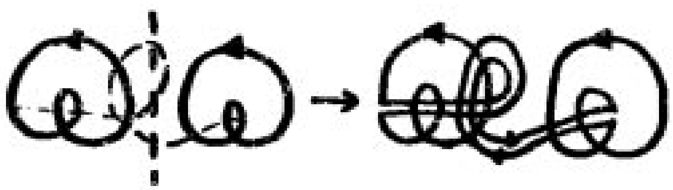
curve can be contracted to a small enough size that it does not affect the half-indices.



Now, $J^{\pm}(C_i)+J^{\pm}(C_j) = \frac{1}{4}\sum i(C_i) \pm \frac{n(C_i)}{2} - 3St(C_i) + \frac{1}{4}\sum i(C_j) \pm \frac{n(C_j)}{2} - 3St(C_j) = \frac{1}{4}(\sum i(C_i) + \sum i(C_j)) \pm \frac{1}{2}(n(C_i) + n(C_j)) - 3(St(C_i) + St(C_j)) = \frac{1}{4}\sum i(C_p) \pm \frac{1}{2}n(C_p) - 3St(C_p) = J^{\pm}(C_p)$. ■

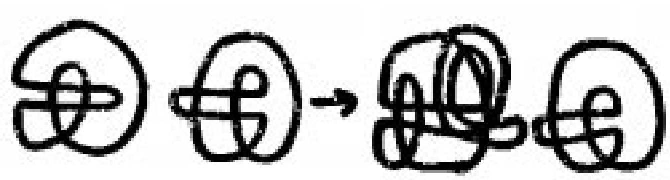
6.2 Strange Sums

Definition 30 *The strange sum of two curves, one in each half of the plane, is defined by a segment joining them such that the given curves orient the segment differently at the endpoints. To get the new curve, double the segment as showm below:*



Theorem 31 *The invariant St is additive under strange summation of immersions.*

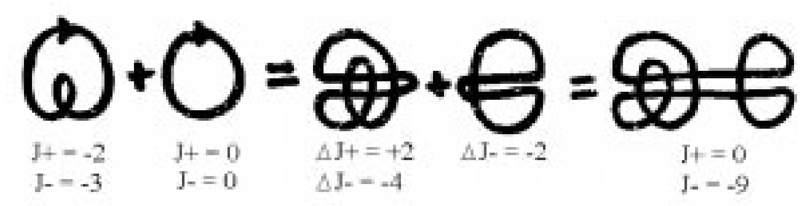
Proof. The strange sum is reduced to the connected sum if first appendices are pushed from each curve toward the other:



The appendices do not create any triple points, and therefore do not change the value of St.

The curves with appendices are then added using the connected sum.

Remark 32 *The appendices are created by positive $J^{\pm}$ moves. This prevents $J^{\pm}$ from being invariant under the connected sum:*

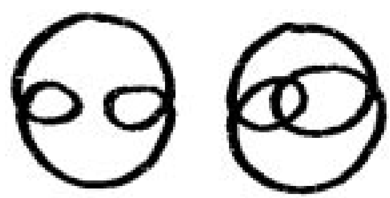


■

7 Tree-like curves

Aicardi reduces the problem of classifying curves by only allowing the tree-like curves. The tree-like curves are of interest because they are "simpler" than non-tree-like curves - many proofs that are all but impossible on non-tree-like curves are attainable on these curves.

Definition 33 *A generic curve is tree-like if every double point divides it into two loops having no other common points.*



A tree-like and a non-tree-like curve.

Lemma 34 *The number of Seifert's circles for a given curve is n + 1.*

Proof. The curve begins as a single loop. Each oriented surgery breaks off another loop, so after n oriented surgeries there are $n+1$ loops, each a Seifert's circle. ■

Definition 35 *The father of a non-external loop L is the last loop on the path from the nearest external loop to L.*

The father of b is a, the father of e is d, and the father of d is c. The father of both a and c is the external loop.

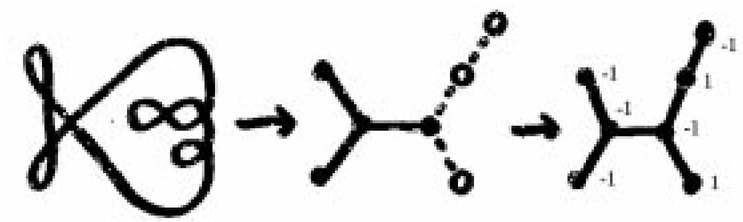
Remark 36 *There are many theorems which can be proved on tree-like curves. For example, Aicardi proves that* $J^+ + 2St = 0$ *for all tree-like curves.*

One can consider $J^+ + 2St = 0$ to be a "zero" of generic curves - the amount by which a curve's invariants differ from this value being a measure of its deviation from a tree-like curve.

7.1 A-Trees

To simplify the problem of classification, Aicardi defines structures called A-trees which represent each tree-like curve [1].

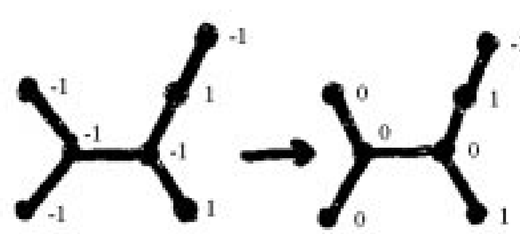
Definition 37 *An A-tree consists of three components:*
a) T - the tree
b) F - the subtree of exterior loops
c) c - the character function, defined as follows:
i) c(v) = -1 if $v \in F$ *is an exterior vertex*
ii) c(v) = 1 if the associated loop of v lies inside its father
iii) c(v) = -1 if the associated loop of v lies outside its father



A given curve, its tree with F indicated in the black vertices, and its character function.

Remark 38 *Since each vertex represents a loop, it follows that the number of vertices for a given curve is n + 1.*

Definition 39 *The t function on a given A-tree is defined as follows, where f(v) denotes the father of v:*
a) t(v) = 0 if $v \in F$
b) t(v) = t(f(v)) + 1 if v lies inside its father
c) t(v) = -t(f(v)) if v lies outside its father



The character function and the t function for the curve above.

Aicardi goes on to prove that St $= \sum t(v)$, $J^+ = -2St$, and $J^- = -2St - n$. This leads to the following theorem:

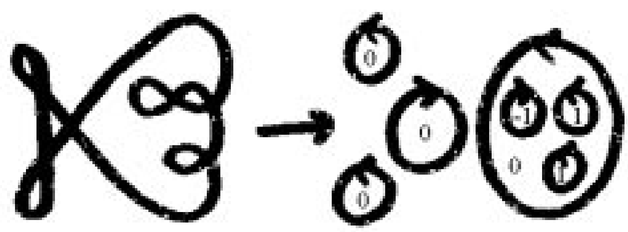
Theorem 40 *The value of St for a given tree-like curve can be calculated from its Seifert's circles, where $s(r)$ is the value of a given circle, and $s(r_0)$ the value of the circle which encloses it:*

a) Let the external circles each have a value of $s(r) = 0$.

b) If a circle has positive orientation, let it have a value $(s(r_0) + 1)$.

c) If a circle has negative orientation, let it have a value $-(s(r_0) + 1)$.

Then $St = \sum s(r)$. Of course, $J^+ = -2St$ and $J^- = -2St - n$ follow from $J^+ + 2St = 0$ and $J^+ - J^- = n$.



The labelling system of the theorem corresponds to the labelling of the t function, so the proof is omitted.

7.2 Extremal values

Two examples of proofs that are difficult to prove on all curves, yet relatively easy on tree-like curves, are those dealing with extremal values of the basic invariants.

Theorem 41 *The maximum value of $St = \frac{n(n+1)}{2}$ and the minimum values of $J+ = -n^2 - 2n$, $J- = -n^2 - n$ occur at one and only one tree-like curve with n crossings.*

Proof. First we will prove $St_{\max} = \frac{n(n+1)}{2}$ by induction.

For $k = 1$ crossing, there are only two possibilities.

K_0 has $St = 0$, and K_2 has $St = 1$. $\frac{1(1+1)}{2} = 1$.

For $k = n+1$ crossings, suppose one is at $St_{\max}$ for n crossings.

Then $St = n(n+1)/2 = 1 + 2 + ... + n$.

There is only one way to get $1 + 2 + ... + n$:

When one crossing is added, a new vertex appears. The greatest possible value for this vertex is desired, to maximize St for the new curve.

The greatest possible value is $n + 1$, which occurs when the new vertex is placed as follows:

$\frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2} = St_{\max}$ for $n+1$ crossings.

Next, because $J^+ + 2St = 0$ for all tree-like curves, and $St \leq \frac{n(n+1)}{2}$, we get $J^+ \geq -n^2 - n$.

Finally, since $J^+ - J^- = n$, $J^- \geq -n^2 - 2n$. ■

Theorem 42 *Let $n = 3d + r$. The minimum value of $St = (-3d^2 + d - 2rd)/2$, and is attained at*

a) one tree-like curve with ind $= 1 - d$ if $r = 0$

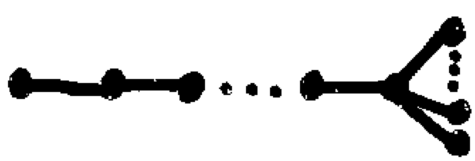
b) one tree-like curve with ind $= -d$ if $r = 1$

c) two tree-like curves with ind $= 1 \pm d$ if $r = 2$

On these curves the maximal values of $J^{\pm}$ are reached; they are $J^+ = 3d^2 - d + 2rd$ and $J^- = 3d^2 - 4d + 2rd - r$.

Proof. Since $St = \sum t(v_i)$, the minimal value of St is obtained at an A-tree where the negative values of $t(v)$ give the maximal contribution. From the definition of $t(v)$, we cannot obtain vertices with $t < 0$ without vertices with $t > 0$. The first idea that comes to mind is to attach as many negative vertices as we can to a vertex with positive character.

Suppose we have k vertices. The minimal value of t is attainable at a negative vertex whose father v has a positive and maximal value of t. This value occurs at the last vertex of a tree with $k-1$ vertices with maximum St, as in the previous theorem. Under these conditions, we end up with an A-tree of the form:



For this tree, $St = \dfrac{n_+(n_+ + 1)}{2} - n_+(n_-) = \dfrac{n_+(n_+ + 1)}{2} - n_+(n - n_-)$, and $ind = 1 + n_+ - n_- = 1 + 2n_+ - n$, where n_+/n_- = the number of positive/negative crossings.

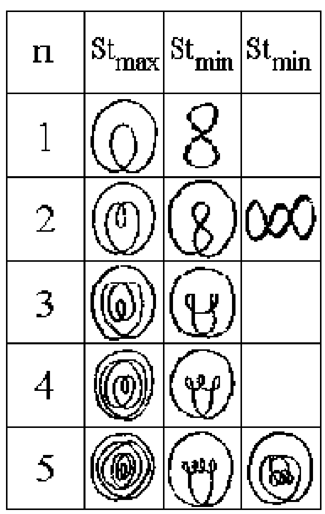
Let $m = n_+$ when St is at its minimum. Then $St_{\min} = \dfrac{m(m+1)}{2} - m(n - m)$, and $ind = 1 + 2m - n$.

Solve the first equation for m to get $m = \frac{1}{6}(\sqrt{(2n-1)^2 + 24St} \pm (2n-1))$.
When St is at a minimum, $m =$ the nearest integer to $\pm\frac{1}{6}(2n-1)$.
Since $m < n$, m is the nearest integer to $\frac{1}{6}(2n-1) = \frac{n}{3} - \frac{1}{6}$.
Let $n = 3d + r$. Some short caculations give the following table:

r	ind	$St_{\min}$	$J^+_{\max}$	$J^-_{\max}$
0	$1 - d$	$(-3d^2 + d)/2$	$3d^2 - d$	$3d^2 - 4d$
1	d	$(-3d^2 - d)/2$	$3d^2 + d$	$3d^2 - 2d - 1$
2	$1 \pm d$	$(-3d^2 - 3d)/2$	$3d^2 + 3d$	$3d^2 - 2d$

The corresponding A-trees are unique up to isomorphism. ■

Remark 43 *The previous theorem simplifies the classification of tree-like curves with extremal values of the invariants. The curves with maximum St are all in the form of concentric loops. To draw the curves with minimum strangeness, one can determine the number of negative loops by noting that there are* $n + 1$ *loops, and that the index is equal to the number of negative loops subtracted from the number of positive loops. Thus the number of positive loops is* $\frac{1}{2}(n+1+ind)$*, and the number of negative loops is equal to* $\frac{1}{2}(n+1-ind)$*. The positive curves are drawn first, as one would draw a curve with maximum St. The negative loops are attached to the innermost positive loop.*



A table of the curves with extremal values of the basic invariants.

References

1. F. Aicardi, *Appendix to [2].*

2. V. I. Arnold, *Plane curves, their invariants, perestroikas, and classifications.* Preprint ETH Zürich. May 1993. 58 pages; *Singularities and curves.* Advances in Soviet math., vol. 21, Amer. Math. Soc., Providence, RI, 1994.

3. H. Whitney, *On regular closed curves in the plane,* Compositio Math. 4 (1937), 276-284.