Lower Bound on Open Knight's Tours

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Abstract

An exponential lower bound on the number of open knight's tours (hamiltonian paths) is constructed for square boards of size $6 \cdot 2^i \times 6 \cdot 2^i$. The method is to establish a lower bound on the number of open king's tours on $2^i \times 2^i$ boards. Then 6×6 knight's tour boards are substituted in place of each square on the king's tour board. The knight's tours on the 6×6 boards may then be connected to follow any king's move from one 6×6 board to the next. The lower bound gives on the order of 1.319^{n^2} tours on $n \times n$ boards for $n = 6 \cdot 2^i$, $i = 1, 2, 3 \ldots$. This holds for closed tours, too.

The exponential lower bound on king's tours for $2^i \times 2^i$ boards is shown, and the exact number of king's tours on $2 \times m$ boards is given informally.

1 Introduction

The problem of finding Knight's tours on chessboards has been studied by mathematicians as early as Euler [1]. There are a variety of questions associated with the problem, and different methods applied in trying to answer these questions. Some discussion about knight's tours in general is followed by a discussion of this research toward the end of this introduction.

1.1 General Knight's Tour Problem

A board is an $n \times m$ grid; a square-shaped area broken up into $n \cdot m$ squares, m going across and n going down, like a chessboard. A knight is the name for something which has a location on a certain square and can move in a 1×2 pattern; one square up and two over, two up and one over, etc. A knight's tour of a board is a sequence of knight's moves from one square to another such that the knight hits each square on the board exactly once; no square may be revisited.

1. Place a knight on an empty board
2. Move the knight from square to square
3. Do this so that every square is hit exactly once

1.2 Terminology

A few terms need to be discussed in order to describe the specifics of the problem.

- A knight's tour graph consists a set of vertices, one for each square on the board, and a set of edges connecting the vertices according to a knight's move.

[0]This research per the REU program in mathmatics at Oregon State University, 1996. Thanks go to: advisor Paul Cull for wearing red shoes; Douglas Squirrel for doing all the programming; program director Dennis Garity for freeing the clouds.

- An open knight's tour is a hamiltonian path on the knight's tour graph.
- A closed knight's tour is a hamiltonian cycle on the knight's tour graph.
- A king's tour is the same as a knight's tour except that the king's move is to any of the eight adjacent squares - one square left, right, up, down, or diagonally.

1.3 Previous Work

It is well established that both open tours [2] and closed tours [3,5] exist on boards meeting certain size criteria. Open tours require 5×5 or greater size; closed tours require 6×6 or greater with an even number of squares. The existence of these tours was shown by constructing a number of base cases. The base cases are boards of certain sizes on which tours start end and end on specific squares. The method of proof is to show that it is possible to patch these boards together and connect the knight's tours to form larger tours.

Additionally, an exponential lower bound has been established on the number of closed knight's tours on $n \times n$ boards, for even $n \geq 12$. The bound guarantees that there are on the order of at least 1.16^{n^2} closed tours. Further, an asymptotic lower bound of 1.35^{n^2} is given [3].

1.4 Interests

There are a number of interests pertaining to the general knight's tour problem. These things are of interest for several reasons. One example is the following.

Lately in computer science, there has been a great deal of interest in graph theory in general since it is essentially the study of how things in a group relate to the other things in the same group; the relation being "is this thing connected to this other thing". This is of obvious interest when one desires to send information from one thing to another. Following from that, an apt question is that of the existence of hamiltonian paths and cycles. The knight's tour graph is simply a nice example to study. It is well known that finding hamiltonian paths/cycles is NP-hard, and generally unapproachable from any analytical sense for random or arbitrary graphs. Hence, it is thought that by studying the highly structured knight's tour graph, a way of approaching the general problem may become apparent.

There are other reasons to be interested in the knight's tour problem ranging from mere curiosity to those, like the above, which relate to useful application. As to the specific topics of interest pertaining to the knight's tour problem, a list is appropriate.

- Proof of the existence of a knight's tour on rectangular boards of arbitrary size which does not rely on the construction of tours from a large number of base cases [4].
- Enumeration of the number of tours - finding the exact number or finding upper and lower bounds - for boards of arbitrary size [3,7].
- Proof of the existence of tours on boards missing squares [6].
- Investigation into how the number of open tours relates to the number of closed tours on a given board.
- Investigation into tours on chessboards with a different definition of a knight's move (such as three up, one over).

- Investigation into tours on higher dimension boards or boards mapped onto closed surfaces.

- Various combinations of the above.

1.5 This Research

The research done here was in trying to enumerate the number of open knight's tours. More specifically, a loose lower bound is given for the number of open tours on square boards where the size of the board is $6 \cdot 2^i \times 6 \cdot 2^i$ for $i = 1, 2, 3...$ Hence, boards like: 12×12, 24×24, 48×48, 96×96 and so on. Also, as a consequence of the method used, a lower bound is given on the number of king's tours on square boards of size $2^i \times 2^i$, as well as the exact number of king's tours on boards of size $2 \times m$ for $m > 1$.

The method used to find the lower bound on knight's tours is to find a number what will be called *structured* knight's tours on 6×6 ***sub-boards***. The structured tours are such that when several sub-boards are blocked together, it is possible to connect a structured tour on one board to a structured tour on any adjacent board, hence constructing larger tours which go through the several sub-boards one at a time. The structured tours are basically ones which both start, and end, near a corner of the sub-board. There is a set of ten (more with symmetry) structured tours which are needed. This will be explained in more detail later. The points of importance are:

- Take a number of 6×6 sub-boards.

- Block the sub-boards together to form a larger board - say a 24×24 board formed from 16 sub-boards.

- There is now a 4×4 grid where each grid-square is a 6×6 sub-board.

- Using structured tours for each sub-board, it is possible to patch together a knight's tour which traverses the entire 24×24 board.

- The tour starts in one of the sub-boards, moves through that sub-board, then jumps to an adjacent sub-board; completes that sub-board and moves on...

Also of importance: it is possible to move from sub-board to sub-board in any of the ways a king can move from one square to the next on a chessboard. Given any series of king's tour moves, it is possible to duplicate that series in moving from one sub-board to the next by using some combination of the ten structured tours. Hence, we were interested in the number of king's tours on rectangular boards.

There was no success in either: 1) determining the exact number of king's tours on general rectangular boards of arbitrary size; or 2) finding anything about number of king's tours in the literature. The best that could be done was to find a lower bound on the number of king's tours on square boards of size $2^i \times 2^i$ for $i = 1, 2, 3...$ which grows like 1.26^{4^i} - that is, 1.26^{N^2} for $N \times N$ size king's boards where $N = 2^i$.

1.5.1 Other Stuff

In connection to the knight's tour problem, several things were looked at beyond what is described above. The structure of the knight's graph was studied in an attempt to find some general classification of types of knight's tours which exist on small boards ($n < 7$). The idea behind this approach is that it may be possible to classify all tours in terms of multiple (nested) symmetries to just a few base tours. This was only looked at briefly since it appeared to be very complicated if not impossible, and time restrictions existed.

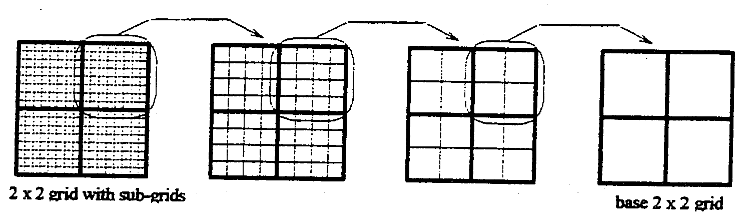
It was not apparent that this more analytic approach to the knight's tour problem would not work, but it appeared very difficult even on small boards.

Finally, a closed form expression was derived giving the exact number of king's tours on a $2 \times m$ board. This is briefly described near the end of this paper.

2 King's Tours - Lower Bound

To find a lower bound on the number of king's tours, we construct larger tours recursively from smaller tours. This is easily done if we only worry about counting a small number of the possible tours. The method avoids the problem of potentially counting a tour twice, or of counting something which is not a tour. The method results in an exponential lower bound on king's tours for certain square boards, namely boards size $2^i \times 2^i$ for $i = 1, 2, 3 \ldots$.

We use an inductive argument to show that it is possible to construct multiple tours recursively. The argument relies on the concept of a 2×2 grid in which each of the four individual grid-squares is also a 2×2 "sub-grid" itself. The figure below is appropriate.



> Base Step: For any given 2×2 grid with given starting and ending points, there are **2** different king's tours of the grid. See Figure 1 for the various cases.
>
> Inductive Step: For any given 2×2 grid, G, which contains four 2×2 sub-grids, g_1, g_2, g_3, g_4, as it's grid-squares, it is possible to find starting and ending points for each of the four 2×2 sub-grids $g_1 \ldots g_4$, so that **2** different king's tours of the original 2×2 grid G exist for any given starting and ending points on the original 2×2 grid G. See Figure 2 for sufficient cases.

With this, we can give a lower bound on the number of king's tours by multiplying together all ways of doing tours on all the sub-grids (and grids containing sub-grids...) contained in a given king's board of size $2^i \times 2^i$. Since there are two different ways for each grid, we will have $2^{H(i)}$, where $H(i)$ is the total number of sub-grids plus one – one large grid containing sub-grids, where each sub-grid is one large grid containing sub-grids, and so forth.

To find $H(i)$, we simply count up how many of these grids and sub-grids there are in a given board size $2^i \times 2^i$, given that $H(1) = 1$ (for the basic 2×2 king's board). From Figure 3, it is obvious that $H(i) = 1 + 4H(i-1)$, or in closed form $H(i) = \frac{1}{3}(4^i - 4)$ (plus one, but it's unimportant).

We have a lower bound on the number of king's tours on $2^i \times 2^i$ boards:

$$king(i) \geq 2^{\frac{1}{3}(4^i - 4)}$$

Noting that there are 4^i squares on a $2^i \times 2^i$ board, let $N^2 = 4^i$. Then for $N = 2^i$, we have on an $N \times N$ board:

$$king(N) \geq 2^{\frac{1}{3}(N^2-4)} \approx 1.26^{N^2} \text{ for } N = 2, 4, 8 \ldots$$

Figure 1

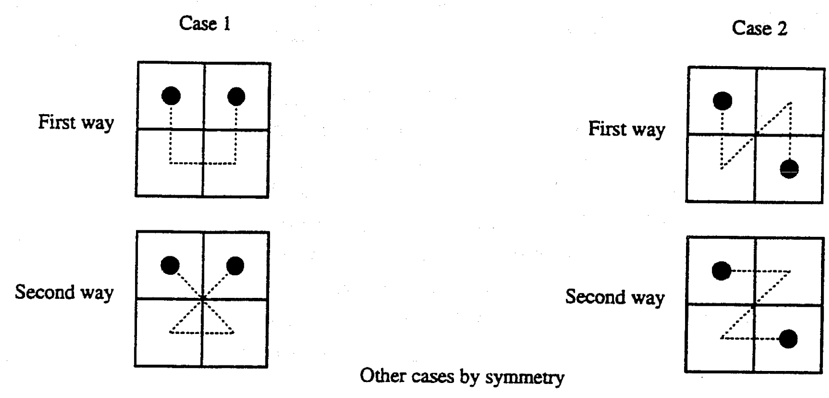


Figure 2

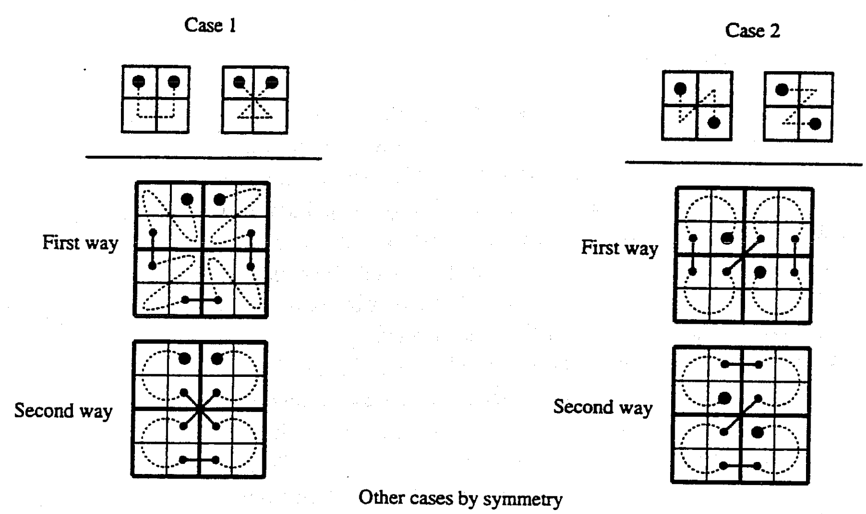
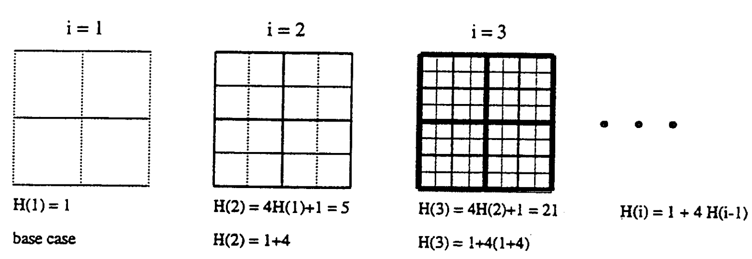


Figure 3



3 Lower Bound on Open Knight's Tours

Using the lower bound on the number of king's tours, we will derive a lower bound on the number of knight's tours for square boards size $6 \cdot 2^i \times 6 \cdot 2^i$ for $i = 1, 2, 3 \ldots$. Knight's tours on small boards will be connected together, from small board to small board, to form a large knight's tour. We will show that this can be done in a pattern following any king's tour.

A few points about the structure of the knight's tour graph and how this relates to exponential bounds on the number of tours seems apt. From any given square on a knight's board, there are at most eight other squares around it where a knight could move. If one considers squares in a knight's tour, there are at most 7 possible moves since the square is entered from one of the eight, but one cannot return to that square leaving only seven. For a board with n^2 squares, an obvious upper bound on the number of tours is 7^{n^2}. The same holds for king's tours. But clearly there cannot be anywhere near 7^{n^2} tours. This is because as the tour proceeds, it is possible to move to a square that is isolated – you get stuck. There are in fact a huge number of ways to disconnect the graph (get stuck). That is what makes it so hard to find knight's tours.

Although we refer only to finding open knight's tour's in this section, the method may be stated equivlenty for closed knight's tours. Both can be given the stated lower bound; when applied to closed tours we actually get a trivialy larger number of tours.

Before showing the construction, it is appropriate to discuss what we call a *structured knight's tour*.

3.1 Structured Knight's Tours

Note: The term *structured knight's tour* is not a good name; it does not refer to a single knight's tour, but rather to *a lot* of knight's tours which are similar.

A structured knight's tour is any one of ten particular types of knight's tours on 6×6 knight's boards. They are particular only in that they each have specified squares where a tour starts and ends. See Figure 4. The structured tours are designed so they can be patched together to form larger tours. For example, four structured tours can be put in a 2×2 grid. The four tours can be connected. What results is a knight's tour of a 24×24 board (actually, *many* knight's tour of a 24×24 board).

There are many ways to do each structured tour. In other words, there are many different knight's tours of a 6×6 board which have the same end points. When the structured tours are patched together, the number of tours multiplies.

For each structured tour, we determined the number of different knight's tours that fit that structure (almost). This was done with a standard backtracking program which searched for knight's tours which started and ended in the correct places[1]. For the ten cases, all had at least 8552 knight's tours. It took several days to check each case running on a Sun Sparc 5. The exact numbers were determined for all but two cases. For those two cases, it was determined that there are more than 8552 tours for each, but the program was halted before it finished due to a file space problem.

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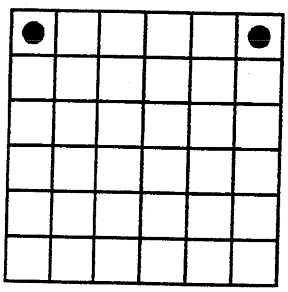
[1]The author is indebted to Douglas Squirrel for writing the backtracking program.

Figure 4

The ten structured knight's tours. Numbers indicate how many different knight's tours there are which go from one point to the other.



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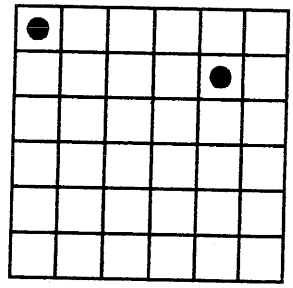

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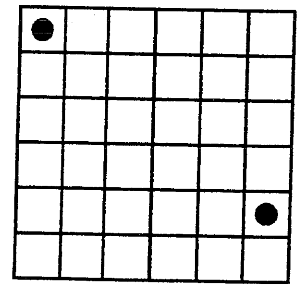

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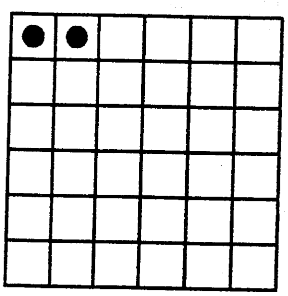

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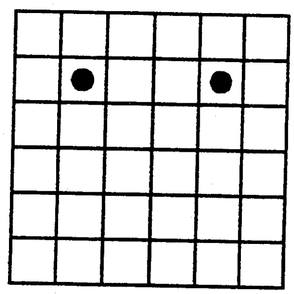

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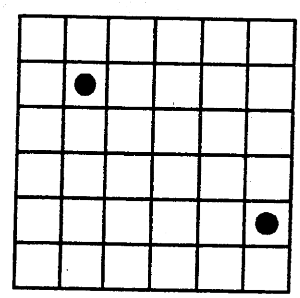

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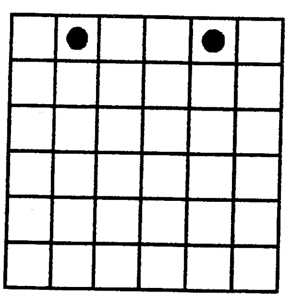

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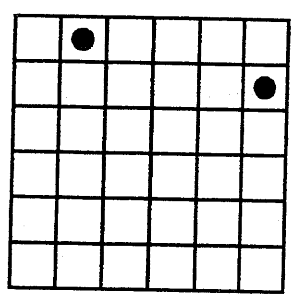

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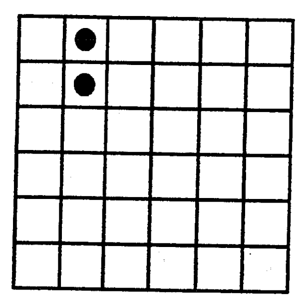

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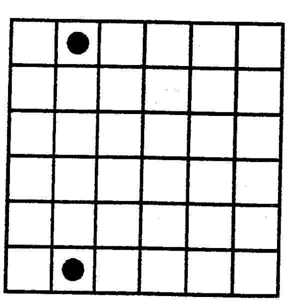

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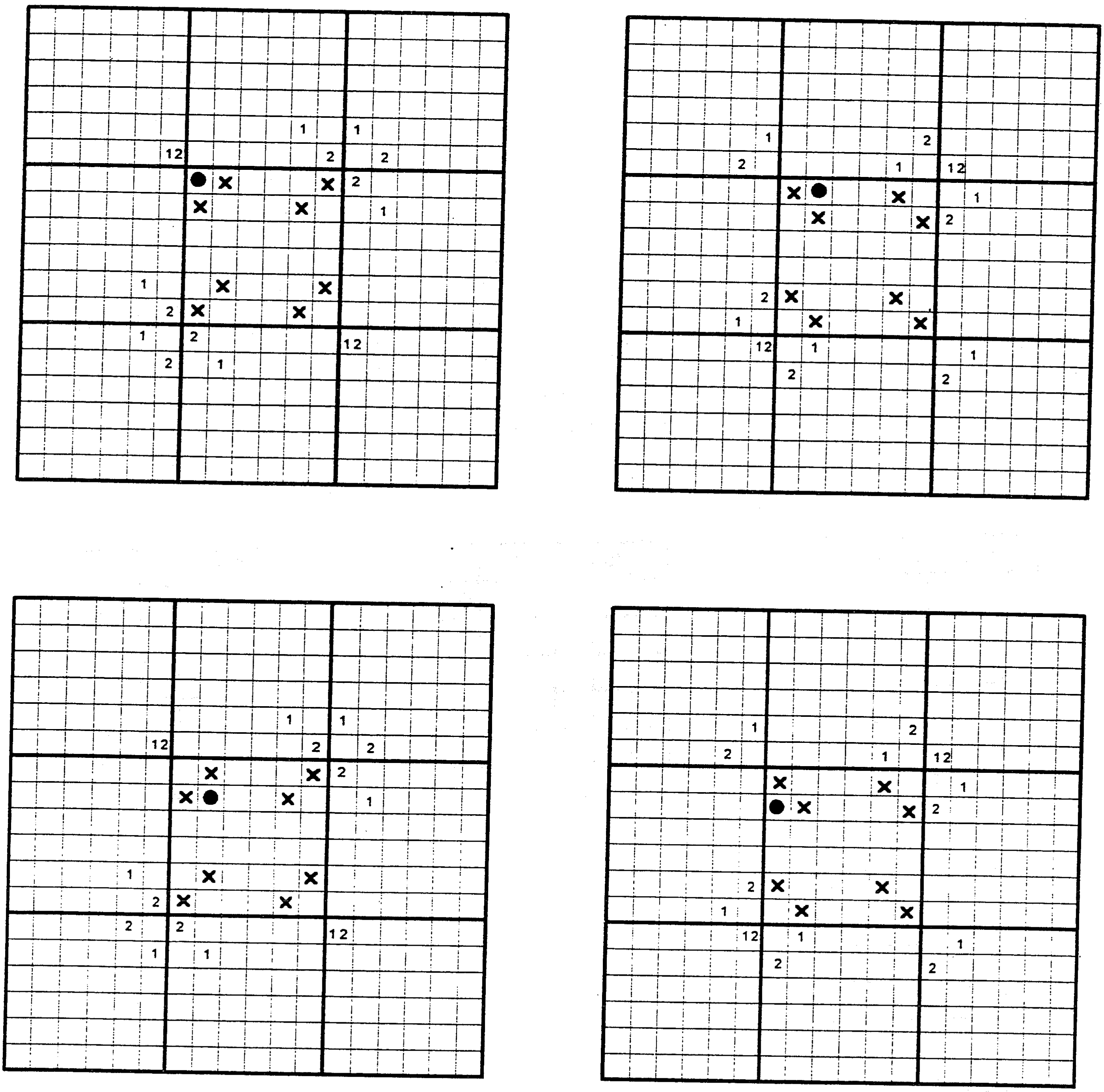
Figure 5

Sufficient cases showing it is possible to follow any king's move at least two ways.

The solid dots represent possible starting squares of a structured tour. The bold **X**'s represent possible ending squares of structured tours with the given starting square. The numbers 1 and 2 in the various squares represent the first and second way to arrive at a starting square of a structured tour on an adjacent board.



3.2 Large Knight's Tours

We show how large knight's tours may be formed from connecting several structured knight's tours together. The idea starts with placing a single 6×6 knight's board in each square of a king's board. Assuming we use king's boards of size $2^i \times 2^i$, the resulting knight's tour board will be of size $6 \cdot 2^i \times 6 \cdot 2^i$. Using some combination of the ten structured tours, the small 6×6 knight's tours can be connected to form a large knight's tour. The tours can be connected from one board to another in any of the ways a king can move from one square to another in a king's tour. Since we have shown there exist lots of king's tours, we have lots of different ways of connecting the structured tours. This will give us lots of different knight's tours.

By using some combination of the ten structured tours, there are **two** different ways of doing the following:

- Begin at the starting point of some structured tour.
- Complete the knight's tour of that 6×6 board; the tour ends on the ending point of a structured tour.
- Make a single knight's move to the starting point of a structured tour on any adjacent 6×6 board.

This is shown case by case in Figure 5.

We are now able to patch together the structured tours in two different ways for each king's tour.

3.3 Summary - Lower Bound on Knight's Tours

We have established the following:

- There are at least $2^{\frac{1}{3}(N^2-4)}$ different king's tours of $N \times N$ king's boards for $N = 2, 4, 8 \ldots$.
- There are at least 8552 different knight's tours for each structured tour.
- There are at least 2 different ways of connecting structured tours for each king's tour move.

It is now trivial to find the combination of all these different ways.

There are $N^2 - 1$ king's moves through the structured knight's tours. For each king's move, there are $2 \cdot 8552$ different knight's tours. This gives $(2 \cdot 8552)^{N^2-1}$ different knight's tours for each king's tour. With $2^{\frac{1}{3}(N^2-4)}$ different king's tours, we have $(2 \cdot 8552)^{N^2-1} \left(2^{1/3}\right)^{N^2-4}$ different knight's tours. The overall size of the knight's tour board is $6 \cdot N \times 6 \cdot N$ and hence there are $36 \cdot N^2$ squares on it. We wish to have our lower bound in terms of n^2, $n = 6 \cdot N$, for an $n \times n$ knight's board. Hence, we replace N^2 by $n^2/36$. This gives:

$$knight(n) \geq (2 \cdot 8552)^{\frac{n^2}{36}-1} \left(2^{\frac{1}{3}}\right)^{\frac{n^2}{36}-4}$$

Simplifies to:

$$knight(n) \geq \frac{\left(17104^{\frac{1}{36}} \cdot 2^{\frac{1}{108}}\right)^{n^2}}{2^{4/3} \cdot 17104} \approx 1.319^{n^2} \text{ for } n = 6 \cdot 2^i$$

These are square knight's tour boards size: 12×12, 24×24, 48×48, $96 \times 96 \ldots$

4 King's Tours on $2 \times m$ Boards - Exact Number

An interesting closed form expression was derived giving the exact number of king's tours on boards of size $2 \times m$ for $m = 2, 3, 4 \ldots$. Two linear recursion relations were derived for the number of tours between 1) points on opposite ends of the strip of given length m, and 2) points both at one end (next to each other). It is possible to then sum over those functions (it's a mess) to get the total number of king's tour between every pair of squares on the board. A complete derivation of the formula will not be given since it is a huge mess. Perhaps there could be found a more elegant way of deriving the formula if more time was spent. Although the accuracy of the formula way verified computationally for $m = 2..13$, it should be noted that the author did not rigorously check it.

The exact number of king's tours on a board size $2 \times m$ is:

$$tours(m) = 2^{m+1}\left(f(m) + \frac{3-m}{4}\right)$$

where $f(m)$ is the m^{th} fibbonacci number. Note: our fibonacci numbers are shifted down one: $f(0) = 1$, $f(1) = 1$, $f(2) = 2$, and so on.

It is "nice" that the fibonacci sequence appears in the formula, but the author does not have any elegant explanation of why.

5 Conclusion

We have derived a lower bound on the number of open knight's tours on $n \times n$ boards, $n = 6 \cdot 2^i$ for $i = 1, 2, 3 \ldots$. The bound increases exponentially like 1.319^{n^2}. In addition, we have a lower bound on king's tours for $N \times N$ boards, $N = 2^i$. The bound increases exponentially like 1.26^{N^2}. The knight's tour bound follows from the king's tour bound. The bounds are by no means tight, and the author feels that by applying the same methods more carefully, better bounds could be found quite easily. It should be noted that the bounds for both the open king's tours and the open knight's tours hold for closed tours. This follows from the fact that starting and ending points are relatively arbitrary, and hence may be one move apart.

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