The Unknotting Number for $2k+3$, $2j+1$, $2k+2$ and Other Knots of the Form a, b, c

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Definitions:

The unknotting number of a knot projection is the number of crossing changes in that projection which are needed to obtain the trivial knot.

The unknotting number of a knot is the minimum of the unknotting numbers of the knot's various projections.

A minimal projection of a knot is a projection with a minimum number of crossings; such a projection has no trivial twists.

Conway Notation:

Throughout this paper we will use Conway Notation [4]. This notation uses the idea of multiplying tangles to produce a knot. For example, multiplying tangles n, m, and p, for integers n, m, and p, we obtain the following diagram, which we call knot n, m, p:

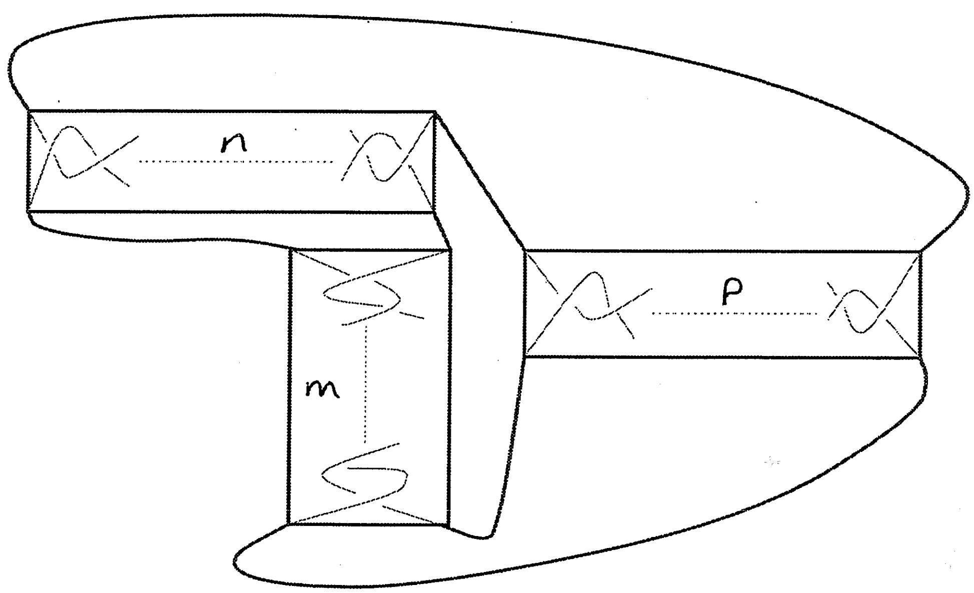


Figure 1

Continued Fractions:

When a knot is written in Conway notation, it can be associated with a continued fraction. The following shows how we associate a continued fraction to any knot K written in Conway notation:

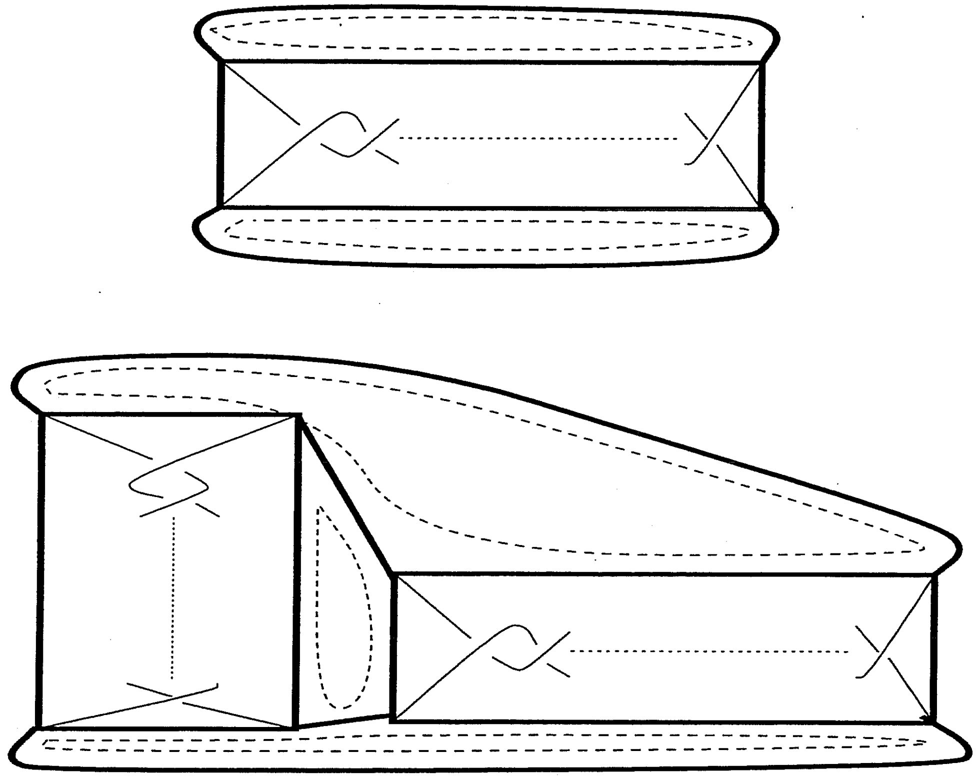
$$K = (c_1,\, c_2,\, \ldots c_{j-1},\, c_j) \implies c_j + \cfrac{1}{c_{j-1} + \cfrac{1}{\ldots + \cfrac{1}{c_2 + \cfrac{1}{c_1}}}}$$

Theorem I: *If the continued fractions of two knots are equal then the knots are equivalent* (Conway).

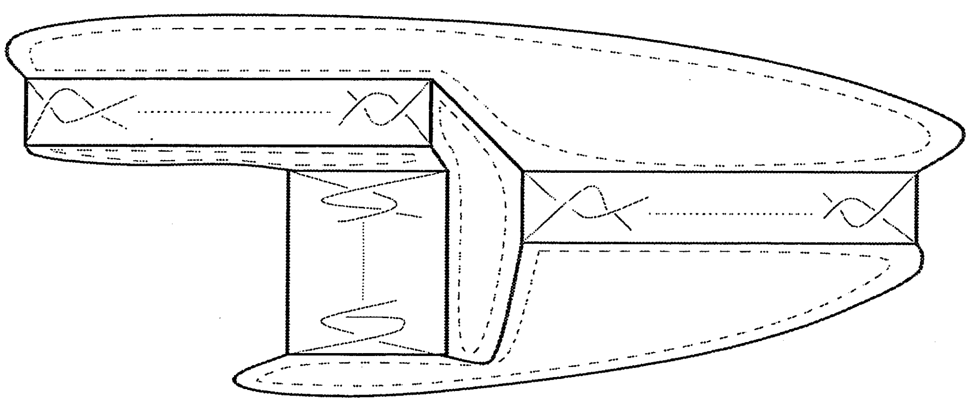
In analyzing specific cases of different knot classes, we noticed patterns for their continued fractions and derived the following general formulas for positive integers a, b and c:

i. For $c > 0, b > 1$, $(-a,\, b,\, c) \equiv (a-1,\, 1,\, b-1,\, c)$.
ii. For $b > 2, c > 1$, $(a,\, -b,\, c) \equiv (a-1,\, 1,\, b-2,\, 1,\, c-1)$.
iii. For $a, c > 2$, $(a,\, -1,\, c) \equiv (a-2,\, 1,\, c-2)$.
iv. For $a > 0, c > 1$, $(a,\, b,\, -c) \equiv -(a,\, b-1,\, 1,\, c-1)$.

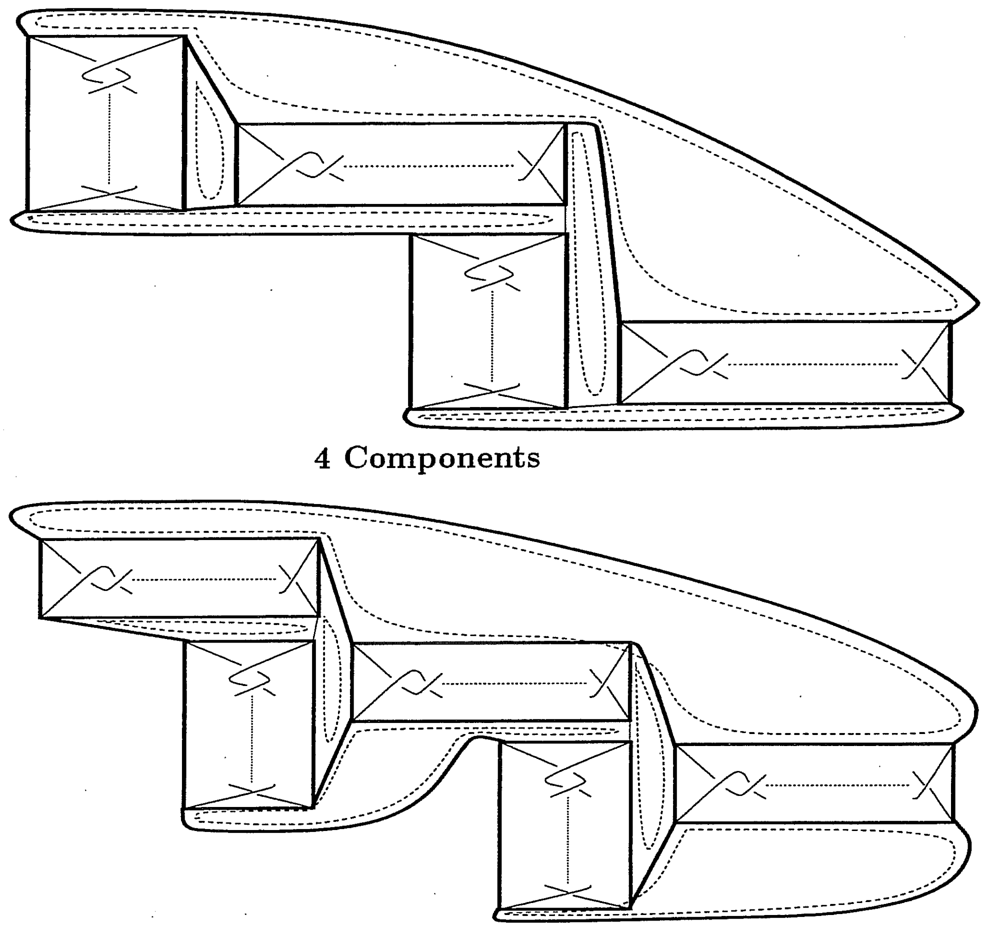
A knot in Conway Notation with all positive or all negative components is a reduced alternating knot and is non-trivial. A projection is alternating if over and under crossings alternate. A projection is non-reduced if there is a simple closed curve that transversely intersects the projection exactly once at a crossing. Figure 2 below illustrates these two concepts for knots with 1, 2, 3, 4, and 5 components.



1 and 2 Components



3 Components



4 Components

5 Components

Figure 2

In all of these alternating knots, no simple closed curve intersects the knot in such a way as to make it non-reduced.

Background Information:

It was thought that the unknotting number of a knot was always realized in a minimal rather than non-minimal projection. However, Bleiler [3] and Nakanishi [6] both produced an example of a knot that contradicted this belief. The example was the knot 5 1 4 whose minimal projection requires 3 crossing changes to become unkotted, but which has a non-minimal projection requiring only 2 crossing changes to become unknotted. Bernhard [2] then expanded on the work of Bleiler and Nakanishi. Using the knot 5 1 4 as the base case, Bernhard produced a proof for the infinite class of knots $2k+1, 1, 2k$ for $k \geq 2$, showing that their unknotting numbers come from a non-minimal rather than a minimal projection. In this paper, we will focus on an even broader class of knots by producing an infinite number of infinite classes of knots whose unknotting numbers result from non-minimal projections.

There are three important theorems to note which help one understand our proof, as well as the origins of the proof. This first theorem concerning continued fractions we have already stated.

Theorem II: *any reduced alternating projection is minimal* (Kauffman [5]).

Theorem III: *any two reduced alternating projections of a knot have the same number of crossings* (Kauffman, Thistlethwaite and Murasugi, see [1]).

As a result of Theorems II and III, if one reduced alternating projection is minimal, all reduced alternating projections are minimal. Thus, when regarding minimal projections, one need only consider one reduced alternating projection. The proof we will develop to generalize this broader class of knots is by induction.

Let the unknotting number of a projection, P, be denoted $u(P)$ and let a minimal projection of P be denoted $(P)_{min}$.

Let (N, M) denote the knot $K = 2N + 3,\ 2M + 1,\ 2N + 2$.

Definition: Statement $S(N, M)$ means that for knot K corresponding to (N, M), where integers $N \geq 1$ and $M \geq 0$,

$$u(K)_{min} > \begin{cases} 2N+1 \text{ changes} & \text{if } N \leq M \\ N+M+1 \text{ changes} & \text{if } N \geq M \end{cases}.$$

Consider the following table of knots:

(k,j)	$j=0$	1	2	3	.	M
$k=1$	5,1,4	5,3,4	5,5,4	5,7,4	.	$(1,M)$
2	7,1,6	7,3,6	7,5,6	.	.	.
3	9,1,8	.	9,5,8	.	.	(k,M)
4	11,1,10	.	.	.	(j,j)	.
.	.	.	.	.	.	.
N	$(N,0)$	$(N,1)$	.	.	.	(N,M)

Table 1

Notice that when $k = j$, the two cases in the definition overlap. The corresponding knots lie on what we shall call the diagonal, the diagonal line in our table which has 5,3,4 at its corner.

We will show that for any knot P on or below the diagonal, the unknotting number of P is not realized directly from its minimal projection. First, we will generate a non-minimal projection of P and find its unknotting number. Then we will use an inductive proof to show that the unknotting number of the minimal projection of P is greater than that of our non-minimal projection.

For any knot Q above the diagonal, we will find $u(Q)_{min}$. Because the middle component of Q has a greater number of crossings than the right component, we cannot use an analagous non-minimal projection of Q to the one we use for P to find the unknotting number of Q. Nonetheless, we will have generated an infinite number of infinite classes of knots whose unknotting numbers cannot be found only by considering their minimal projections.

Unknotting the Non-Minimal Projection

Let P be a knot on or below the diagonal of Table 1, so $P = 2k+3,\ 2j+1,\ 2k+2$ for integers $j \geq 0$ and $k \geq 1$ such that $k \geq j$. Make j changes to the middle component of P and we get the knot $2k+3,\ 1,\ 2k+2$. Pull alternating loops in the right component under the string connecting the middle and right components, as depicted in Figure 3 below, and we obtain a non-minimal projection of P corresponding to the non-minimal projections of the knot class $2k+1,\ 1,\ 2k$, for $k \geq 2$, generated by Bernhard. Writing these knots in the form $2k+3,\ 1,\ 2k+2$ for $k \geq 1$, Bernhard found that these non-minimal projections have unknotting number $\leq k+1$. So the unknotting number for the non-minimal projection of $2k+3,\ 2j+1,\ 2k+2$ is $\leq k+j+1$.

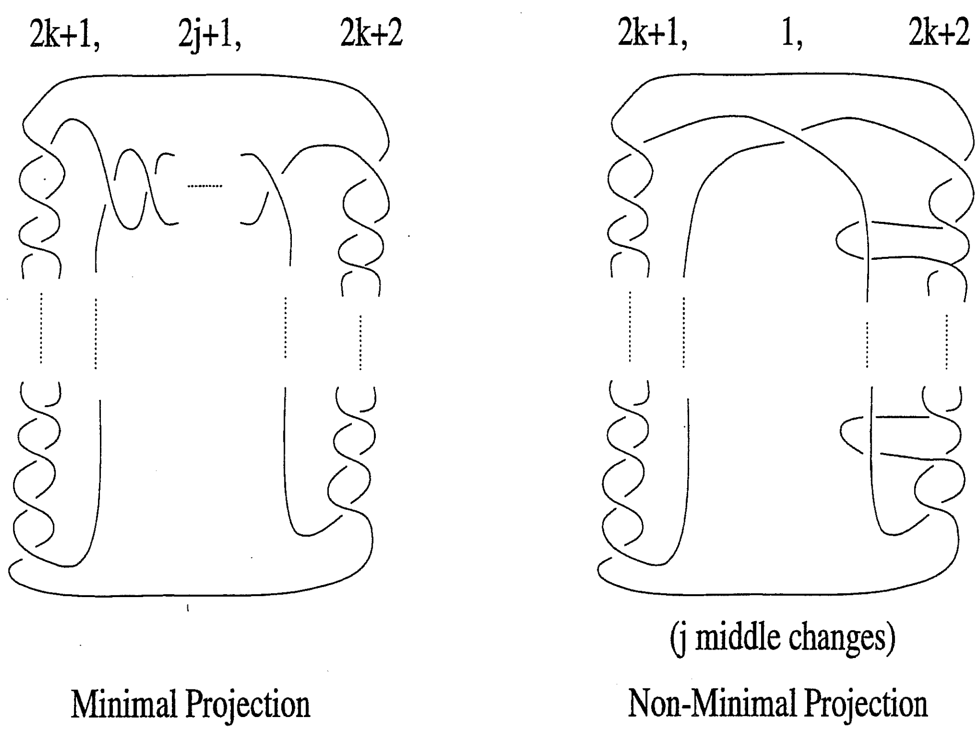


Figure 3

Unknotting the Minimal Projection

First we will show that the minimal projections of knots above the diagonal can be unknotted in $2N+2$ changes and the minimal projections of knots on and below the diagonal can be unknotted in $N+M+2$ changes. Then by proving $S(k,j)$ for all integers $k \geq 1$ and $j \geq 0$, we will find the unknotting number for the minimal projection of any knot in our table.

Throughout this paper, we will call the left tangle of our knot l and the right tangle r.

Consider knot $K = 2k+3,\ 2j+1,\ 2k+2$. If $k < j$, make $k+1$ changes on both l and r and the resulting knot is $1,\ 2j+1,\ 0 \equiv 0$. So for $k < j$, the trivial knot is obtained after making $2k+2$ changes. Likewise, if $k \geq j$, make $k+1$ changes on l and $j+1$ changes in the middle and the resulting knot is $1,\ -1,\ 2k+2 \equiv 0$; so for $k \geq j$, the trivial knot is obtained after making $k+j+2$ changes.

To show that $S(k,j)$ is true for all integers $k \geq 1$ and $j \geq 0$, that is, for any knot in Table 1, we will use a proof by induction, with the induction going in two directions: across rows and down columns in the table.

Bernhard's proof of $S(k,0)$ leads to our proof of $S(k,1)$, providing us with a base case for columns in the table, which follows. To generate a base case for rows in the table, we will prove $S(1,j)$. We will use these base cases to inductively prove $S(k,j)$ for all integers $k \geq 1$ and $j \geq 0$.

Prove $S(1,j)$: i.e., show that
i) for $j = 0$, $5,\ 2j+1,\ 4$ cannot be unknotted in 2 or fewer changes.
ii) for $j \geq 1$, $5,\ 2j+1,\ 4$ cannot be unknotted in 3 or fewer changes.

The proof of part (i), $S(1,0)$, is completed by Bernhard and is our base case for proving part (ii). We will use an inductive proof:

Assume: $S(1,h)$ is true for $0 \leq h \leq j$.
Show: $S(1,j+1)$ is true, i.e., $u(5,\ 2j+3,\ 4)_{min} > 3$, for $j \geq 1$.

Again, we will examine changes to the middle $(2j+3)$ section of the knot.

0 middle changes

case 1: at least 1 change in both l and r
Our knot becomes $3,\ 2j+3,\ 2$ with ≤ 1 change remaining.

a) 0 changes: we have a knot with all positive components which is therefore reduced alternating and non-trivial.
b) 1 change l: we have the knot $1,\ 2j+3,\ 2$ which is non-trivial by the same reason as (a).
c) 1 change r: we have the knot $3,\ 2j+3,\ 0 \equiv 3$, which is non-trivial.

case 2: 0 changes l, $1 \leq n \leq 3$ changes r
Our knot becomes $5,\ 2j+3,\ 4-2n$.

a) if $n = 1$, we have a knot with positive components which is reduced alternating and non-trivial.

b) if $n = 2$, our knot becomes 5, $2j + 3$, $0 \equiv 5$, a non-trivial knot.

c) if $n = 3$, our knot becomes 5, $2j+3$, $-2 \equiv -5$, $-(2j+2)$, -1, -1 since their continued fractions are equal. However, this is a knot with all negative components which is therefore reduced alternating and non-trivial.

case 3: 0 changes r, $1 \leq n \leq 3$ changes l
Our knot becomes $5 - 2n$, $2j + 3$, 4.

a) if $1 \leq n \leq 2$, we have a knot with positive components which is non-trivial.

b) if $n = 3$, our knot becomes -1, $2j+3$, $4 \equiv 2j+2$, 4 since their continued fractions are equal. The latter knot is non-trivial since it is reduced alternating with positive components.

<u>1 middle change</u>

We have the knot 5, $2(j-1)+3$, 4 with ≤ 2 changes left, which is non-trivial by our inductive hypothesis.

<u>2 middle changes</u>

We have the knot 5, $2(j-2)+3$, 4 with ≤ 1 changes left to make. Making the changes on l or r, we still have a knot with all positive components which is reduced alternating and non-trivial.

<u>3 middle changes</u>

We have the knot 5, $2(j-3)+3$, 4 with 0 changes left.

a) if $j > 1$, the knot is reduced alternating with all positive components.

b) if $j = 1$, the knot becomes 5, -1, $4 \equiv 3$, 1, 2, a reduced alternating knot with all positive components, since their continued fractions are equal.

Thus, we have proven statement $S(1, j)$.

Prove $S(k, 1)$:

i.e., show that $2k + 3$, 3, $2k$ cannot be unknotted in $k + 2$ or fewer changes, for $k \geq 1$.

This will be a proof by induction.

<u>Assume:</u> $S(h, 1)$ is true for $1 \leq h \leq k$. Our base case is $S(1, 1)$.
<u>Show:</u> $S(k + 1, 1)$ is true, i.e., $u(2k + 5,\ 3,\ 2k + 4 > k + 3$, for $k \geq 1$.

To begin the induction, we need $u(5,\ 3,\ 4)_{min} \geq 3$. This base case is done in $S(1, j)$. We will examine changes to the middle (3) section of the knot and will consider all possibilities of making at most $k + 3$ changes to the knot.

0 middle changes

case 1: at least 1 change in both l and r.

The resulting knot is $2k+3$, 3, $2k+2$, with $\leq k+1$ changes left to make. This knot is non-trivial by our inductive hypothesis.

case 2: 0 changes in l, $1 \leq n \leq k+3$ changes in r.
Our knot becomes $2k+5$, 3, $2(k-n)+4$.

a) if $k-n > -2$, we still have a reduced alternating projection with all positive components.
b) if $k-n = -2$, we have $2k+5$, 3, $0 \equiv 2k+5$, a reduced alternating projection.
c) if $k-n < -2$, then we have a knot in the form $K = 2k+5$, 3, $-c$, where $c > 1$. Since their continued fractions are equal, $K \equiv -(2k+5)$, -2, -1, $-(c-1)$. All components of this knot are negative, making it non-trivial; thus K is non-trivial.

case 3: 0 changes in r, $1 \leq n \leq k+3$ changes in l.
Our knot becomes $2(k-n)+5$, 3, $2k+4$.

a) if $k-n \geq -2$, we have a reduced alternating projection with all positive components.
b) if $k-n < -2$, then $k-n = -1$ and we have a knot in the form $Q = -1$, 3, $2k+4$. Since their continued fractions are equal, $Q \equiv 2$, $2k+4$, a reduced alternating and non-trivial knot.

1 middle change

We are left with the knot $2k+5$, 1, $2k+4$ with at most $k+2$ changes left to make. Writing this knot as $(N, M) = (k, 0)$, we know that $S(k, 0)$ is true, thus our knot is non-trivial.

2 middle changes

We are left with the knot $2k+5$, -1, $2k+4$ with at most $k+1$ changes left to make. This knot is one possible reduction of having the knot $2k+5$, 1, $2k+4$ with at most $k+2$ changes left to make. For the same reason as when we made 1 middle change, this knot is non-trivial.

3 middle changes

We are left with the knot $2k+5$, -3, $2k+4$ and at most k changes to make.
Regardless of whether we make changes on l and/or r, both l and r are positive, so our knot is in the form $Q = a$, -3, c, where $a, c > 1$. Since their continued fractions are equal, $Q \equiv a-1$, 1, 1, 1, $c-1$. This is a reduced alternating and non-trivial projection since the components are all positive.

Thus we have proven statement $S(k, 1)$.

Prove $S(k,j)$ for $k,j \geq 1$:

We will break our proof of $S(k,j)$ into two parts:

i) if $k \leq j$, show that $2k+3$, $2j+1$, $2k+2$ will not unknot in $2k+1$ or fewer changes.
ii) if $k \geq j$, show that $2k+3$, $2j+1$, $2k+2$ will not unknot in $k+j+1$ or fewer changes.

Part (i):

Assume:
i. $S(k,j)$, $\forall j < M$, i.e., our statement is true for columns to the left of our current position in Table 1. Our base case is $S(k,1)$.
ii. $S(k,M)$, $\forall k < M$, i.e., our statement is true for entries in column M above our current position in the table. Our base case is $S(1,j)$.

Show:
$S(N,M)$ i.e., $S(k+1, j+1)$ from our assumptions. Since $N \leq M$, this implies that $2N+3, 2M+1, 2N+2$ cannot be unknotted in $2N+1$ or fewer changes.

We will focus on making changes to the middle, or $2M+1$ section of the knot and will break our examination into the following cases: 0 middles changes and from 1 to $2N+1$ middle changes, with any remaining changes occurring on the outer tangles.

Case I: 0 middle changes with at least one change on l and one on r

The resulting knot is $2(N-1)+3, 2M+1, 2(N-1)+2$ with $\leq 2(N-1)+1$ changes left to make.

i. For $N \geq 2$, this case follows by induction from our inductive hypothesis (i).

ii. However, we must also check for $N = 1$. The result is the projection $3, 2M+1, 2$. This projection has all positive components; therefore, it is reduced alternating, and nontrivial.

Case II: O middle changes with $\leq 2N+1$ changes on l and 0 changes on r.

i. Choose k, for $1 \leq k < N+1$
Making k changes on l results in the projection $2(N-k)+3, 2M+1, 2N+2$. For $N \geq 1, k \geq 1, M \geq 1$, this knot is all positive, reduced alternating, and nontrivial.

ii. For $k = N+1$
Making $N+1$ changes on l results in the projection $1, 2M+1, 2N+2$ For $N \geq 1$ and $M \geq 1$ this knot has all positive components, so it is reduced alternating, and nontrivial.

iii. For $k = N+2$
Making $N+2$ changes on l results in the projection $-1, 2M+1, 2N+2$ After checking the continued fraction for this knot we get the equivalent projection $2M, 2N+2$ which is all positive, reduced alternating, and nontrivial.

iv. For $N+2 < k \leq 2N+1$

Making k changes on l results in the knot $2(N-k)+3,\ 2M+1,\ 2N+2$. For $N-k > -2$, the resulting knot has all positive components and is non-trivial. For $N = -2$, the resulting knot is $-1,\ 2M+1,\ 2N+2 \equiv 2M,\ 2N+2$. For $N < -2$, we have a knot in the form $-a,\ 2M+1,\ 2N+2 \equiv a-1,\ 1,\ 2M,\ 2N+2$, a knot with all positive components which is nontrivial.

Case III: 0 middle changes with $\leq 2N+1$ changes on r and 0 changes on l.

i. Choose k, for $1 \leq k < N+1$

Making k changes on r results in the projection $2N+3, 2M+1, 2(N-k)+2$, a knot which is all positive and nontrivial.

ii. For $k = N+1$

Making $N+1$ changes on r results in the projection $2N+3, 2M+1, 0 \equiv 2N+3$, which has all positive components and is nontrivial.

iii. For $k = N+2$

Making $N+2$ changes on r results in the projection $2N+3, 2M+1, -2$. After checking the continued fraction for this knot we find it is equivalent to $-(2N+3), -2M, -1, -1$ which is all negative, and nontrivial.

iv. For $N+2 < k \leq 2N+1$

Making k changes on r gives us the knot $2N+3,\ 2M+1,\ 2(N-k)+2$. For $N-k > -1$, we get a knot with all positive components which is nontrivial. For $N-k = -1$, we get the knot $2N+3,\ 2M+1,\ 0 \equiv 2N+3$, a nontrivial knot. For $N-k < -1$, we get a knot of the form $2N+3, 2M+1, -c \equiv -[2N+3, 2M, 1, c-1]$ for $c > 1$ after checking the continued fraction, which is a nontrivial situation.

Case IV: β middle changes for $1 \leq \beta < 2N+1$

i. Making $\beta = 1$ changes in the middle results in the projection $2N+3, 2(M-1)+1, 2N+2$ with $\leq 2N$ changes to make. This case is non-trivial by inductive hypothesis (i).

ii. Making $1 < \beta < 2N+1$ changes in the middle results in the projection $2N+3, 2(M-\beta)+1, 2N+2$. For $M-\beta > -1$ we get a knot which is all positive and nontrivial. For $M-\beta = -1$ we get the projection $2N+3, -1, 2N+2$. After checking the continued fraction we find it is equivalent to $2N+1,\ 1,\ 2N$ which is all positive, thus the knot is nontrivial.

iii. Making $\beta = 2N+1$ changes in the middle results in the projection $2N+3, 2(M-2N)-1, 2N+2$. For $M-2N > 0$, we have a knot with all positive components which is non-trivial. For $M-2N = 0$, we have the knot $2N+3,\ -1,\ 2N+2 \equiv 2N+1,\ 1,\ 2N$, a non-trivial knot as we saw in part (ii). For $M-2N < 0$, we get a knot in the form $2N+3,\ -b,\ 2N+2 \equiv 2N+2,\ 1,\ b-2,\ 1,\ 2N+1$, which is positive and non-trivial.

This completes the proof of part (i).

Part (ii)

Pick $j \geq 1$ and let N, M be positive integers such that $N > M$ and $M \geq 1$ so $N \geq 2$.

Assume:
i) $S(j, j)$, i.e., our statement is true for the diagonal elements, which is proved in part (i).
ii) $S(k, j)$, $\forall j < M$, i.e., our statement is true for columns to the left of our current position in the table. Our base case is $S(k, 1)$.
iii) $S(k, M)$, $\forall k < N$ i.e., our statement is true for elements in the row above our current position in the table. Our base case is $S(j, j)$.

Show:
$S(N, M)$, i.e., $S(k+1, j+1)$ from our assumptions. Since $N > M$, this implies that the knot $2N+3$, $2M+1$, $2N+2$ cannot be unknotted in $N+M+1$ or fewer changes.

We will focus on making changes to the middle, or $2M+1$, section of the knot and will break our examination into three cases: 0 middle changes, from 1 to $2M$ middle changes, and $2M+1$ middle changes, with any remaining changes occurring on the outer tangles. We will call the left tangle $(2N+3)$ of the knot, l, and the right tangle $(2N+2)$, r.

0 middle changes

case 1: at least 1 change in both l and r.

We are left with the knot $2(N-1)+3$, $2M+1$, $2(N-1)+2$ with $\leq N+M-1$ changes left to make. Write this knot as $S(N-1, M)$ and by inductive hypothesis (iii), it is non-trivial.

case 2: 0 changes l, $1 \leq n \leq N+M+1$ changes r.
We are left with the knot $2N+3$, $2M+1$, $2(N-n)+2$.

a) if $2(N-n) > -2$, we have a knot with all positive components which is therefore reduced alternating and non-trivial.
b) if $2(N-n) = -2$, we have the knot $2N+3$, $2M+1$, $0 \equiv 2N+3$, a non-trivial knot.
c) if $2(N-n) < -2$, we have a knot of the form $2N+3$, $2M+1$, $-c$ where $c > 1$. Because their continued fractions are equal, this knot is equivalent to the knot $-(2N+3)$, $-2M$, -1, $-(c-1)$, a knot with all negative components which is reduced alternating and non-trivial.

case 3: 0 changes r, $1 \leq n \leq N+M+1$ changes l.
We are left with the knot $2(N-n)+3$, $2M+1$, $2N+2$.

a) if $2(N-n) > -3$, we have a knot with positive components which is reduced alternating and non-trivial.
b) $2(N-n)$ will not equal -3 since N and n are integers.
c) if $2(N-n) < -3$, we have a knot in the form $K = -a$, $2M+1$, $2N+2$. We will use the equality of continued fractions to find a knot to which K is equivalent.

i. if $a = 1, K \equiv 2M$, $2N+2$, a reduced alternating and non-trivial knot.

ii. if $a > 1, K \equiv a-1,\ 1,\ 2M,\ 2N+2$, again, a reduced alternating and non-trivial knot.

$1 \leq n \leq 2M$ middle changes

Our knot becomes $2N+3,\ 2(M-n)+1,\ 2N+2$ with $\leq N+(M-n)+1$ changes left to make on l and r.

case 1: $1 \leq n \leq M-1$.
Write this knot as $(N, M-n)$ and by inductive hypothesis (ii), it is non-trivial.

case 2: $n = M$.
Our knot becomes $2N+3,\ 1,\ 2N+2$ with $\leq N+1$ changes left to make. Write this knot as $(N, 0)$ and we know that $S(N, 0)$ is true, so our knot is non-trivial.

case 3: $n = M+1$.
Our knot becomes $2N+3,\ -1,\ 2N+2$ with $\leq N$ changes left to make. This knot is one reduction of having the knot $2N+3,\ 1,\ 2N+2$ with $\leq N+1$ changes left. For the same reason as case (ii), however, it is non-trivial.

case 4: $M+2 \leq n \leq 2M$.
Our knot will have a negative middle component and we will use similar reasoning as in case (ii) to show it is one reduction of a knot with all positive components whose unknotting number we know to be greater than the number of possible changes we have left to make.
Let $-L = M-n$. Then for $M+2 \leq n \leq 2M$, $2 \leq L \leq M$. So our knot becomes $2N+3,\ -(2L-1),\ 2N+2$ with $\leq N-L+1$ changes left. This knot is one possible reduction of the knot $2N+3,\ 2(L-1)+1,\ 2N+2$ with $\leq N-L+1+(2L-1) = N+L$ changes left to make. Write this knot as $(N, L-1)$ and by inductive hypotheses (ii) and (iii), it is non-trivial.

$2M+1$ middle changes

We are left with the knot $2N+3,\ -(2M+1),\ 2N+2$ with $\leq N+M+1-(2M+1) = N-M$ changes to make. Suppose we make n changes on l where $0 \leq n \leq N-M$ and q changes on r where $0 \leq q \leq N-M-n$. Our knot becomes $2N+3-2n,\ -(2M+1),\ 2N+2-2q$.
By substituting the upper bounds of n and q into the left and right sections of this knot, we see that these sections always have more than 1 crossing. Since $M \geq 1$, the middle section has more than 2 crossings. Thus, since their continued fractions are equal, our knot is equivalent to $2N+2-2n,\ 1,\ 2M-1,\ 1,\ 2N+1-2q$, a reduced alternating knot with all positive components which is therefore non-trivial.

Therefore, $2N+3,\ 2M+1,\ 2N+2$ cannot be unknotted in $\leq N+M+1$ changes when $N > M$, so we have proved $S(N, M)$ where $M \neq 0$.

Including the result of Bernhard's proof of $S(N, 0)$, we can find the unknotting number for the minimal projection of any knot in our table, so for any integers $k \geq 1$ and $j \geq 0$, if knot $K = 2k+3,\ 2j+1,\ 2j+2$,

$$u(K)_{min} = \begin{cases} 2k+2 \text{ changes} & \text{if } k \leq j \\ k+j+2 \text{ changes} & \text{if } k \geq j \end{cases}.$$

We also know that for any knot Q on or below the diagonal, $u(Q) \leq u(Q)_{min} - 1$.

KNOTS OF THE FORM a, b, c

We will now begin an analysis of the unknotting number of knots with Conway notation a, b, c. The knots we will examine will all have an odd middle component. We will say that a knot P displays the Non-Reduced Unknotting Property (NRU Property) if the unknotting number of P results from a non-minimal projection of P. Let the gap of P ($gap(P)$) be the difference between the unknotting number of a minimal projection of P and the unknotting number of P. Our analysis will consider 3 knot classes. Let Q, R and S represent knots of the form:

$Q = odd,\ z,\ even = even,\ z,\ odd$ for any odd integer z.
$R = odd,\ 1,\ odd$
$S = even,\ 1,\ even$

I) *odd, odd, even*

We have shown that for any knot Q given above, the NRU Property holds if $z > even$ and fails otherwise, if $|odd - even| = 1$. When the property holds, $gap(Q) \geq 1$.

II) *odd, 1, odd*

For any knot R given above, the NRU Property fails for the specific non-minimal projection we consider.

Unknotting the Minimal Projection

Let $K = x,\ 1,\ y$ be a knot for odd integers $x, y \geq 5$.

Assume $x \leq y$. Then $u(K)_{min} \leq \frac{x+1}{2}$ since by changing $\frac{x+1}{2}$ crossing on l, the resulting knot is $-1,\ 1,\ y \equiv 0$.

We will show that $u(K)_{non-min} \geq \frac{x+1}{2}$; i.e., $K_{non-min}$ cannot be unknotted in $\leq \frac{x+1}{2} - 1 = \frac{x-1}{2}$ changes and the NRU Property fails.

Unknotting the Non-Minimal Projection

If $x = y$, we may pull over the loops for our non-minimal projection on either outer tangle for the same result, so we will choose side x.

If $x < y$, we will pull over loops on side x for an analagous non-minimal projection to the one previously used in which we pull over loops on the outer tangle having the fewest crossings.

After making 1 change to the middle component, we have $\leq \frac{x-3}{2}$ changes left to make.

On side x we have $x-1$ loops. Changing 1 crossing on each pulled over loop, as we did in previous cases, we will make changes on $\leq \frac{x-3}{2}$ loops. Thus, there will be at least 2 adjacent loops which are not pulled over and whose crossings will not be changed, whereas when we begin with an even number of crossings, as with knots of the form *odd*, *odd*, *even*, we pulled over alternating loops and had no 2 adjacent loops not having been pulled over. This leaves us with at least 1 extra crossing on the x side needing to be changed to obtain the unknot.

We saw that any knot of the form *odd*, *odd*, *even* reduced to $3,-1,2 \equiv 0$ using our non-minimal projection. However, for *odd*, 1, *odd* knots, using an analagous non-minimal projection reduces to the knot $3,-1,3 \equiv 3$, so we have not achieved the unknot in $\leq \frac{x+1}{2} - 1$ changes.

Therefore, for the proposed non-minimal projection, the NRU Property fails for knots of the form *odd*, 1, *odd*. We conjecture that the property fails for any knot of the form *odd*, *odd*, *odd*.

III) *even, 1, even*

Any knot $S = 2k,\ 1,\ 2j$ is a 2 link knot. We will show that the NRU Property holds for all such S where $k \geq j \geq 2$, and that $gap(S) = j-1$. This means that we can find an arbitrarily large difference between the unknotting number of a minimal and non-minimal projection for an infinite number of 2 link knots, as depicted in the following table:

$4,1,4$							
$6,1,4$	$6,1,6$						
$8,1,4$	$8,1,6$	$8,1,8$					
$10,1,4$	$10,1,6$	$10,1,8$	$10,1,10$				
$12,1,4$	$12,1,6$	$12,1,8$	$12,1,10$	$12,1,12$			
.	.	.	.	.	.		
.	$2k,1,2j$	.	.	.	.	$2k,1,2k$	
.	.	.	.	.	.	.	
$Min =$ $Non + 1$	$Min =$ $Non + 2$	$Min =$ $Non + 3$	$Min =$ $Non + 4$	.	.	$Min =$ $Non + j - 1$	
.	.	.	.	.	.	.	.

Table 2

Unknotting the Non-Minimal Projection

We will use an analagous non-minimal projection to the one we used for knots $2k+3,\ 2j+1,\ 2k+2$. Whereas these knots eventually reduce to $3,-1,2 \equiv 0$ in k changes, knots of the form $2k$, 1, $2j$ reduce to $2, -1, 2 \equiv 0, 0$. If $k > j$, make $k - j$ changes on l and the resulting knot is $2j$, 1, $2j$. Make 1 middle change and $j - 1$ changes on pulled over loops of r. Then the unknotting number for this non-minimal projection is k.

Unknotting the Minimal Projection

We claim that for $S = 2k$, 1, $2j$, where $k \geq j \geq 2$, $u(S)_{min} = k + j - 1$. Notice that if we make 1 middle change, $k - 1$ changes on l and $j - 1$ changes on r, we get the knot $2, -1, 1 \equiv 0, 0$, so $u(S)_{min} \leq k + j - 1$. Our conjecture is that no fewer than $k + j - 1$ changes unknot $2k$, 1, $2j$. What follows is the beginning of this proof; we will generate base cases of the left column and the diagonal in Table 2 which we believe lead to an inductive proof that $u(S)_{min} = k + j - 1$. Our base case for both the left column and the diagonal is 4,1,4.

Let $(N, M)'$ denote the knot $2N$, 1, $2M$.

Definition: Statement $S(N, M)'$ means that for knot K corresponding to $(N, M)'$, $u(K)_{min} = N + M - 1$.

Prove $S(2, 2)'$: i.e., show that $u(4, 1, 4)_{min} = 3$.

By making 1 middle change and 1 change on both l and r, we get the knot $2, -1, 2 \equiv 0, 0$; thus we know that the minimal projection of 4,1,4 can be unknotted in 3 changes. We will show that it cannot be unknotted in fewer than 3 changes.

0 middle changes

case 1: at least 1 change both l and r
We are left with the non-trivial knot 2,1,2 with ≤ 1 change left. Making this change on either l or r, we are left with the two link knot 2, which is non-trivial.

case 2: 0 changes l, $1 \leq n \leq 2$ changes r
(This is equivalent to making 0 changes r, $1 \leq n \leq 2$ changes l.)

a) for $n = 1$, the resulting knot has all positive components and is non-trivial.
b) for $n = 2$, the resulting knot is $0, 1, 4 \equiv 4$, which is non-trivial.

1 middle change
We are left with the knot 4,-1,4 with 0 or 1 changes left to make.

case 1: making 0 changes, $4, -1, 4 \equiv 2, 1, 2$, a non-trivial knot.

case 2: 1 change l, which is equivalent to making 1 change r.
The resulting knot is $2, -1, 4 \equiv 0, 1, 2 \equiv 2$, which is non-trivial when we have 2 links.

Thus we have proven $S(2, 2)'$.

Prove $S(k, k)'$: i.e., show that $u(2k,\ 1,\ 2k)_{min} = 2k - 1$

Assume: $S(h, h)'$ is true for $2 \leq h \leq k$. Our base case is 4,1,4.
Show: $S(k + 1, k + 1)'$ is true, i.e., $u(2k + 2, 2k + 2)_{min} = 2k + 1$.

We see that $2k + 1$, 1, $2k + 1$ can be unknotted in $2k + 1$ changes by making 1 middle change and k changes on both l and r to obtain the knot $2, -1, 2 \equiv 0, 0$. We will now show that making $\leq 2k$ changes does not produce the unknot.

<u>0 middle changes</u>

case 1: at least 1 change both l and r

We are left with the knot $2k$, 1, $2k$ with $\leq 2k-2$ changes left to make. From our inductive hypothesis, this knot is non-trivial.

case 2: 0 changes r, $1 \leq n \leq 2k$ changes l
(This is equivalent to making 0 changes l and $1 \leq n \leq 2k$ changes r.)

We are left with the knot $2(k-n)+2$, 1, $2k+2$.

a) if $k-n > -1$, our knot has all positive components and is non-trivial.

b) if $k-n = -1$, the resulting knot is 0, 1, $2k+2 \equiv 2k+2$, a non-trivial knot.

c) if $k-n < -1$, the resulting knot is in the form $-a$, 1, $2k+2 \equiv a-1$, $2k+3$, a knot with all positive components which is non-trivial.

<u>1 middle change</u>

The resulting knot is $2k+2$, -1, $2k+2$ with $\leq 2k-1$ changes left.

case 1: at least 1 change on both l and r.
We have the knot $2k$, -1, $2k$ with $\leq 2k-3$ changes left. This knot is one possible reduction of having the knot $2k$, 1, $2k$ with $\leq 2k-2$ changes left, which is non-trivial by our hypothesis.

case 2: 0 changes l, $1 \leq n \leq 2k-1$ changes r.
(This is equivalent to making 0 changes r and $1 \leq n \leq 2k-1$ changes l.) We are left with the knot $K = 2k+2$, -1, $2(k-n)+2$.

a) if $k-n > 0$, K is in the form $2k+2$, -1, $c \equiv 2k$, 1, $c-2$, a reduced alternating and non-trivial knot.

b) if $k-n = 0$, $K = 2k+2$, -1, $2 \equiv 2k$, a non-trivial knot.

c) if $k-n = -1$, $K = 2k+2$, -1, $0 \equiv 2k+2$, a non-trivial knot.

d) if $k-n < -1$, K is in the form $2k+2$, -1, $-c \equiv -(2k+1)$, $-(c+1)$, a reduced alternating and non-trivial knot.

This completes our proof of $S(k,k)'$.

Prove $S(k,2)'$: i.e., show that $u(2k,\ 1,\ 4)_{min} = k+1$

<u>Assume:</u> $S(h,2)'$ is true for $1 < h \leq k$. Our base case is $S(2,2)'$.
<u>Show:</u> $S(k+1,2)'$ is true, i.e., $u(2k+2,\ 1,\ 4)_{min} = k+2$.

We see that the minimal projection of $2k+2$, 1, 4 can be unknotted in $k+2$ changes by changing the middle crossing, 1 on r and k on l to obtain the knot $2, -1, 2 \equiv 0, 0$. We will now show that no fewer than $k+2$ changes unknot this projection by checking the possibilities of making $\leq k+1$ changes on our knot.

0 middle changes

case 1: at least 1 change on both l and r.

We are left with the knot $2k$, 1, 2 with $\leq k-1$ changes to make. This is one possible reduction of having the knot $2k$, 1, 4 with $\leq k$ changes left to make, which is non-trivial by our hypothesis.

case 2: 0 changes l, $1 \leq n \leq k+1$ changes r.

The resulting knot is $2k+2$, 1, $4-2n$.

a) if $n < 2$, we get a reduced alternating knot with all positive components which is non-trivial.

b) if $n = 2$, we get the knot $2k+2$, 1, $0 \equiv 2k+2$, a reduced alternating and non-trivial knot.

c) if $n > 2$, we get a knot in the form $2k+2$, 1, $-c \equiv -(2k+3)$, $-(c-1)$. Such a knot has all negative components and is non-trivial.

case 3: 0 changes r, $1 \leq n \leq k+1$ changes l.

The resulting knot is $2(k-n)+2$, 1, 4.

a) if $k-n > 0$, we have a reduced alternating knot with all positive components which is non-trivial.

b) if $k-n = 0$, our knot reduces to $0, 1, 4 \equiv 4$, a non-trivial knot.

c) if $k-n < 0$, our knot is in the form $-a$, 1, $4 \equiv a-1$, 5, a reduced alternating and non-trivial knot.

1 middle change

The resulting knot is $2k+2$, -1, 4 with $\leq k$ changes left.

case 1: at least 1 change both l and r.

We are left with the knot $2k$, -1, 2 with $\leq k-2$ changes to make. This knot is one possible reduction of having the knot $2k$, 1, 4 with $\leq k$ changes to make, which is non-trivial by our hypothesis.

case 2: 0 changes l, $1 \leq n \leq k+1$ changes r.

We are left with the knot $K = 2k+2$, -1, $4-2n$.

a) if $n = 1$, $K = 2k+2$, -1, $2 \equiv 2k$, a reduced alternating and non-trivial knot.

b) if $n = 2$, $K = 2k+2$, -1, $0 \equiv 2k+2$, again, a reduced alternating and non-trivial knot.

c) if $n > 2$, K is in the form $2k+2$, -1, $-c \equiv -(2k+1)$, $-(c+1)$, a knot with all negative components which is reduced alternating and non-trivial.

case 3: 0 changes r, $1 \leq n \leq k+1$ changes l.

We are left with the knot $K = 2(k-n)+2$, -1, 4.

a) if $k-n > 0$, K is in the form $a, -1, 4 \equiv a-2, 1, 2$, a knot with all positive components which is non-trivial.

b) if $k-n = 0$, $K = 2, -1, 4 \equiv 4$, a non-trivial knot.

c) if $k-n=-1$, $K=0,-1,4 \equiv 4$, a non-trivial knot.

d) if $k-n<-1$, K is in the form $-a,-1,4 \equiv a+1$, 3, a knot with all positive components which is non-trivial.

This completes our proof of $S(k,2)'$ for any integer $k \geq 2$.

Using these base cases, we believe one can prove $S(k,j)'$ for integers $k \geq j \geq 2$. The proof will be inductive and will be quite similar to the proof of $S(k,j)$ for integers $k \geq 1$ and $j \geq 0$. It will then follow that for $S = 2k$, 1, $2j$, $\text{gap}(S) = k+j-1$.

UNANSWERED QUESTIONS

The work we have done leaves many unanswered questions. We would like to be able to provide a full analysis of the unknotting numbers of knots written a, b, c in Conway notation. Thus we must examine all remaining cases where b is an odd integer, as well as the knots for which b is an even integer. For $b=$ odd integer, our analysis of knots having 1 odd and 1 even outer component only consisted of knots in which the outer components differed by 1. We feel that completion of this analysis is well within reach.

There are also open questions relating to the topics discussed in this paper. Is there a 1 link knot which generates a class of knots, a component of which we will call Q, for which $gap(Q) > 1$? Can $gap(Q)$ be arbitrarily large? Are there different non-minimal projections than the ones we use which will lead to other knots exhibiting the NRU Property? We think it likely that after further study, one will find the answers to these questions to be yes.

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