Euclidean Neighborhood Retracts

- $X \subset R^n$ is an **ENR** if X is retract of a neighborhood of X in R^n.
- X an ENR $\Rightarrow X = C \cap O$ for C closed in R^n and O open in R^n (i.e. X is locally closed).
- X locally closed in R^n implies $X \cong Y$ where Y is closed in R^{n+1}.
- TFAE for $X \subset R^n$:
  - (i) X is locally closed
  - (ii) Each $p \in X$ has an open nbhd U in R^n such that $X \cap U$ is closed in U
  - (iii) X is locally compact

Relation to ANRs

Corollary: If $X \subset R^k$ is locally closed in R^k, $Y \cong X$, and $Y \subset R^n$, then Y is locally closed in R^n.

Theorem: If $X \subset R^n$ is a neighborhood retract, $Y \cong X$, and $Y \subset Z$, then Y is a neighborhood retract in Z.

Corollary: Any ENR is an ANR for the class of separable metric spaces and for the class of normal spaces.

Homotopies

Theorem: If X is an ENR, and $f, g : Y \to X$ are maps that agree on a subspace $B \subset Y$, then there is an open neighborhood of B in Y and a homotopy H between $f|_W$ and $g|_W$ with $H(b,t) = f(b) = g(b)$ for all t and all $b \in B$.

Compare with Homotopy Extension Theorem.

Corollary: If $B \subset X$ are ENRs, then B is a neighborhood retract in X, and given such a retraction $r : V \to B$, there is an open nbhd W of B in V such that $i \circ r|_W \sim$ inclusion of W in V.

Unions of ANRs

Theorem: If X is Hausdorff and is a union of finitely many locally compact open subsets, each of which is $\cong$ a subset of some Euclidean space, then $X \cong$ to a closed subspace of some Euclidean space.

Theorem: If a Hausdorff X is a finite union of open subspaces, each of which is an ENR, then X is an ENR.

Generalizations to countable unions.

Characterizing

Theorem: If X is a compact ENR, and $\varepsilon > 0$, then there is a $\delta > 0$ such that any two maps $f, g : Y \to X$ that are δ close are ε homotopic.

Characterization: Let $X \subset R^n$ (equivalently, let X be finite dimensional, separable metric). Then X is an *ENR* if and only if X is locally compact and locally contractible if and only if X is locally compact and locally $n-1$ connected.

Definition: Locally k connected.

Properties of ANRs

- Every retract of an ANR (AR) is an ANR (AR)
- Every open subset of an ANR is an ANR
- The countable product of ARs is an AR
- The finite product of ANRs is an ANR

Local Properties:
Neighborhood retracts preserve local compactness, local connectedness, local path connectedness, and local contractibility.

References

- Algebraic Topology by Dold
- Decomposition Theory by Daverman
- Theory of Retracts by Borsuk