A Rate of Convergence for the LANSα Regularization of Navier-Stokes Equations

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Abstract

A rate of convergence of the solutions of the LANSα equations with periodic boundary to the solutions of the Navier-Stokes equations as $\alpha \downarrow 0$ is obtained in a mixed $L^1 - L^2$ time-space norm for small initial data in Besov-type function spaces in which global existence and uniqueness of solutions can also be established.

Key Words: Lagrangian averaged Navier-Stokes, turbulence, convergence rate, multiplicative stochastic cascade, branching random walk.

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1 Introduction

The analysis and simulation of fluid flows continue to pose challenging problems whose solutions have important scientific and practical consequences. The LANSα (Lagrangian

Averaged Navier-Stokes-α) model was developed in an effort to provide an efficient numerical simulation of three-dimensional turbulence on a periodic domain; see [9], [8], [4], [7], [14], [13], [3] and references therein for recent results and perspectives. These equations may be viewed as modifications of a regularization introduced in [2].

For $\alpha > 0$, this particular model corresponds to a regularization of the ($\alpha = 0$) Navier-Stokes equations that satisfies Kelvin's circulation theorem, and in this way avoids some of the physical limitations present in other proposed regularizations (see [6]). Consequently, a considerable amount of mathematical analysis, including physical interpretations, global well-posedness in time, and finite dimensionality of the global attractor has been undertaken for this system of equations. Much less has been obtained for convergence, especially rates of convergence, to solutions of incompressible Navier-Stokes equations as $\alpha \to 0$. In fact, the rate of convergence problem was cited as an essential open problem for alpha models in the recent paper on MHD-α models in the conclusion section to [12].

In this paper we focus on the rate of convergence as α approaches zero of solutions of the $\mathsf{LANS}\alpha$ system of equations using a probabilistic approach introduced for incompressible Navier-Stokes equations in [10] and extended in [1]; a general survey is provided in [16]. Formally, one expects that the $\alpha \downarrow 0$ limit should satisfy the Navier Stokes equation whenever these equations have a solution. Most results obtained so far are based upon a functional analytic approach, which, in turn, produces global existence of a weak solution and also convergence of a subsequence to a weak solution of Navier-Stokes as $\alpha \to 0$, see [5]. Of course under further conditions (to be determined) for uniqueness, this will provide convergence of the full limit as $\alpha \to 0$. However obtaining such conditions is also part of the general problem. Once well-posedness in terms of convergence for unique global solutions is obtained in an appropriate function space, then we can address the question of *convergence rate*. In this paper we develop a mix of probabilistic and analytic approaches to these problems. The basic objective is to introduce and explore the multiplicative branching random walk cascade within the context of $\mathsf{LANS}\alpha$. From a probabilistic perspective it is noteworthy that solutions to all of the $\mathsf{LANS}\alpha$ equations (for $\alpha \geq 0$) can be accommodated by a single probability model (branching random walk cascade), distinguished entirely by the (deterministic) α-dependent multiplicative factors used in the expected value representation. This may have advantages for Monte-Carlo simulation approaches; see [15] for illustrative Monte-Carlo numerical applications of the branching random walk cascade to Burgers equation.

From the theoretical point of view, probabilistic considerations lead to natural function spaces for which one has representations of unique global solutions to $\mathsf{LANS}\alpha$ (with periodic boundary) for each $\alpha \geq 0$ as an expected value of a stochastic cascade defined on a common probability space. In particular, this includes the limiting $\alpha = 0$ case of incompressible Navier-Stokes equations with periodic boundary. Within this function space we then obtain a rate on a mixed $L^1 - L^2$ space-time norm in which $\mathsf{LANS}\alpha$ solutions converge to those of Navier-Stokes.

2 The Mild $\mathsf{LANS}\alpha$ Equations

Recall that the $\mathsf{LANS}\alpha$ equations on a periodic domain $D = [-L, L]^3, L > 0$ in $\mathbf{R}^3$, can be written as

$$\begin{aligned} \frac{\partial \mathsf{v}^{(\alpha)}}{\partial t} + \nabla\cdot(\mathsf{u}^{(\alpha)} \otimes \mathsf{v}^{(\alpha)}) + (\nabla \mathsf{u}^{(\alpha)})^{\mathsf{T}} \mathsf{v}^{(\alpha)} &= \nu \Delta \mathsf{v}^{(\alpha)} - \nabla p + \mathsf{g} \\ \nabla\cdot \mathsf{v}^{(\alpha)} = 0, \qquad & (1 - \alpha^2 \Delta)\mathsf{u}^{(\alpha)} = \mathsf{v}^{(\alpha)} \end{aligned}$$

with initial data $\mathsf{v}^{(\alpha)}(\mathsf{x}, 0) = \mathsf{v}_0(\mathsf{x})$. Here $\mathsf{v}^{(\alpha)} = (v_1^{(\alpha)}, v_2^{(\alpha)}, v_3^{(\alpha)})$ denotes the velocity field, p the pressure, ν is a positive constant representing the viscosity, and g represents an external body force, respectively. The superscript T denotes matrix transpose. The initial velocity $\mathsf{v}_0(\mathsf{x})$ does not depend on α and we assume, without loss of generality, that both it and g are divergence free and that they have zero mean,

$$\int_D \mathsf{v}_0(\mathsf{x}) d\mathsf{x} = \int_D \mathsf{g}(\mathsf{x}, t) d\mathsf{x} = 0, \quad t \geq 0. \tag{2.1}$$

In this paper we will consider mild solutions of the Fourier transform of this equation. These solutions correspond to solutions of the integral equation obtained from the differential equation, see for example [11] for an up to date use of mild solutions in the context of the Navier-Stokes equations. In this section we present details of an equivalent alternative formulation of the LANSα equation that is amenable to a probabilistic representation.

First, let's introduce some notation that will be used in this paper. Define the aspect ratio by $\beta = \frac{2\pi}{2L}$, and let $\mathsf{k} = (k_1, k_2, k_3)$ denote the integer lattice vector with integer coordinates k_1, k_2, k_3. The superscript (α) will be suppressed in the remainder of this section in which $\alpha \geq 0$ is fixed but arbitrary. With $\hat{\mathsf{v}}$ defined by $\hat{\mathsf{v}}(\mathsf{k},t) = (2L)^{-3}\int_D \mathsf{v}(\mathsf{x},t)e^{-i\beta\mathsf{k}\cdot\mathsf{x}}d\mathsf{x}$ (or equivalently $\mathsf{v}(\mathsf{x},t) = \sum_{\mathsf{k}\in\mathbf{Z}^3}\hat{\mathsf{v}}(\mathsf{k},t)e^{i\beta\mathsf{k}\cdot\mathsf{x}}$), the Fourier transform of the LANSα equation takes the form

$$\begin{aligned}\frac{\partial\hat{\mathsf{v}}(\mathsf{k},t)}{\partial t} \quad + \quad i\beta\left(\mathsf{k}\sum_{\mathsf{j}}\hat{\mathsf{u}}(\mathsf{j},t)\otimes\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t) + \sum_{\mathsf{j}}\mathsf{j}\hat{\mathsf{u}}(\mathsf{j},t)\cdot\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t)\right)\\ = -\nu|\beta\mathsf{k}|^2\hat{\mathsf{v}} - i\beta\mathsf{k}\hat{p}(\mathsf{k},t) + \hat{\mathsf{g}}.\end{aligned}$$

Observe that the mean zero property is preserved for all time in this evolution. In particular

$$\hat{\mathsf{v}}(\mathbf{0},t) = \hat{\mathsf{v}}_0(\mathbf{0}) = 0, \quad \forall t \geq 0. \tag{2.2}$$

Here, we express each coefficient vector $\hat{\mathsf{v}}(\mathsf{k},t) = (\hat{v}_1(\mathsf{k},t), \hat{v}_2(\mathsf{k},t), \hat{v}_3(\mathsf{k},t))$ componentwise. Similarly we define $\hat{\mathsf{u}}, \hat{p}, \hat{g}$ and their components, whereas $\mathsf{j} = (j_1, j_2, j_3)$ also denotes an integer lattice vector and throughout the paper the summation is always performed over $\mathbf{Z}^3$ unless otherwise indicated. Now integrate with respect to time from 0 to t, and utilize the (transformed) regularization

$$\hat{\mathsf{u}}(\mathsf{k},t) = \frac{\hat{\mathsf{v}}(\mathsf{k},t)}{1+\alpha^2|\beta\mathsf{k}|^2}$$

to obtain,

$$\hat{\mathsf{v}}(\mathsf{k},t) \quad = \quad \exp[-\nu|\beta\mathsf{k}|^2 t]\hat{\mathsf{v}}_0(\mathsf{k}) - i\int_0^t \exp[-\nu|\beta\mathsf{k}|^2 s]\sum_{\mathsf{j}}\frac{\beta\mathsf{k}\cdot\hat{\mathsf{v}}(\mathsf{j},t-s)}{1+\alpha^2|\beta\mathsf{j}|^2}\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t-s)ds$$

$$
-i\int_0^t \exp[-\nu|\beta \mathsf{k}|^2 s]\sum_{\mathsf{j}}\beta\mathsf{j}\frac{\hat{\mathsf{v}}(\mathsf{j},t-s)\cdot\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t-s)}{1+\alpha^2|\beta\mathsf{j}|^2}ds
$$

$$
-i\beta\mathsf{k}\int_0^t \exp[-\nu|\beta\mathsf{k}|^2 s]\hat{p}(\mathsf{k},t-s)ds+\int_0^t \exp[-\nu|\beta\mathsf{k}|^2 s]\hat{\mathsf{g}}(\mathsf{k},t-s)ds \tag{2.3}
$$

Remark Using the properties of convolutions, one can rewrite the summations appearing in the third term of (2.3) as

$$
\begin{aligned}
\sum_{\mathsf{j}}\hat{\mathsf{v}}(\mathsf{j},t-s)\cdot\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t-s)\frac{\beta\mathsf{j}}{1+\alpha^2|\beta\mathsf{j}|^2} &= \frac{1}{2}\sum_{\mathsf{j}}\hat{\mathsf{v}}(\mathsf{j},t-s)\cdot\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t-s)\frac{\beta\mathsf{j}}{1+\alpha^2|\beta\mathsf{j}|^2}\\
&+\frac{1}{2}\sum_{\mathsf{j}}\hat{\mathsf{v}}(\mathsf{j},t-s)\cdot\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t-s)\frac{\beta(\mathsf{k}-\mathsf{j})}{1+\alpha^2|\beta(\mathsf{k}-\mathsf{j})|^2}\\
&=\frac{|\beta\mathsf{k}|^2\alpha^2}{2}\sum_{\mathsf{j}}\beta\mathsf{j}\frac{\hat{\mathsf{v}}(\mathsf{j},t-s)\cdot\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t-s)}{(1+\alpha^2|\beta\mathsf{j}|^2)(1+\alpha^2|\beta(\mathsf{k}-\mathsf{j})|^2)}\\
&-\alpha^2\sum_{\mathsf{j}}\beta\mathsf{j}\frac{\hat{\mathsf{v}}(\mathsf{j},t-s)\cdot\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t-s)}{(1+\alpha^2|\beta\mathsf{j}|^2)(1+\alpha^2|\beta(\mathsf{k}-\mathsf{j})|^2)}\beta\mathsf{k}\cdot\beta\mathsf{j}\\
&+\beta\mathsf{k}\sum_{\mathsf{j}}\frac{\hat{\mathsf{v}}(\mathsf{j},t-s)\cdot\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t-s)}{1+\alpha^2|\beta(\mathsf{k}-\mathsf{j})|^2}.
\end{aligned} \tag{2.4}
$$

Observe that the projection onto the plane perpendicular to k of this last term vanishes. It should also be noted that ones obtains the Navier-Stokes equation from (2.3) by taking $\alpha = 0$.

To proceed, we introduce a multiplier function $h : \mathbf{Z}^3 \to \mathbf{R}^+$ such that $h(\mathsf{k}) > 0$ for $\mathsf{k} \neq 0$. For convenience we take $h(\mathbf{0}) = 0$. This is permissible in what follows since, in view of the mean-zero property (2.2), the terms $\mathsf{j} = 0$ and $\mathsf{j} = \mathsf{k}$ can be dropped from the convolution. For $\mathsf{k} \neq 0$, let $\mathsf{e}_{\mathsf{k}} = \mathsf{k}/|\mathsf{k}|$ and denote the projection on the plane perpendicular to k by π_{k}. Then, eliminating the pressure using this projection, dividing by $h(\mathsf{k})$ and balancing, for $\mathsf{k} \neq 0$ one has for arbitrary $0 \le q_j (j = 0, 1, 2, 3)$, $q_0 + q_1 + q_2 + q_3 = 1$,

$$
\begin{aligned}
\frac{\hat{\mathsf{v}}(\mathsf{k},t)}{h(\mathsf{k})} &= \exp[-\nu|\beta\mathsf{k}|^2 t]\frac{\hat{\mathsf{v}}_0(\mathsf{k})}{h(\mathsf{k})}\\
&+ q_0\int_0^t \nu|\beta\mathsf{k}|^2\exp[-\nu|\beta\mathsf{k}|^2 s]\ \sum_{\mathsf{j}}\left[\frac{h*h(\mathsf{k})|\beta\mathsf{k}|}{\nu|\beta\mathsf{k}|^2 h(\mathsf{k})(1+\alpha^2|\beta\mathsf{j}|^2)}\frac{1}{q_0}\right]
\end{aligned}
$$

$$\left[\frac{(\mathsf{e}_{\mathsf{k}}\cdot\hat{\mathsf{v}}(\mathsf{j},t-s))(\pi_{\mathsf{k}}\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t-s))}{i\ h(\mathsf{j})h(\mathsf{k}-\mathsf{j})}\right]\ \frac{h(\mathsf{j})h(\mathsf{k}-\mathsf{j})}{h*h(\mathsf{k})}\ ds$$

$$+\ q_1\int_0^t \nu|\beta\mathsf{k}|^2\exp[-\nu|\beta\mathsf{k}|^2 s]\sum_{\mathsf{j}}\left[\frac{\alpha^2 h*h(\mathsf{k})|\beta\mathsf{k}|^2}{2\nu|\beta\mathsf{k}|^2 h(\mathsf{k})}\frac{1}{q_1}\frac{|\beta\mathsf{j}|}{(1+\alpha^2|\beta\mathsf{j}|^2)(1+\alpha^2|\beta\mathsf{k}-\beta\mathsf{j}|^2)}\right]$$

$$\left[\pi_{\mathsf{k}}(\mathsf{e}_{\mathsf{j}})\frac{\hat{\mathsf{v}}(\mathsf{j},t-s)\cdot\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t-s)}{i\ h(\mathsf{j})h(\mathsf{k}-\mathsf{j})}\right]\left[\frac{h(\mathsf{j})h(\mathsf{k}-\mathsf{j})}{h*h(\mathsf{k})}\right]\ ds$$

$$+\ q_2\int_0^t \nu|\beta\mathsf{k}|^2\exp[-\nu|\beta\mathsf{k}|^2 s]\ \sum_{\mathsf{j}}\left[\frac{\alpha^2 h*h(\mathsf{k})|\beta\mathsf{k}|}{\nu|\beta\mathsf{k}|^2 h(\mathsf{k})}\frac{|\beta\mathsf{j}|^2}{(1+\alpha^2|\beta\mathsf{j}|^2)(1+\alpha^2|\beta\mathsf{k}-\beta\mathsf{j}|^2)}\frac{1}{q_2}\right]$$

$$\left[\pi_{\mathsf{k}}(\mathsf{e}_{\mathsf{j}})(\mathsf{e}_{\mathsf{k}}\cdot\mathsf{e}_{\mathsf{j}})\frac{i\ \hat{\mathsf{v}}(\mathsf{j},t-s)\cdot\hat{\mathsf{v}}(\mathsf{k}-\mathsf{j},t-s)}{h(\mathsf{j})h(\mathsf{k}-\mathsf{j})}\right]\ \frac{h(\mathsf{j})h(\mathsf{k}-\mathsf{j})}{h*h(\mathsf{k})}\ ds$$

$$+\ q_3\int_0^t \nu|\beta\mathsf{k}|^2\exp[-\nu|\beta\mathsf{k}|^2 s]\left[\frac{\hat{\mathsf{g}}(\mathsf{k},t-s)}{\nu|\beta\mathsf{k}|^2 h(\mathsf{k})}\frac{1}{q_3}\right]ds. \tag{2.5}$$

To simplify this expression let

$$\mathsf{m}(\mathsf{k})=\frac{h*h(\mathsf{k})}{h(\mathsf{k})\nu|\beta\mathsf{k}|},\quad W(\mathsf{j},\mathsf{n};\mathsf{k})=\frac{h(\mathsf{j})h(\mathsf{n})}{h*h(\mathsf{k})}\delta_{\mathsf{k}}(\mathsf{j},\mathsf{n}), \tag{2.6}$$

where $\delta_{\mathsf{k}}(\mathsf{j},\mathsf{n})=1$ if $\mathsf{j}+\mathsf{n}=\mathsf{k}$, and vanishes otherwise. Define, with $\mathsf{k}=\mathsf{j}+\mathsf{n}$,

$$\mathsf{m}_0^{(\alpha)}(\mathsf{j},\mathsf{n})=\mathsf{m}(\mathsf{k})\frac{1}{q_0(1+\alpha^2|\beta\mathsf{j}|^2)}\le\frac{\mathsf{m}(\mathsf{k})}{q_0}, \tag{2.7}$$

and for $l=1,2$,

$$\mathsf{m}_l^{(\alpha)}(\mathsf{j},\mathsf{n})=\mathsf{m}(\mathsf{k})\frac{\alpha^2|\beta\mathsf{j}|^l|\frac{\beta\mathsf{k}}{2}|^{2-l}}{(1+\alpha^2|\beta\mathsf{j}|^2)(1+\alpha^2|\beta\mathsf{n}|^2)q_l}. \tag{2.8}$$

Also, for j and n in $\mathbf{Z}^3$, and with $\mathsf{k}=\mathsf{j}+\mathsf{n}$, define the bilinear forms on $\mathbf{Z}^3\times\mathbf{Z}^3$

$$Q_0(\mathsf{a},\mathsf{b};\mathsf{j},\mathsf{n})=-i(\mathsf{e}_{\mathsf{k}}\cdot\mathsf{a})\pi_{\mathsf{k}}(\mathsf{b}),\quad Q_1(\mathsf{a},\mathsf{b};\mathsf{j},\mathsf{n})=-i\pi_{\mathsf{k}}(\mathsf{e}_{\mathsf{j}})(\mathsf{a}\cdot\mathsf{b}),\quad Q_2(\mathsf{a},\mathsf{b};\mathsf{j},\mathsf{n})=i\pi_{\mathsf{k}}(\mathsf{e}_{\mathsf{j}})(\mathsf{e}_{\mathsf{j}}\cdot\mathsf{e}_{\mathsf{k}})(\mathsf{a}\cdot\mathsf{b}). \tag{2.9}$$

It should be noted that

$$|Q_l(\mathsf{a},\mathsf{b};\mathsf{j},\mathsf{n})|\le|\mathsf{a}||\mathsf{b}|. \tag{2.10}$$

Introducing the re-scaled Fourier coefficients

$$\chi(\mathsf{k},t)=\frac{\hat{\mathsf{v}}(\mathsf{k},t)}{h(\mathsf{k})},\qquad \varphi(\mathsf{k},t)=\frac{\hat{\mathsf{g}}(\mathsf{k},t)}{\nu|\beta\mathsf{k}|^2 h(\mathsf{k})q_3} \tag{2.11}$$

the equation reduces to the following equivalent form:

$$\begin{aligned}
\chi(\mathsf{k},t) \;&=\; \exp[-\nu|\beta\mathsf{k}|^2 t]\chi_0(\mathsf{k}) \\
&+\sum_{l=0}^{2} q_l \int_0^t \nu|\beta\mathsf{k}|^2 \exp[-\nu|\beta\mathsf{k}|^2 s] \sum_{\mathsf{j},\mathsf{n}} \mathsf{m}_l^{(\alpha)}(\mathsf{j},\mathsf{n}) Q_l(\chi(\mathsf{j},t-s),\chi(\mathsf{n},t-s);\mathsf{j},\mathsf{n}) W(\mathsf{j},\mathsf{n};\mathsf{k})\, ds \\
&+ q_3 \int_0^t \nu|\beta\mathsf{k}|^2 \exp[-\nu|\beta\mathsf{k}|^2 s]\; \varphi(\mathsf{k},t-s) ds \qquad (2.12)
\end{aligned}$$

The probabilistic interpretation of this equation given in the next section is made possible by observing that $W(\mathsf{j},\mathsf{n};\mathsf{k})$ (for fixed k) is a probability mass functions with support contained in the set $\{(\mathsf{j},\mathsf{n}) \in \mathbf{Z}^3 \times \mathbf{Z}^3 : \mathsf{j}+\mathsf{n} = \mathsf{k}\}$. In particular, these will provide transition probabilities for which a wavenumber k will branch into a pair of wavenumbers $\mathsf{j}, \mathsf{k}-\mathsf{j}$. This view of the underlying equations will be exploited in the next section to obtain conditions for global existence, uniqueness, and convergence.

3 Stochastic Cascade Representation: Global existence, uniqueness and convergence

Equation (2.12) can be interpreted in terms of an expected value of a multiplicative functional defined on a branching binary tree. Indeed, each of the terms of this equation has been weighted to explicitly reflect this form. For example, if one considers a random variable $S_\emptyset$ with an exponential distribution with parameter $\nu|\beta\mathsf{k}|^2$, the first term in this equation can be written as

$$\mathbf{E}\left[\chi_0(\mathsf{k})\mathbf{1}[S_\emptyset > t]\right].$$

On the other hand, if $S_\emptyset < t$ one thinks of either terminating the process at time $t - S_\emptyset$ with probability q_3, or branching into two particles (j,n) chosen according to the probability mass function $W(\mathsf{j},\mathsf{n};\mathsf{k})$, and (independently) assigned multipliers $\mathsf{m}_l^{(\alpha)}(\mathsf{j},\mathsf{n})$ with probability $q_l (l = 0, 1, 2)$. The process then continues with each branch following the same process independently of each other.

To provide the details of this construction, let $\mathcal{V}$ denote the vertex of a complete binary tree rooted at $\emptyset$,

$$\mathcal{V} = \cup_{j=0}^{\infty}\{1,2\}^j = \{\emptyset, \langle 1\rangle, \langle 2\rangle, \langle 11\rangle, \langle 12\rangle, ...\}.$$

As standard, a vertex $\langle \mathsf{v}\rangle = \langle v_1, v_2, ...v_k\rangle$ of the binary tree is said to be of length $|\langle \mathsf{v}\rangle| = k$, with $|\emptyset| = 0$. For $l = 1,2$, denote by $\langle \mathsf{v}l\rangle = \langle v_1, v_2, ..., v_k, l\rangle$ the vertex of length $k+1$ obtained from $\langle \mathsf{v}\rangle$ by concatenating the value l.

Let $\{\kappa_{\langle \mathsf{v}\rangle} : \langle \mathsf{v}\rangle \in \mathcal{V}\}$ be a collection of independent and identically distributed random variables with

$$\mathbf{P}(\kappa_{\langle \mathsf{v}\rangle} = l) = q_l, \quad l = 0,1,2,3.$$

Let $\{\overline{S}_{\langle \mathsf{v}\rangle} : \langle \mathsf{v}\rangle \in \mathcal{V}\}$ be a collection of i.i.d. mean-one exponentially distributed random variables, and independent of $\{\kappa_{\langle \mathsf{v}\rangle} : \langle \mathsf{v}\rangle \in \mathcal{V}\}$. Then for each $\langle \mathsf{v}\rangle \in \mathcal{V}$ and nonzero wavenumber $\mathsf{k}_{\langle \mathsf{v}\rangle}$,

$$S_{\langle \mathsf{v}\rangle} = \frac{1}{\nu|\beta \mathsf{k}_{\langle \mathsf{v}\rangle}|^2}\overline{S}_{\langle \mathsf{v}\rangle}$$

is a random variable, independent of $\{\kappa_{\langle \mathsf{v}\rangle} : \langle \mathsf{v}\rangle \in \mathcal{V}\}$, having an exponential distribution with parameter $\nu|\beta \mathsf{k}_{\langle \mathsf{v}\rangle}|^2$. Finally, conditioned on $\mathsf{k}_{\langle \mathsf{v}\rangle}$ and $\kappa_{\langle \mathsf{v}\rangle} = l$ for $l \neq 3$, the ordered pair $(\mathsf{k}_{\langle \mathsf{v}1\rangle}, \mathsf{k}_{\langle \mathsf{v}2\rangle})$ is chosen according to the probability mass function $W(\mathsf{j}, \mathsf{n}; \mathsf{k}_{\langle \mathsf{v}\rangle})$. The resulting family of wavenumbers $\{\mathsf{k}_{\langle \mathsf{v}\rangle} : \langle \mathsf{v}\rangle \in \mathcal{V}\}$ defines a tree-indexed Markov chain starting at $\mathsf{k}_\emptyset = \mathsf{k}$ whose distribution does not depend on α.

Next we recursively define the cascade functional by

$$\chi^{(\alpha)}(\mathsf{k}_{\langle \mathsf{v}\rangle}, t) = \begin{cases} \chi_0(\mathsf{k}_{\langle \mathsf{v}\rangle}) & \text{if } S_{\langle \mathsf{v}\rangle} \geq t \\ \\ \varphi(\mathsf{k}_{\langle \mathsf{v}\rangle}, t - S_{\langle \mathsf{v}\rangle}) & \text{if } S_{\langle \mathsf{v}\rangle} < t, \text{ and } \kappa_{\langle \mathsf{v}\rangle} = 3 \\ \\ \mathsf{m}_l^{(\alpha)}(\mathsf{k}_{\langle \mathsf{v}1\rangle}, \mathsf{k}_{\langle \mathsf{v}2\rangle}) Q_l\left(\chi^{(\alpha)}(\mathsf{k}_{\langle \mathsf{v}1\rangle}, t - S_{\langle \mathsf{v}\rangle}), \chi^{(\alpha)}(\mathsf{k}_{\langle \mathsf{v}2\rangle}, t - S_{\langle \mathsf{v}\rangle}); \mathsf{k}_{\langle \mathsf{v}1\rangle}, \mathsf{k}_{\langle \mathsf{v}2\rangle}\right) \\ \quad \text{if } S_{\langle \mathsf{v}\rangle} < t, \text{ and } \kappa_{\langle \mathsf{v}\rangle} = l \neq 3. \end{cases} \tag{3.13}$$

Note that with $q_3 = 1/2$, the expected number of branches at any given vertex equals 1. Consequently, the recursion is well defined, since with probability one it terminates after a finite number of branchings.

This recursion provides a stochastic representation of the solution of equation (2.12) according to the following theorem.

Theorem 3.1 *Assume that* $\hat{\mathsf{v}}_0(k), \hat{\mathsf{g}}(k,s)$ *and* $h(k)$ *are such that* $\mathbf{E}(|\chi^{(\alpha)}(k,t)|)$ *is finite for all* $k \in \mathbf{Z}^3$, $0 \le t \le T$. *Then* $\hat{\mathsf{v}}^{(\alpha)}(k,t) = h(k)\mathbf{E}(\chi^{(\alpha)}(k,t))$ *is a mild solution of the LANS*α *equation.*

Proof: It suffices to show that $\chi(\mathsf{k},t) = \hat{\mathsf{v}}(\mathsf{k},t)/h(\mathsf{k})$ satisfies equation (2.12). Under the assumption of finite expectation, one has

$$\begin{aligned}
\mathbf{E}(\chi^{(\alpha)}(\mathsf{k},t)) &= \mathbf{E}(\chi^{(\alpha)}(\mathsf{k},t)\mathbf{1}[S_\emptyset \ge t]) + \sum_{l=0}^{3} q_l \mathbf{E}(\chi^{(\alpha)}(\mathsf{k},t)\mathbf{1}[S_\emptyset < t]|\kappa_\emptyset = l) \\
&= \mathbf{P}(S_\emptyset \ge t)\chi_0(\mathsf{k}) + q_3 \int_0^t \nu|\beta\mathsf{k}|^2 \exp[-\nu|\beta\mathsf{k}|^2 s]\varphi(\mathsf{k}, t-s)ds \\
&+ \sum_{l=0}^{2} q_l \int_0^t \nu|\beta\mathsf{k}|^2 \exp[-\nu|\beta\mathsf{k}|^2 s] \\
&\qquad \mathbf{E}\left[\mathsf{m}_l^{(\alpha)}(\mathsf{k}_{\langle 1\rangle}, \mathsf{k}_{\langle 2\rangle}) Q_l(\chi^{(\alpha)}(\mathsf{k}_{\langle 1\rangle}, t-s), \chi^{(\alpha)}(\mathsf{k}_{\langle 2\rangle}, t-s); \mathsf{k}_{\langle 1\rangle}, \mathsf{k}_{\langle 2\rangle})|\kappa_{\langle \mathsf{v}\rangle} = l\right] ds
\end{aligned}$$

The theorem follows since by construction,

$$\begin{aligned}
&\mathbf{E}\left[Q_l(\chi^{(\alpha)}(\mathsf{k}_{\langle 1\rangle}, t-s), \chi^{(\alpha)}(\mathsf{k}_{\langle 2\rangle}, t-s); \mathsf{k}_{\langle 1\rangle}, \mathsf{k}_{\langle 2\rangle})|\kappa_\emptyset = l\right] \\
= &\sum_{\mathsf{j}+\mathsf{n}=\mathsf{k}} Q_l(\chi^{(\alpha)}(\mathsf{j}, t-s), \chi^{(\alpha)}(\mathsf{n}, t-s); \mathsf{j}, \mathsf{n}) W(\mathsf{j}, \mathsf{n}; \mathsf{k})
\end{aligned}$$

□

Theorem 3.1 provides the basis for the determination of function spaces appropriate to this theory by considering conditions on the initial data and forcing such that the hypothesis hold. Give $C^\infty(T^3)$ the Fréchet space topology defined by the seminorms $||g||_m = \sup_{x\in T^3}\{|\partial^{(m)} g(x)|\}$ and let $\mathcal{D}'$ denote the usual space of bounded continuous linear functionals on $C^\infty(T^3)$ with the weak*topology. Given a majorizing kernel h one defines a function space $\mathcal{F}_h$ of distributions (in the sense of Schwartz) by

$$\mathcal{F}_h = \{v \in \mathcal{D}' : ||v||_h \equiv \sup_{0\le t\le T, \mathsf{k}\ne \mathbf{0}} \frac{|\hat{\mathsf{v}}(\mathsf{k},t)|}{h(\mathsf{k})} < \infty\}. \tag{3.14}$$

Such function spaces can be viewed as a generalization of Besov spaces; see [1].

Observe for any positive constant $c > 0$ that,

$$\mathcal{F}_h = \mathcal{F}_{ch}, \quad ||v||_{ch} = \frac{1}{c}||v||_h. \tag{3.15}$$

Thus we will formulate results in terms of a *standardized majorizing kernel* defined by

$$h * h(\mathsf{k}) \leq |\mathsf{k}| h(\mathsf{k}), \quad \mathsf{k} \neq \mathbf{0}.$$

In particular results may be stated for standardized kernels in defining $\mathcal{F}_h$. On the other hand, such constants are reflected in the size of the ball in the space $\mathcal{F}_h$ for which the global existence/uniqueness/convergence results are obtained.

To be precise, suppose that $h(\mathsf{k})$ is a standardized majorizing kernel. The approach is to show that one may choose $R > 0$ such that in the stochastic representation defined by Rh one has

$$\mathsf{m}_l^{(\alpha)}(\mathsf{k}, \mathsf{j}) \leq 1, \quad l = 0, 1, 2.$$

In this way, if the initial data $\mathsf{v}_0 \in \mathcal{F}_h (= \mathcal{F}_{Rh})$, and forcing are subject to the conditions

$$|\hat{v}_0(\mathsf{k})| \leq Rh(\mathsf{k}), \qquad |\hat{g}(\mathsf{k}, t)| \leq \nu |\beta \mathsf{k}|^2 Rh(\mathsf{k}) q_3,$$

then one obtains

$$|\chi^{(\alpha)}(\mathsf{k}, t)| \leq 1.$$

In particular, the hypothesis of Theorem 3.1 are trivially satisfied. One may note that the condition on the forcing may be equivalently expressed by a condition on the inverse Laplacian of g (noting the role of β in the definition of the Fourier coefficients):

$$\Delta^{-1} g \in B_{q_3 \nu R}.$$

The following lemma will be used for the determination of the radius R in the proof of the theorem to follow.

Lemma 3.1 *The following inequality holds for any $\alpha, \beta > 0$ and $\mathsf{k} \in \mathbf{Z}^3$.*

$$\frac{\alpha^2|\beta\mathsf{k}||\beta\mathsf{j}|}{(1+\alpha^2|\beta\mathsf{j}|^2)(1+\alpha^2|\beta\mathsf{k}-\beta\mathsf{j}|^2)} \leq 1.$$

Proof: Introducing a new variable $\gamma = \alpha\beta > 0$ we need only prove the following:

$$\frac{|\mathsf{j}|}{(1+\gamma^2|\mathsf{j}|^2)(1+\gamma^2|\mathsf{k}-\mathsf{j}|^2)} - \frac{1}{\gamma^2|\mathsf{k}|} \leq 0$$

Consider two cases $|\mathsf{j}| \geq |\mathsf{k}|$ and $|\mathsf{j}| < |\mathsf{k}|$ for a given k. For the first case we simply find

$$\frac{|\mathsf{j}|}{(1+\gamma^2|\mathsf{j}|^2)(1+\gamma^2|\mathsf{k}-\mathsf{j}|^2)} \leq \frac{|\mathsf{j}|}{1+\gamma^2|\mathsf{j}|^2} \leq \frac{1}{\gamma^2|\mathsf{j}|} \leq \frac{1}{\gamma^2|\mathsf{k}|}$$

and done. For the second case, using the fact $|\mathsf{k}-\mathsf{j}|^2 \geq (|\mathsf{k}|-|\mathsf{j}|)^2$ we will show, for $0 \leq t \leq a$,

$$\frac{t}{(1+\gamma^2t^2)(1+\gamma^2(t-a)^2)} \leq \frac{1}{\gamma^2 a}.$$

But by the substitution $u = \gamma t, v = \gamma(a-t)$, this is equivalent to another inequality for $0 \leq u \leq \gamma a$,

$$\frac{u(u+v)}{(1+u^2)(1+v^2)} \leq 1.$$

which, if expanded, is the same as

$$(uv)^2 - uv + v^2 + 1 = (uv - \frac{1}{2})^2 + v^2 + \frac{3}{4} > 0.$$

Thus the proof is complete

$\square$

.

Conditions for global existence, uniqueness and convergence of mild solutions (as $\alpha \downarrow 0$) may now be expressed in terms of a suitably small ball in the space $\mathcal{F}_h$ of the form:

$$B_R = \{v \in \mathcal{F}_h : ||v||_h \leq R\}, \quad R = \nu\beta/6. \tag{3.16}$$

Theorem 3.2 *Let h be a standardized majorizing kernel. Take $q_3 = \frac{1}{2}$, and $q_0 = q_1 = q_2 = \frac{1}{6}$. Let $B_R \subseteq \mathcal{F}_h$ denote the ball of radius R centered at 0, where $R = \frac{\nu\beta}{6}$. If the $\mathsf{v}_0 \in B_R$ and $\Delta^{-1}g \in B_{\frac{\nu R}{2}}$ then the solution of each $\mathsf{LANS}\alpha$, $\hat{\mathsf{v}}_\alpha(\mathsf{k}, t)$ exists and is unique for all $t > 0$. Moreover, for each $\mathsf{k} \in \mathbf{Z}^3$ one has*

$$\lim_{\alpha \to 0} \mathsf{v}^{(\alpha)}(\mathsf{k}, t) = \mathsf{v}^{(0)}(\mathsf{k}, t).$$

Proof: As noted earlier the choice $q_3 = \frac{1}{2}$ insures that the cascade terminates, i.e., the branching is critical. The existence of a common function space is based on the observation that the multiplier for the Navier-Stokes equation ($\alpha = 0$) is found to be an upper bound of the multipliers of the $\mathsf{LANS}\alpha$ for $\alpha > 0$. We easily observe with respect to $h_R := Rh$ one has

$$(h_R * h_R)(\mathsf{k}) \leq R|\mathsf{k}|h_R(\mathsf{k}).$$

Thus,

$$||\chi||_{h_R} \leq 1, \quad ||\Delta^{-1}g||_{h_R} \leq 1.$$

So, applying the stochastic cascade representation for the majorizing kernel h_R, one has

$$\mathsf{m}(\mathsf{k}) \leq \frac{R}{\nu\beta},$$

and therefore

$$\mathsf{m}_0^{(\alpha)}(\mathsf{j}, \mathsf{k}-\mathsf{j}) \leq \frac{\mathsf{m}(\mathsf{k})}{q_0} = \frac{R}{q_0\nu\beta}.$$

Next from Lemma 3.1, we have

$$\mathsf{m}_1^{(\alpha)}(\mathsf{j}, \mathsf{k}-\mathsf{j}) \leq \frac{\mathsf{m}(\mathsf{k})}{2q_1} \leq \frac{R}{2q_1\nu\beta}.$$

Finally

$$\mathsf{m}_2^{(\alpha)}(\mathsf{j}, \mathsf{k}-\mathsf{j}) \leq \frac{\mathsf{m}(\mathsf{k})}{q_2} \leq \frac{R}{q_2\nu\beta}.$$

Thus one may take

$$R = \nu\beta \min\{q_0, 2q_1, q_2\} = \frac{\nu\beta}{6}$$

to bound each multiplier by unity.

The uniqueness follows from the martingale argument originating in [10], and also used in [1].

The pointwise convergence of the Fourier coefficients is an immediate consequence of the expected value representation and Lebesgue's dominated convergence theorem.

$\square$

Remark Recalling that $q_3 = \frac{1}{2}$, it is evident from the above proof with regard to the size R of the ball, that the optimal choice for $0 < q_0, q_1, q_2 < \frac{1}{2}$ with sum $\frac{1}{2}$ is the assumed $q_0 = q_1 = q_2 = \frac{1}{6}$.

In the next section we indicate some general methods for constructing majorizing kernels to which the above theory applies. These are extensions of methods given in [1]. Also, a primary goal of this paper is to obtain a rate of convergence in a mixed $L^1 - L^2$ space-time norm in function spaces defined by a certain class of majorizing kernels whose construction is also provided in the next section.

4 Majorizing Kernels for $\mathbf{LANS}\alpha$

Following the definition used in [1], a non negative function $h(\mathsf{k})$ defined on $\mathbf{Z}^3$ is said to be a majorizing kernel for the Navier-Stokes equation ($\alpha = 0$) if there exists $C > 0$ such that

$$h * h(\mathsf{k}) \leq C|\beta \mathsf{k}| h(\mathsf{k}), \mathsf{k} \neq \mathbf{0}. \tag{4.17}$$

Here $h * h$ is the discrete convolution $h * h(\mathsf{k}) = \sum_{\mathsf{j}} h(\mathsf{j}) h(\mathsf{j} - \mathsf{k})$. Since all finite-dimensional norms are equivalent, the choice of norm defining $|\mathsf{k}|$ is often taken as l_1 for convenience. Observe that if h is a majorizing kernel then so is ch for any positive constant c.

Remark The majorizing theory developed for incompressible Navier-Stokes equations on $\mathbf{R}^3$ in [1] is without boundary. In particular convolution is an integral formula in that setting.

While it was recognized that the general ideas would go over to periodic boundary, until now there has not been a need to exhibit examples of majorizing kernels in this context. One point for distinction is the singularity as the wavenumber $\xi \to 0$ required for (4.17) on $\mathbf{R}^3$. No such constraint arises on the lattice $\mathbf{R}^3$. Convolution requires that functions be defined on the entire group but, in view of the mean-zero property (2.2), the value at zero may be arbitrarily specified. The present choice $h(\mathbf{0}) = 0$ is a matter of convenience.

From the point of view of analysis it is occasionally more convenient to obtain majorizing kernels for integral convolutions. The following proposition firms up a useful connection.

Proposition 4.1 *For measurable $h : \mathbf{R}^3 \to [0, \infty)$, define*

$$h *_c h(\xi) := \int_{\mathbf{R}^d} h(\xi - \eta) h(\eta) d\eta, \quad \xi \in \mathbf{R}^3,$$

and

$$h *_d h(\mathsf{k}) := \sum_{\mathsf{k} \in \mathbf{Z}^3} h(\mathsf{k} - \mathsf{j}) h(\mathsf{j}), \quad \mathsf{k} \in \mathbf{Z}^3.$$

Suppose

$$h *_c h(\xi) \leq c|\xi| h(\xi), \quad \xi \in \mathbf{R}^3.$$

Let $Q_{\mathsf{k}}(1)$ denote the unit cube centered at $\mathsf{k} \in \mathbf{Z}^3$. If there are constants c_1, c_2 such that

$$c_2 h(\mathsf{k}) \leq h(\eta) \leq c_1 h(\mathsf{k}), \quad \forall \eta \in Q_{\mathsf{k}}(1),$$

then

$$c_2^2 h *_d h(\mathsf{k}) \leq h *_c h(\mathsf{k}) \leq c_1^2 h *_d h(\mathsf{k}), \quad \mathsf{k} \in \mathbf{Z}^3.$$

In particular,

$$h *_d h(\mathsf{k}) \leq \frac{c}{c_2^2} |\mathsf{k}| h(\mathsf{k}), \quad \mathsf{k} \neq \mathbf{0}.$$

Proof: Use the l^1-norm on $\mathbf{R}^3$ to get for $\eta \in Q_{\mathsf{j}}(1)$, and $\mathsf{k} \in \mathbf{Z}$ that $\mathsf{k} - \eta \in Q_{\mathsf{k}-\mathsf{j}}$. Therefore,

$$h *_c h(\mathsf{k}) \quad = \quad \int_{\mathbf{R}} h(\eta) h(\mathsf{k} - \eta) d\eta$$

$$
\begin{aligned}
&= \sum_{\mathsf{j}\in\mathbf{Z}} \int_{Q_{\mathsf{j}}(1)} h(\eta)h(\mathsf{k}-\eta)d\eta \\
&\geq \sum_{\mathsf{j}\in\mathbf{Z}} \int_{Q_{\mathsf{j}}(1)} c_2^2 h(\mathsf{j})h(\mathsf{k}-\mathsf{j})d\eta = c_2^2\ h *_d h(\mathsf{k}). \qquad (4.18)
\end{aligned}
$$

The upper bound is proved in the same way. From the majorizing property one obtains

$$
h *_d h(\mathsf{k}) \leq \frac{1}{c_2^2} h *_c h(\mathsf{k}) \leq \frac{c}{c_2^2}|\mathsf{k}|h(\mathsf{k}).
$$

□

Remark The special notations for convolution ($*_c$, $*_d$) introduced in the lemma will dropped when the meaning of $*$ is clear from the context.

As an application it follows that

Corollary 4.1 *The function*

$$
h(k) = \frac{e^{-|k|}}{|k|}, \quad k \in \mathbf{Z}^3, k \neq \mathbf{0}, \quad h(\mathbf{0}) = 0,
$$

defines a majorizing kernel. In fact, $h \in l_1$ is normalizable to a probability.

The majorizing kernel provided by Corollary 4.1 is significant in providing an example of a majorizing kernel satisfying the moment conditions under which *rates* of convergence will be obtained in the last section of this paper. As noted in [1], the continuous version of this kernel implicitly appears in the analysis in [10]. Also, one may check that the lattice potential it is asymptotically equivalent to a three-dimensional Bessel potential on $\{\mathsf{k} \in \mathbf{Z} : |\mathsf{k}| \geq 1\}$.

However, the regularity and uniqueness, and (full) convergence of solutions as $\alpha \to 0$ can be obtained for a larger class of majorizing kernels. So we include general approaches to the construction of such functions in the remainder of this section. This is largely an extension of ideas developed in [1] to the (lattice) case for periodic boundary conditions.

We first note the following.

Lemma 4.1 *Let $g_1, g_2 : \mathbf{Z}^d \to \mathbf{R}^+$. Assume that $g_1(\mathsf{k}) \sim g_2(\mathsf{k})$ for large k. Then there exists $c > 0, C > 0$ such that for all $\mathsf{k} \in \mathbf{Z}^d$*

$$cg_2(\mathsf{k}) \le g_1(\mathsf{k}) \le Cg_2(\mathsf{k})$$

Proof From the asymptotic behavior, it follows that there exists M such that $|\mathsf{k}| > M$, $(1/2)g_2(\mathsf{k}) \le g_1(\mathsf{k}) \le 2g_2(\mathsf{k})$. Since the $g_j(\mathsf{k})$ are assumed to be strictly positive, the lemma follows by taking $c = \min\{1/2, g_1(\mathsf{k})/g_2(\mathsf{k}), |\mathsf{k}| \le M\}$, and $C = \max\{2, g_1(\mathsf{k})/g_2(\mathsf{k}), |\mathsf{k}| \le M\}$.

□

Lemma 4.2 *Let $g_j : \mathbf{Z} \to (0, \infty)$ $j = 1, 2, 3$ be such that*

$$c_j|k|^{\delta_j} g_j(k) \le g_j * g_j(k) \le C_j|k|^{\delta_j} g_j(k), \quad \forall k \ne 0, \quad g_j * g_j(0) = m_j < \infty$$

and let $\mathsf{k} = (k_1, k_2, k_3)$, $\delta = \sum_j \delta_j$. Then, $h(\mathsf{k}) = \Pi_{j=1}^3 g_j(k_j)$, satisfies for appropriate c, C,

$$c|\mathsf{k}|^\delta h(\mathsf{k}) \le h * h(\mathsf{k}) \le C|\mathsf{k}|^\delta h(\mathsf{k}), \quad h * h(0) = M < \infty$$

Proof This is immediate since the variables separate. Note if $\tilde{C}_j$ takes the values C_j or m_j, then $C = \max \Pi \tilde{C}_j$, and $M = \Pi_j m_j$. Similar considerations apply to c.

□

The following is an immediate consequence of these lemmas.

Lemma 4.3 *Let $1 > \theta > 1/2$ and let $g(k) = |k|^{-\theta}$ for $0 \ne k \in \mathbf{Z}$, $g(0) = 1$. Then there exists $c, C > 0$ such that for $k \ne 0$,*

$$c|k|^{1-\theta} g(k) \le g * g(k) \le C|k|^{1-\theta} g(k).$$

Proof: For arbitrary $k > 0$ one has

$$\begin{aligned} g * g(k) &= \sum_{j=1}^{k-1} \frac{1}{j^\theta (k-j)^\theta} + 2 \sum_{j=k+1}^{\infty} \frac{1}{j^\theta (j-k)^\theta} + 2\frac{1}{k^\theta} \\ &= \frac{k}{k^{2\theta}} \left[\sum_{j=1}^{k-1} \frac{1}{(j/k)^\theta (1-(j/k))^\theta} \frac{1}{k} + 2 \sum_{j=k+1}^{\infty} \frac{1}{(j/k)^\theta ((j/k)-1)^\theta} \frac{1}{k} \right] + 2\frac{1}{k^\theta} \end{aligned}$$

Since for large k one has

$$\sum_{j=1}^{k-1} \frac{1}{(j/k)^\theta(1-(j/k))^\theta}\frac{1}{k} \sim \int_0^1 \frac{1}{x^\theta(1-x)^\theta}dx = B(1-\theta, 1-\theta),$$

and

$$\sum_{j=k+1}^{\infty} \frac{1}{(j/k)^\theta((j/k)-1)^\theta}\frac{1}{k} \sim \int_1^\infty \frac{1}{x^\theta(x-1)^\theta}dx = B(2\theta-1, 1-\theta),$$

the conditions on θ imply that these integrals and hence the series are convergent. The lemma follows applying Lemma 4.1 with $g_1(k) = g * g(k)$ and $g_2(k) = k^{1-2\theta}$.

□

A first example of a majorizing kernel is a consequence of Lemma 4.2 and 4.3.

Corollary 4.2 *Let* $g(k) = |k|^{-2/3}$ *for* $0 \neq k \in \mathbf{Z}$, $g(0) = 1$ *and for* $\mathsf{k} \in \mathbf{Z}^3$ *let* $h(\mathsf{k}) = g(k_1)g(k_2)g(k_3)$. *Then there exists* $c, C > 0$ *such that*

$$c|\mathsf{k}|h(\mathsf{k}) \leq h * h(\mathsf{k}) \leq C|\mathsf{k}|h(\mathsf{k}).$$

A further example of a majorizing kernel is given by the following proposition.

Lemma 4.4 *For* $0 \neq \mathsf{k} \in \mathbf{Z}^3$, *let* $h(\mathsf{k}) = |\mathsf{k}|^{-2}$ *and set* $h(0) = 1$. *Then there exists positive* c_1, c_2 *such that*

$$c_1|\mathsf{k}|h(\mathsf{k}) \leq h * h(\mathsf{k}) \leq c_2|\mathsf{k}|h(\mathsf{k})$$

Proof: We follow similar steps as in the proof of Lemma 4.3. Indeed, for $\mathsf{k} \neq 0$,

$$\begin{aligned} h * h(\mathsf{k}) &= \frac{2}{|\mathsf{k}|^2} + \sum_{0\neq \mathsf{j} \neq \mathsf{k}} \frac{1}{|\mathsf{j}|^2|\mathsf{k}-\mathsf{j}|^2} \\ &= \frac{1}{|\mathsf{k}|}\left[\frac{2}{|\mathsf{k}|} + \sum_{0\neq \mathsf{j} \neq \mathsf{k}} \frac{1}{|\mathsf{j}/|\mathsf{k}||^2|\mathsf{e}_\mathsf{k} - \mathsf{j}/|\mathsf{k}||^2}\frac{1}{|\mathsf{k}|^3}\right] \sim \frac{A}{|\mathsf{k}|} \end{aligned}$$

were $\mathsf{e}_\mathsf{k} = \mathsf{k}/|\mathsf{k}|$ and

$$A = \int_{\mathbf{R}^3} \frac{1}{|\mathsf{x}|^2|\mathsf{e}_1 - \mathsf{x}|^2} d\mathsf{x}.$$

The proof is completed by virtue of Lemma 4.1.

□

5 Rates of Convergence in Physical Space

In this section we pursue the convergence obtained in the preceding section further by analyzing the rate of convergence for a particular class of majorizing kernels.

Having identified natural function spaces for this problem, we will analyze the evolution of the differences denoted by

$$\delta(\mathsf{k},t) = \mathsf{v}^{(\alpha)}(\mathsf{k},t) - \mathsf{v}^{(0)}(\mathsf{k},t), \qquad \mathsf{k} \in \mathbf{Z}^3, \quad \Delta(t) := \sup_{\mathsf{k}} |\delta(\mathsf{k},t)|, t \geq 0.$$

Note that the forcing term $\hat{g}$ will cancel in this difference. In particular one has

Proposition 5.1 *Let $h \in l^1(\mathbf{Z}^3)$ be a standardized majorizing kernel satisfying the following further moment conditions:*

$$\sum_{j} |j| h(j) < \infty, \quad \sum_{j} |j|^l h(j) h(k-j) < \infty, \quad k \in \mathbf{Z}^3, l = 2, 3.$$

Let $\gamma = \frac{\nu\beta^2}{2}$. If $||v_0||_h \leq M$ then there is a constant $C(T) > 0$, not depending on α, such that

$$\int_0^T e^{\gamma s} \Delta(s) ds \leq C(T) \alpha^2.$$

Proof: Consider the projection on the plane perpendicular to k of (2.3). One then has,

$$\begin{aligned}
& i\ \delta(\mathsf{k},t) \\
= & \int_0^t \sum_{\mathsf{j}} \{ \frac{\beta\mathsf{k} \cdot \hat{\mathsf{v}}^{(\alpha)}(\mathsf{j}, t-s)}{1+\alpha^2|\beta\mathsf{j}|^2} \pi_{\mathsf{k}} \hat{\mathsf{v}}^{(\alpha)}(\mathsf{k}-\mathsf{j}, t-s) \\
& -\beta\mathsf{k} \cdot \hat{\mathsf{v}}^{(0)}(\mathsf{j}, t-s)\ \pi_{\mathsf{k}} \hat{\mathsf{v}}^{(0)}(\mathsf{k}-\mathsf{j}, t-s) \} e^{-\nu|\beta\mathsf{k}|^2 s} \\
& + \frac{1}{2} \int_0^t \sum_{\mathsf{j}} D^{(\alpha}(\mathsf{j},\mathsf{k}) (\hat{\mathsf{v}}^{(\alpha)}(\mathsf{j}.t-s) \cdot \hat{\mathsf{v}}^{(\alpha)}(\mathsf{k}-\mathsf{j}, t-s)) e^{-\nu|\beta\mathsf{k}|^2 s} ds \\
= & \int_0^t \sum_{\mathsf{j}} \{ \beta\mathsf{k} \cdot \delta(\mathsf{j}, t-s) \pi_{\mathsf{k}} \hat{\mathsf{v}}^{(\alpha)}(\mathsf{k}-\mathsf{j}, t-s) \\
& + \beta\mathsf{k} \cdot \hat{\mathsf{v}}^{(0)}(\mathsf{j}, t-s) \pi_{\mathsf{k}} \delta(\mathsf{k}-\mathsf{j}, t-s) \} \frac{1}{1+\alpha^2|\beta\mathsf{j}|^2} e^{-\nu|\beta\mathsf{k}|^2 s} ds
\end{aligned}$$

$$-\int_0^t e^{-\nu|\beta \mathsf{k}|^2 s} \sum_{\mathsf{j}} \frac{\alpha^2|\beta \mathsf{j}|^2}{1+\alpha^2|\beta \mathsf{j}|^2} \beta \mathsf{k} \cdot \hat{\mathsf{v}}^{(0)}(\mathsf{j}, t-s) \pi_{\mathsf{k}} \hat{\mathsf{v}}^{(0)}(\mathsf{k}-\mathsf{j}, t-s) ds$$
$$+\frac{1}{2} \int_0^t e^{-\nu|\beta \mathsf{k}|^2 s} \sum_{\mathsf{j}} D^{(\alpha)}(\mathsf{j}, \mathsf{k}) \{\delta(\mathsf{j}, t-s) \hat{\mathsf{v}}^{(\alpha)}(\mathsf{k}-\mathsf{j}, t-s) + \hat{\mathsf{v}}^{(0)}(\mathsf{j}, t-s) \cdot \delta(\mathsf{k}-\mathsf{j}, t-s)$$
$$+\hat{\mathsf{v}}^{(0)}(\mathsf{j}, t-s) \hat{\mathsf{v}}^{(0)}(\mathsf{k}-\mathsf{j}, t-s)\} ds, \tag{5.19}$$

where

$$D^{(\alpha)}(\mathsf{j}, \mathsf{k}) = \pi_{\mathsf{k}}(\beta \mathsf{j}) \left[\frac{1}{1+\alpha^2|\beta \mathsf{j}|^2} - \frac{1}{1+\alpha^2|\beta(\mathsf{k}-\mathsf{j})|^2} \right].$$

Suppose that

$$|\hat{\mathsf{v}}^{(0)}(\mathsf{k})| \leq M h(\mathsf{k}), \quad \mathsf{k} \neq \mathbf{0}.$$

Then

$$|\hat{\mathsf{v}}^{(\alpha)}(\mathsf{k}, t)| \leq M h(\mathsf{k}), \quad \mathsf{k} \neq \mathbf{0}, t \geq 0.$$

Straightforward estimates yield the following integral inequality.

$$\begin{aligned} \Delta(t) \quad \leq \quad & \sup_{\mathsf{k}} \{ 2M [\sum_{\mathsf{j}} \frac{h(\mathsf{j})}{1+\alpha^2|\beta \mathsf{j}|^2}] \int_0^t |\beta \mathsf{k}| e^{-\nu|\beta \mathsf{k}|^2 s} \Delta(t-s) ds \\ & + M^2 \alpha^2 \sum_{\mathsf{j}} \frac{|\beta \mathsf{j}|^2 h(\mathsf{j}) h(\mathsf{k}-\mathsf{j})}{1+\alpha^2|\beta \mathsf{j}|^2} \int_0^t |\beta \mathsf{k}| e^{-\nu|\beta \mathsf{k}|^2 s} ds \\ & + \frac{1}{2} 2M \sum_{\mathsf{j}} \left| D^{(\alpha)}(\mathsf{j}, \mathsf{k}) \right| h(\mathsf{j}) \int_0^t e^{-\nu|\beta \mathsf{k}|^2 s} \Delta(t-s) ds \\ & + \frac{1}{2} M^2 \sum_{\mathsf{j}} \left| D^{(\alpha)}(\mathsf{j}, \mathsf{k}) \right| h(\mathsf{j}) h(\mathsf{k}-\mathsf{j}) \int_0^t e^{-\nu|\beta \mathsf{k}|^2 s} ds \}. \end{aligned} \tag{5.20}$$

It is convenient to label the four terms appearing in the supremum, alternatively according to their "Gronwall" roles as homogeneous or forcing terms, as H_1, F_1, H_2, F_2. Then

$$\Delta(t) \leq \sup_{\mathsf{k}} \{H_1 + F_1 + H_2 + F_2\}.$$

Observing that

$$|\pi_{\mathsf{k}}(\beta \mathsf{j})| = |\pi_{\mathsf{k}}(\beta(\mathsf{k}-\mathsf{j}))| \quad \mathsf{k}, \mathsf{j} \in \mathbf{Z}^3,$$

one has

$$\left|D^{(\alpha)}(\mathsf{j},\mathsf{k})\right| \leq \frac{|\pi_k(\beta\mathsf{j})|\,||\beta(\mathsf{k}-\mathsf{j})|^2-|\beta\mathsf{j}|^2|}{(1+\alpha^2|\beta\mathsf{j}|^2)(1+\alpha^2|\beta(\mathsf{k}-\mathsf{j})^2)}\alpha^2 \leq \frac{2\alpha^2|\pi_{\mathsf{k}}(\beta\mathsf{j})||\beta\mathsf{j}|^2}{(1+\alpha^2|\beta\mathsf{j}|^2)(1+\alpha^2|\beta(\mathsf{k}-\mathsf{j})^2)}, \quad (5.21)$$

$$\int_0^t |\beta\mathsf{k}|e^{-\nu|\beta\mathsf{k}|^2 s}ds \leq \frac{1}{\nu|\beta\mathsf{k}|} \leq \frac{1}{\beta\nu}, \quad (5.22)$$

$$\int_0^t e^{-\nu|\beta\mathsf{k}|^2 s}ds \leq \frac{1}{\nu|\beta\mathsf{k}|^2} \leq \frac{1}{\nu\beta^2}, \quad (5.23)$$

and

$$|\beta\mathsf{k}|e^{-\nu|\beta\mathsf{k}|^2 s} \leq \frac{1}{\sqrt{\nu s}}e^{-\frac{1}{2}\nu|\beta\mathsf{k}|^2 s} \leq \frac{e^{-\frac{1}{2}\nu\beta^2 s}}{\sqrt{\nu s}}. \quad (5.24)$$

Using (5.22), the first forcing term is bounded as

$$F_1 \leq \frac{\beta\alpha^2 M^2}{\nu}\sum_{\mathsf{j}}|\mathsf{j}|^2 h(\mathsf{j})h(\mathsf{k}-\mathsf{j}).$$

Using (5.21) and (5.23), one obtains

$$F_2 \leq \frac{\beta\alpha^2 M^2}{\nu}\sum_{\mathsf{j}}|\mathsf{j}|^3 h(\mathsf{j})h(\mathsf{k}-\mathsf{j}).$$

Let

$$m_0 = \sum_{\mathsf{j}}\frac{h(\mathsf{j})}{1+\alpha^2|\beta\mathsf{j}|^2}, \quad m_1 = \sum_{\mathsf{j}}|\mathsf{j}|h(\mathsf{j}),$$

and

$$m_\ell = \sum_{\mathsf{j}}|\mathsf{j}|^\ell\frac{h(\mathsf{j})h(\mathsf{k}-\mathsf{j})}{h*h(\mathsf{k})}, \quad l=2,3.$$

The forcing term contribution is bounded by

$$F_1+F_2 \leq \alpha^2\frac{M^2\beta}{\nu}(m_2+m_3).$$

One has using (5.24) and remembering $|\mathsf{k}|\geq 1$,

$$H_1 \leq 2Mm_0\int_0^t e^{-\gamma s}\frac{1}{\sqrt{\nu s}}\Delta(t-s)ds.$$

To bound the second homogeneous term it is enough to use the obvious bound

$$|D^{(\alpha)}(\mathsf{j},\mathsf{k})| \leq 2|\beta \mathsf{j}|. \tag{5.25}$$

Using this and (5.24), one has

$$H_2 \leq 2M\beta \sum_{\mathsf{j}} |\mathsf{j}| h(\mathsf{j}) \int_0^t e^{-\gamma s} \frac{1}{\sqrt{\nu s}} \Delta(t-s) ds.$$

Combining these estimates yields

$$\Delta(t) \leq M^* \left(\alpha^2 + \int_0^t \frac{e^{-\gamma(t-s)}}{\sqrt{\nu(t-s)}} \Delta(s) ds \right),$$

where

$$M^* = \max\{ \frac{M^2 \beta}{\nu} (m_2 + m_3), 2Mm_1\beta, 2Mm_0 \}.$$

It is convenient to consider

$$\tilde{\Delta}(t) = e^{\gamma t} \Delta(t), \quad t \geq 0.$$

Then

$$\tilde{\Delta}(t) \leq M^* \left(\alpha^2 e^{\gamma t} + \int_0^t \frac{\tilde{\Delta}(s) ds}{\sqrt{\nu(t-s)}} \right), \quad t \geq 0. \tag{5.26}$$

The special "square-root" case of the Abel transform term appearing in this inequality is "invertible" by multiplying (5.26) by $\frac{1}{\sqrt{u-t}}$ and integrating over $(0, u)$, say, to obtain

$$\begin{aligned}
\int_0^u \frac{\tilde{\Delta}(t)}{\sqrt{u-t}} dt &\leq \alpha^2 M^* \int_0^u \frac{e^{\gamma t}}{\sqrt{u-t}} dt + \frac{M^*}{\sqrt{\nu}} \int_0^u \int_0^t \frac{\tilde{\Delta}(s)}{\sqrt{(u-t)(t-s)}} ds dt \\
&= \alpha^2 M^* \int_0^u \frac{e^{\gamma t}}{\sqrt{u-t}} dt + \frac{M^*}{\sqrt{\nu}} \int_0^u \int_s^u \frac{\tilde{\Delta}(s)}{\sqrt{(u-t)(t-s)}} dt ds \\
&= \alpha^2 M^* e^{\gamma u} \int_0^u \frac{e^{-\gamma t}}{\sqrt{t}} dt + \frac{M^*}{\sqrt{\nu}} \int_0^u \tilde{\Delta}(s) \int_0^1 \frac{1}{\sqrt{\tau(1-\tau)}} d\tau ds \\
&= \alpha^2 c(u) + \frac{M^* \pi}{\sqrt{\nu}} \int_0^u \tilde{\Delta}(s) ds,
\end{aligned} \tag{5.27}$$

where

$$c(u) = M^* e^{\gamma u} \int_0^u \frac{e^{-\gamma t}}{\sqrt{t}} dt.$$

Substituting this bound into (5.26) yields

$$\tilde{\Delta}(t) \leq \alpha^2 M^* \left(e^{\gamma t} + \frac{1}{\sqrt{\nu}} c(t) \right) + \frac{M^{*2}\pi}{\nu} \int_0^t \tilde{\Delta}(s) ds. \tag{5.28}$$

Thus, viewed as a differential inequality for $I(t) = \int_0^t \tilde{\Delta}(s)ds$, one may introduce the appropriate integrating factor (or apply a trivial version of Gronwall's inequality), to obtain

$$\int_0^t e^{\gamma s} \Delta(s) ds = \int_0^t \tilde{\Delta}(s) ds \leq \alpha^2 C(t), \tag{5.29}$$

where

$$C(t) = M^* e^{\frac{M^{*2}\pi}{\nu} t} \int_0^t (e^{\gamma s} + \frac{1}{\sqrt{\nu}} c(s)) ds.$$

$\square$

We are now in a position to prove the main theorem in which we obtain convergence at a rate of order α in a mixed L^1-norm in time of the spatial (energy) L^2-norm.

Theorem 5.1 *Let $h \in l^1(\mathbf{Z}^3)$ be a standardized majorizing kernel satisfying the following further moment conditions:*

$$\sum_{\mathsf{j}} |\mathsf{j}| h(\mathsf{j}) < \infty, \quad \sum_{\mathsf{j}} |\mathsf{j}|^l h(\mathsf{j}) h(\mathsf{k} - \mathsf{j}) < \infty, \quad \mathsf{k} \in \mathbf{Z}^3, l = 2, 3.$$

Take $q_0 = q_1 = q_2 = \frac{1}{6}$ and $q_3 = \frac{1}{2}$. Let $\gamma = \frac{\nu\beta^2}{2}$. Let $R = \frac{\nu\beta}{6}$ and suppose $\mathsf{v}_0 \in B_R$, $\Delta^{-1} g \in B_{\frac{\nu R}{2}}$. Then there is a positive constant $A(T)$, not depending on α, such that

$$\int_0^T ||\mathsf{v}^{(\alpha)}(\cdot, t) - \mathsf{v}^{(0)}(\cdot, t)||_{L^2(T^3)} dt \leq A(T)\alpha,$$

where for each $\alpha \geq 0$, $v^{(\alpha)}$ denotes the unique global solution to $\mathsf{LANS}\alpha$.

Proof: The global existence and uniqueness follow from the representation Theorem 3.2. Observe that for $\mathsf{v}_0 \in B_R$ one has

$$|\hat{\mathsf{v}}^{(\alpha)}(\mathsf{k}, t)| \leq R h(\mathsf{k}), \quad \alpha \geq 0.$$

Also, using the Plancherel identity, Cauchy-Schwarz inequality and this bound, one has

$$\begin{aligned}
& \int_0^T ||\mathsf{v}^{(\alpha)}(\cdot,s) - \mathsf{v}^{(0)}(\cdot,s)||_{L^2(T^3)} ds \\
= & \int_0^T e^{-\frac{\gamma s}{2}} e^{\frac{\gamma s}{2}} ||\hat{\mathsf{v}}^{(\alpha)}(\cdot,s) - \hat{\mathsf{v}}^{(0)}(\cdot,s)||_{l^2(\mathbf{Z}^3)} ds \\
\leq & \left(\int_0^T e^{-\gamma s}\right)^{\frac{1}{2}} (\int_0^T e^{\gamma s} \sup_{\mathsf{k}} |\hat{\mathsf{v}}^{(\alpha)}(\mathsf{k},s) - \hat{\mathsf{v}}^{(0)}(\mathsf{k},s)| \sum_{\mathsf{k}} \left|\hat{\mathsf{v}}^{(\alpha)}(\mathsf{k},s) - \hat{\mathsf{v}}^{(0)}(\mathsf{k},s)\right|) ds)^{\frac{1}{2}} \\
\leq & \sqrt{\frac{1-e^{-\gamma T}}{\gamma}} \left(2R \int_0^T e^{\gamma s} \Delta(s) ds\right)^{\frac{1}{2}} \\
\leq & \alpha \sqrt{2RC(T)\frac{1-e^{-\gamma T}}{\gamma}} = \alpha A(T),
\end{aligned}$$

where the constant $C(T) > 0$ was determined in the Proposition 5.1.

□

The small ball condition required here is both for well-posedness of the rate of convergence problem, i.e., existence and uniqueness, as well as the actual rate itself. It will be interesting to see if comparable rates can be determined in other function spaces based on other methods, e.g., energy estimates.

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