Care to Compare: Eliciting Mathematics Discourse in a Professional Development Geometry Course for K-12 Teachers

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Abstract

This article describes how the idea of comparing Euclidean geometry with two non-Euclidean geometries (taxicab and spherical) provides participants with engaging mathematical tasks in a professional development geometry course for K-12 teachers. We illustrate with three different examples how the combination of rich mathematical activities centered on comparison, well-orchestrated group work, and skilled facilitators makes for productive classroom discourse and provides the teacher participants with a model to be emulated in their own classrooms. This model demonstrates for the teacher participants the process of generating and supporting student learning through mathematics discourse in the classroom.

A FRAMEWORK FOR MATHEMATICS DISCOURSE

The National Council of Teachers of Mathematics (NCTM, 2000) considers communication a fundamental part of doing and of learning mathematics. Rich mathematics discourse is at the center of constructing and connecting knowledge in mathematics. In the November 2007 issue of *The Mathematics Teacher*, which is specifically dedicated to mathematics discourse, Himmelberger and Schwartz (2007) and Staples and Colonis (2007) describe three indispensable components behind classroom discourse that promote sense-making and growth in mathematics: an engaging task or activity; collaborative group effort; and a proficient facilitator who knows when to question, when to listen, and when to summarize or clarify ideas. The role of the mathematics instructor is then not only to select challenging and conceptually rich tasks for her class, but also to decide how to organize student work and student reporting in a way that centers on sense-making and on justification of the main mathematical ideas (Krussel, Springer & Edwards, 2004; Groves & Doig, 2004).

Nathan, Elliam, and Kim (2006) describe how the interplay of convergence and divergence of ideas in the debriefing process generates the type of discourse that helps build students' understanding. Comparison is a natural way of creating this interplay of convergence and divergence of ideas. Rittle-Johnson and Jon R. Star (2007) and Star (2008) argue that comparison is the key component in allowing students to become flexible users of mathematical ideas and techniques and in making connections among concepts. Thus, we believe that grounding instruction in comparison together with the three indispensable components described above can provide new depth to mathematical discourse.

Recent research confirms the expectation that teachers' learning experiences in professional development programs directly impact their own teaching (Heck, Banilower, Weiss & Rosenberg; 2008). Thus, the design and use of mathematical tasks centered on comparison can also supply teacher participants with an effective model to follow in their own teaching practice as a means of eliciting mathematics discourse.

THE OREGON MATHEMATICS LEADERSHIP INSTITUTE (OMLI) AND MATHEMATICS DISCOURSE

The Oregon Mathematics Leadership Institute (OMLI) is an NSF-funded Mathematics/Science Partnership that aims at increasing mathematics achievement of K-12 students by providing professional development for in-service teachers from ten participating Oregon school districts. Participants have taken part in three 3-week intensive summer institutes (2005-2007), covering mathematics content as well as leadership skills. Six content courses have been developed, covering Number and Operations, Geometry, Abstract Algebra, Probability and Statistics, Measurement and Change, and Discrete Mathematics; each was offered in 28-hour sessions and designed and delivered by an entire team of mathematics teacher educators.

One of the main research and pedagogical foci of OMLI has been mathematics discourse, and especially how the quantity and quality of discourse affect student achievement. As a result, the OMLI courses were designed to focus heavily on mathematics discourse, and modeling the use of that discourse, as a method not only for improving the teachers' own conceptual understanding of a variety of mathematical topics, but also as a vehicle for expanding their use of discourse in their own classroom. To accomplish this, all OMLI faculty members were given the opportunity to receive training in discourse facilitation and in collaborative learning as well as in best practices in teaching.

COMPARING DIFFERENT GEOMETRIES COURSE AT OMLI

The geometry course team consisted of five mathematics educators with diverse backgrounds: a research mathematician, a master teacher, a mathematics educator at an undergraduate institution and two mathematics instructors, one with a Ph.D. in mathematics education, the other with an interest in obtaining that degree.

An interest from the participating OMLI school districts in non-Euclidean geometry led to selecting taxicab and spherical geometries as the two major topics in this course. The two main reasons for choosing these two geometries were that 1) both of these topics allowed for hands-on explorations and applications, and 2) both geometries were unfamiliar to virtually all teacher participants. Thus the selection of these topics allowed all K-12 participants to begin the course on an equal footing.

One compelling approach to teaching non-Euclidean geometry is to use comparison with Euclidean geometry. This became the fundamental idea behind the course—thus the capstone of the course consisted of creating extensive comparison charts, one for taxicab and Euclidean geometry, and one for spherical and Euclidean geometry. These charts allowed teachers to synthesize their newly acquired knowledge of spherical and taxicab geometries, and to connect it to ideas in the school geometry curriculum.

The *Comparing Different Geometries* course consisted of two separate units—one corresponding to spherical geometry and one corresponding to taxicab geometry, and then a project unit. The learning goals and objectives for the course were for participants to improve their content knowledge of Euclidean geometry, especially as it pertained to undefined terms, axioms, definitions, and justification of geometric results; to focus on effective mathematical reasoning and mathematics discourse; and to make connections between Euclidean and non-Euclidean geometries by examining similarities and differences between them.

In the first unit, spherical geometry, participants investigated the concepts of lines, parallel and perpendicular lines, the set of points equidistant from two given points, common perpendiculars, triangles, squares, and circles.

In the second unit, taxicab geometry, participants explored distance, midpoints, the set of points equidistant from two given points, squares, circles, triangles, and congruence of line segments and triangles.

During the project unit, participants completed independent out-of-class projects. These group projects were investigations of unexplored topics from either spherical or taxicab geometry. Participants made posters of their work and presented the most important parts of their discoveries during the last two days of the course.

Manipulatives were used throughout the geometry course, including colored pencils, rulers, compasses, protractors and grid paper, Etch-a-Sketch toys for demonstrating taxicab distance and Lénárt Spheres with spherical tools for visualizing geometric shapes in spherical geometry.

Each participant was assigned to a group of three or four teachers from different grade levels. Groups were reassigned several times during the course. In conjunction with the groups, a number of norms and protocols were used that were intended to ensure everyone's participation and to encourage risk-taking. These protocols included private time to think, followed by specific instructions about sharing ideas first within a group, and then with the class as a whole.

The three classroom episodes described below were chosen as illustrations of productive mathematics discourse. All of these cases included the generation of new (to the participants) mathematical ideas, making connections among concepts, using precise definitions, and justification of statements. Each of these scenarios is meant to highlight one of the three specific components in the discourse framework described above (engaging task, effective cooperative learning, and skilled facilitation). We believe all three of these episodes (as well as all the remaining ones from the course) were enriched by the comparison nature of the tasks. The comparison technique allowed the teacher participants to see clearly the interconnections among geometric ideas, and the uniquely Euclidean properties of school geometry. Due to the fact that the instructors did not obtain the explicit permission of the participants to use the exact dialogue, the conversations were paraphrased in order to preserve the anonymity of the teachers.

FUN WITH "SQUIRCLES": FOCUS ON AN ENGAGING MATHEMATICS ACTIVITY

During the unit on taxicab geometry, teacher participants were presented with a task that required them to solve a real-world problem using taxicab geometry. Using an urban context as a starting place led to a rich discussion of the definition of circle, congruent line segments, squares, distance, and area of a circle, allowing participants an opportunity to deepen their understanding of these concepts in both the Euclidean and the taxicab settings, by comparing and contrasting the definitions in the respective geometries. The final part of the activity was for participants to investigate their own questions about circles in taxicab geometry—participants were especially engaged in the task of responding to each other.

The task consisted of the following problems:

a. *Because Clark works early in the morning, he wants to live within three blocks of his workplace. Where could he and Lois live?*
b. *Lois does not want to live more than ten blocks from her workplace. For this scenario, where could they live? (*See Figure 1 in Appendix A.)
c. *How would you describe the boundaries of the regions you drew in problems #1 and #2?*

First, the facilitator gave the teacher participants several minutes to think privately about the three questions in the task, after which they shared, in a small-group go-around, their beginning thoughts, ideas, strategies, and/or answers. The small groups then tried to reach a consensus about each of the three questions in the task. The realistic setting of the task captured the attention of the teachers right away, and the unexpected shape of the boundaries they found seemed to intrigue them. A lot of the teacher participants shared that they had no idea that the resulting shape will be built entirely of line segments.

All of the groups were able to agree about the answers to all three parts of the task fairly quickly. Then, the facilitator instructed them to work together to try to write a definition for the set of points that make up the boundary of the regions they found in the task. Because the class had worked to write a precise definition of a circle in Euclidean geometry in a previous unit, most of the groups readily came up with the needed definition as the set of points equidistant from one given point (called the center of the circle).

For a whole-class debrief of this part of the task, the instructor asked a selected participant from one group that had been successful to share their group's definition for a set of points, in taxicab geometry, that looked like the shape shown in Appendix A, Figure 2 in taxicab geometry (the instructor recorded these definitions on a poster).

The first participant offered this definition from the group: The circle with center *O* and radius *OP* is the set of all points *Q*, such that *OQ* is congruent to *OP*. This group had engaged in the task of writing a definition beyond what was asked and begun following their "I wonder…" questions. The teacher participant shared that their group was wondering whether their circle definition would be satisfied by a square oriented as shown in Figure 3 of Appendix A:

The instructor asked, "What do others think? Is their definition satisfied by this shape?"

Another participant demonstrated that this does not fit the definition of circle because in taxicab geometry OQ_1 is not the same length as OQ_2 (See Appendix A, Figure 4).

There was still some class confusion about the notion of congruence, in the taxicab sense—this was because the congruent radii of the taxicab circle do not "look" congruent, in the Euclidean sense. The idea that congruence is directly related to the way distances are measured continued to cause disequilibrium in some of the teacher participants for the duration of this activity. But the concrete set-up helped in creating a mental image of the notion of congruence in taxicab geometry.

This discussion was wrapped up when a several participants offered their observations: (1) Euclidean circles are not circles in taxicab geometry, and (2) Not all squares in taxicab geometry are taxicab circles—only the ones whose diagonals line up with the grid.

The class facilitator then asked, “What are some questions that you have about circles? What are you wondering about right now?” Participants took a couple of minutes to write at least two different questions. The instructor noted that, as anticipated, several people had written down the question, “What about *C/d,* the ratio of the circumference to the diameter of a taxi-circle?” and asked groups to start by trying to answer this question. Participants were instructed to try to answer some of the other questions they wrote down only if there was still time after working on the *C/d* question. The facilitator made this instructional move since calculating this ratio allowed for direct comparison with the Euclidean geometry ratio π, a topic likely to lead to productive discourse.

To wrap up the discussion of the ratio of circumference to diameter of any taxicab circle, the facilitator first selected two participants to share their results. These participants were chosen based on the quality of their work, the first being concrete and somewhat incomplete while the second was more abstract and precisely justified. The first participant showed one specific example of how to calculate the ratio and then conjectured, without proof, that C/d = 4 in general. The second participant gave a general proof by explaining that if the diameter of a taxicab circle is $2r$, then its circumference would be $8r$, and then $8r/2r = 4$. The facilitator at this point proceeded to let the participants decide in their groups on one more question that they would explore.

The discussion that followed in each of the small groups was especially engaging because participants were pursuing their own questions, following whatever path they and their group mates agreed to. After the initial shock that the analogue of Euclidean π is in fact equal to 4 in taxicab geometry wore off, most groups chose to examine the formulas for the areas of a circle and a square, again comparing these to the familiar formulas for Euclidean circles and squares.

The class facilitator again selected a participant whose group’s question focused on comparison: What is the formula for the area of a taxicab geometry circle, a “squircle,” and is it the same as for a Euclidean circle? In response, the group first conjectured that the area of a taxicab circle would be $A = 4r^2$, in complete analogy to the formula for the area of a Euclidean circle. Several different participants disagreed quite vocally and two participants proceeded to offer geometric proofs that the formula for the area of a taxicab circle was in fact: $A = 2r^2$. This was another surprise for the participants and it generated a lot of high-energy discourse not only during class, but also, as we later learned, beyond our classroom, during the subsequent office hour periods.

This classroom vignette presents a concrete illustration of how an interesting applied mathematics task can be combined with effective group protocols and judicious facilitator moves to generate meaningful mathematics discourse centered on precise mathematical definitions and their implications. Simply finding and preparing an interesting mathematics activity was not sufficient for ensuring student learning. The classroom needed to be organized in a way that made each student responsible and responsive, and cooperative learning protocols were one way to achieve this. Every participant was given plenty of time to think by herself, and then there was sharing in the small groups, before sharing with the entire class, the groups were given freedom to pursue their own mathematical questions and the sense of ownership made for high quality work. At each point during the task, everyone was engaged either in doing mathematics, or listening and responding to his classmates.

Furthermore, the class facilitator had to know how to sequence student responses and when to pose a new question. Due to the freedom of participants to explore their own ideas, this activity required a lot of immediate reaction to student ideas and on the fly decisions of which ones to keep and develop further. Moreover, it must be noted that it was the focus (original definition of a taxicab circle) and the re-focus on comparison of circles and their properties (ratio of circumference to diameter, and finding the formula for the area of a taxicab circle) that allowed the participants' discourse to go further and it prompted for justification of conjectures (such as the area of a taxicab circle which was generated in complete analogy to the Euclidean case).

JIGSAW GROUPS: FOCUS ON EFFECTIVE GROUP WORK

During the final activity on spherical geometry, participants revisited core mathematical ideas from previous class tasks. They completed a chart making explicit the similarities and differences between spherical geometry and Euclidean geometry. The purpose of making these comparisons was to further deepen participants' content knowledge of Euclidean geometry. One way to promote meaningful discourse during this activity was to use a jigsaw protocol. This protocol lent itself nicely to this activity because there were many subtasks to complete, each one corresponding roughly to one geometric object or property to be compared (and thus completing one row in the comparison chart).

The activity began with the facilitator randomly assigning one of the following terms to each small group: lines and parallel lines, perpendicular lines, distance and units, triangles and angle sums, circles and π, and squares. Appendix B contains the task card that the participants received.

After thinking privately first, group members discussed with each other and came to a consensus about any differences and similarities between Euclidean and spherical geometries for their given term. This information was recorded on the appropriate row of every participant's comparison chart. Participants, then, went through two rounds of rotation to share what their home group came up with, as well as to learn what the other groups did. Each participant took notes on the information he/she gathered. Note that there were 6 topics in the comparison chart (lines and parallel lines, distance, triangles, circles and π, perpendicular lines, and squares), one per home group. After two rotations, participants returned to their home group and shared what they had learned from the experts in the other groups. They then filled in the remaining rows of the comparison chart.

Due to its structure, this entire activity was discourse driven. It gave each participant a chance to become an expert on a single concept and thus solidify one major geometric topic in her mind. Through discussion in their home groups, participants had to come to a consensus about any similarities and differences they found. Not only did the participants have the opportunity to discuss their ideas, but they also had to listen carefully and then articulate this information to new group members. The use of questioning and justification were emphasized throughout the task.

Furthermore, this mathematical task provided a rich cooperative learning structure. Participants were interdependent and engaged the entire time both in their home and in their subsequent groups by not only becoming experts on one topic but also bringing back new information from the rest of the groups. The comparison theme of this task provided the conceptual thread which made the generated discourse effective.

SQUARES ON THE SPHERE: FOCUS ON SKILLED FACILITATION

As part of the initial spherical geometry unit, teachers were given the following open-ended task card:

> *Investigate squares on the sphere. Justify any conclusions you reach during your investigations.*

The rationale for using this mathematics activity came from the following considerations. Squares are objects familiar to all teacher participants and they provide an example of how the Euclidean definition of a parallelogram with congruent sides and 90° angles needs to be modified to fit with the non-existence of parallel lines on the sphere. The corresponding quadrilaterals on the sphere will have congruent sides and congruent angles, but the angles are necessarily larger than 90°. In the process of attempting to construct squares in spherical geometry, participants focused on the importance of precise and concise definitions in geometry.

This activity was also a suitable choice for group work because it was non-prescriptive and it allowed for a variety of approaches and subtasks to be used in completing it. In our experience, the more open-ended an activity is, the higher the probability of it generating productive discourse during the class discussion.

For this task, teachers sat in groups of three to four members with both early elementary school and late elementary or middle school teachers present in each group. High school teachers were working on a separate investigation at the time, but the entire class participated in the debriefing session.

Participants thought about this task privately for about 10-15 minutes. Then teachers worked with their group on completing the task and making a poster of their results. The posters were displayed on the wall of the classroom or nearby hallway, and groups did a silent walk-through, with one group at a time observing each poster. One person was chosen per group to answer questions about her group's poster, and the rest of the group members read the other posters, got answers about any points that need clarification, and left individual feedback about each poster by writing comments on sticky notes. Then teacher participants went back to their groups and modified their own posters based on the feedback provided by their classmates.

While teachers were busy completing this activity, all the instructors not only observed, asked probing questions (e.g. "Is there another way to think about this type of quadrilateral?", "Do you think this is always true on the sphere?" "Do you have a justification of this statement?") and took detailed notes, but they also reported to the class facilitator for this activity who made a decision about how to organize the debriefing session with the entire class.

The following class began with a careful and deliberate sequencing of the poster presentations based on the facilitator's sense of the flow of ideas that would lead to the most complete understanding of the analogue of squares on the sphere. It was also very helpful that there was enough time for the facilitator to share her ideas with the other instructors of the course and to think some more about her moves. A representative from each selected group was chosen randomly.

The first participant chosen came from a group that got stuck trying to adapt the Euclidean definition of a square as a special parallelogram with congruent sides and 90 degree angles to the sphere.

The discussion proceeded as follows. (Actual dialogue is paraphrased in order to preserve the anonymity of teacher participants.)

> Facilitator: Could you explain how your group approached the square task?
> Student 1: Squares have equal sides and 90 degree angles. But there are no parallel lines on the sphere, so we thought for a long time that there are no squares on the sphere. Then we realized that we were on the wrong track. So the sides of a quadrilateral on the sphere will not be parallel but they might still be equal… and maybe the angles might be equal too.
> Facilitator: What did your group do then as a result of this realization?
> Student 1: We tried to make a quadrilateral with 4 equal sides and 4 equal angles. We did not know how to do this at first. Suddenly a person in our group realized that perhaps we can start at the North Pole and see if we can find 4 points that are the same distance from this pole. So we wanted the pole to be the center of our spherical square. So we drew a circle centered at the North Pole and by measuring with the spherical compass we found 4 points on the circle that split the circle up into 4 equal parts. Then we connected the 4 points with straight line segments to get our square.
> Facilitator: Does anyone have any comments or questions?
> Participant: Did you get all your squares in this fashion? We did a similar thing too.
> Student 1: Yes, we did.
> Facilitator: Now I would like to ask the representative from this (she points) group to come and explain what their group did.
> Student 2: We wanted to make a square on the sphere. We started with two perpendicular great circles intersecting at say the North Pole, and then just like the previous group we constructed a series of circles with the North Pole as their center. These circles touched the two great circles in four points that became the vertices of our squares. We constructed a whole bunch of these spherical squares and then measured their angles. We were surprised to find out the angles were not the same—they got bigger, the bigger the square was.

> Facilitator: Why did you begin your process with two perpendicular circles?
> Student 2: These great circles end up forming the diagonals of the square. I think maybe diagonals of spherical squares have to be perpendicular.
> Facilitator: Does anyone see how we can use this?
> Participant: Can we define a spherical square as a quadrilateral with congruent perpendicular diagonals?
> Facilitator: I encourage everyone to go home and to see if this new proposed definition that a spherical square is a quadrilateral with two equal perpendicular diagonals is equivalent to the definition of a quadrilateral with congruent sides and congruent angles. Now I am inviting the representative of this other group (she points) to come up.
> Student 3: We did the same thing as the previous group, but we actually made a chart with the angles of the spherical squares. We made a conjecture that the interior angles are always larger than 90 degrees and smaller than 270 degrees. And you can see here that the larger the area of the square, the larger its angles. This seems to work the same way as for spherical triangles and it is so weird thinking about it together with the usual squares on the plane.

This classroom vignette is just one of many examples of productive mathematics discourse that happened in the course of teaching *Comparing Different Geometries*. We chose to highlight this particular discussion because, due to circumstances that allowed time to think about the sequencing of the participants' presentations, the debriefing part of the discourse went almost flawlessly. The class discussion also illustrated the effectiveness of sequencing in the presentation of mathematical ideas: the class facilitator had a concrete plan of where she wanted to take the discussion and she chose participants to share their ideas in ways that would reinforce the importance of precise mathematical definitions and the extension of mathematical ideas from one paradigm to another.

Even though at first glance it appears that this mathematical task was not centered on comparison, teacher participants began their explorations based on their experiences with Euclidean squares. They attempted to extract the geometric properties that made a square into a square and to apply these properties to the spherical geometry case. Invariably, some of the participants realized that they needed to abandon some of these properties and assumptions in order to fit the situation on the sphere.

Also, the square task was open enough to allow for multiple approaches and ways to succeed, and thus it was truly suitable for independent group investigation. The initial discourse in the small groups or in front of the posters, together with the written poster feedback created opportunities for connecting mathematical ideas and then revisiting them during the revision of the posters. When the final debriefing session was underway, the teacher participants were ready to understand what their classmates did and to clarify in their minds how it related to their own work and to the definition of the spherical square. But it is obvious that comparison underlay this mathematical task and made it possible for the discourse to expand both in magnitude and depth.

CONCLUSION

We believe that teachers who take mathematics content courses not only improve their knowledge of the discipline but they also tend to absorb the pedagogical tools of their instructors.

The three examples presented in this paper are offered to provide a glimpse of how the OMLI *Comparing Geometry* course was a hands-on discovery-driven course. Mathematics discourse was at the heart of student learning in this course.

We also strongly believe that the comparison method is a very effective tool in teaching non-Euclidean geometry for gains on conceptual understanding. All the participants had some knowledge of Euclidean geometry. Before making comparisons among the concepts of the new geometries, they needed to have a precise understanding of the concepts of Euclidean geometry including the undefined terms, axioms and definitions. Each time a new concept was explored in one of the new geometries, it was important then for them to revisit their knowledge of Euclidean geometry and to have a clear grasp of the same concept in Euclidean geometry, thus, deepening their knowledge of Euclidean geometry while at the same time discovering those same concepts in a new geometry.

But above all, the described classroom examples unraveled in depth mathematics discourse in the experience of all five geometry team members in terms of teacher participants' observations, questioning, generalizations and justification. Throughout the course, participants would make a conjecture based on their knowledge of Euclidean geometry. They would then explore their conjectures and possibly make new conjectures or make generalizations based on their explorations. Finally, they would provide a justification or a proof of their conjecture. This is illustrated particularly in the first activity discussed above. Through exploration, participants made a conjecture that C/d in taxicab geometry is 4. A participant then provided a proof of this fact. In this same activity, the participants first conjectured that the area of a taxicab circle would be $A = 4r^2$ based on the fact that $A = \pi r^2$ in Euclidean geometry. Then through exploration, they conjectured that the area was in fact $A = 2r^2$. It was then noted that two participants offered a geometric justification for the area of a taxicab circle.

These classroom episodes confirmed our belief that the comparison approach allows the discourse in the classroom to deepen, especially when accompanied by rich mathematical tasks, effective participant collaboration, and skilled facilitation. One key to the success of using the comparison approach is that the participants all came with pre-conceived notions about concepts in Euclidean geometry. As they studied other geometries, it was natural for the participants to try to use previous knowledge to investigate new ideas. So the comparison approach was a natural choice to encourage discovery of new ideas since there was a foundation on which to build. Since taxicab and spherical geometries were virtually unfamiliar to all the participants, the opportunities for discourse were greater than if some of the participants had prior knowledge of these geometries. In addition, each participant had varying backgrounds in Euclidean geometry and varying teaching experiences. This led to the opportunity for exciting discourse among the group members. So a foundation on which to build, new geometries, and varying backgrounds of participants all contributed to the success of using the comparison method to encourage discourse.

Throughout the course, the instructors witnessed engaging discourse among the participants. They had three summers in which to make changes in the tasks. After the first summer, it became apparent that open-ended tasks encouraged more discourse and discovery than those tasks which were scaffolded. The geometry instructors also learned to pose questions to the participants when they asked a question rather than immediately answering their questions. This led the participants to have rich discourse among their group members. Thirdly, the geometry instructors learned how to sequence participant presentations in a way that focused on tying together all the different approaches and ideas discovered by the participants. These presentations provided more opportunity for the participants to question and discuss beyond their group discussions. In the end, all of the geometry instructors learned as much about discourse as the teacher participants.

Appendix A

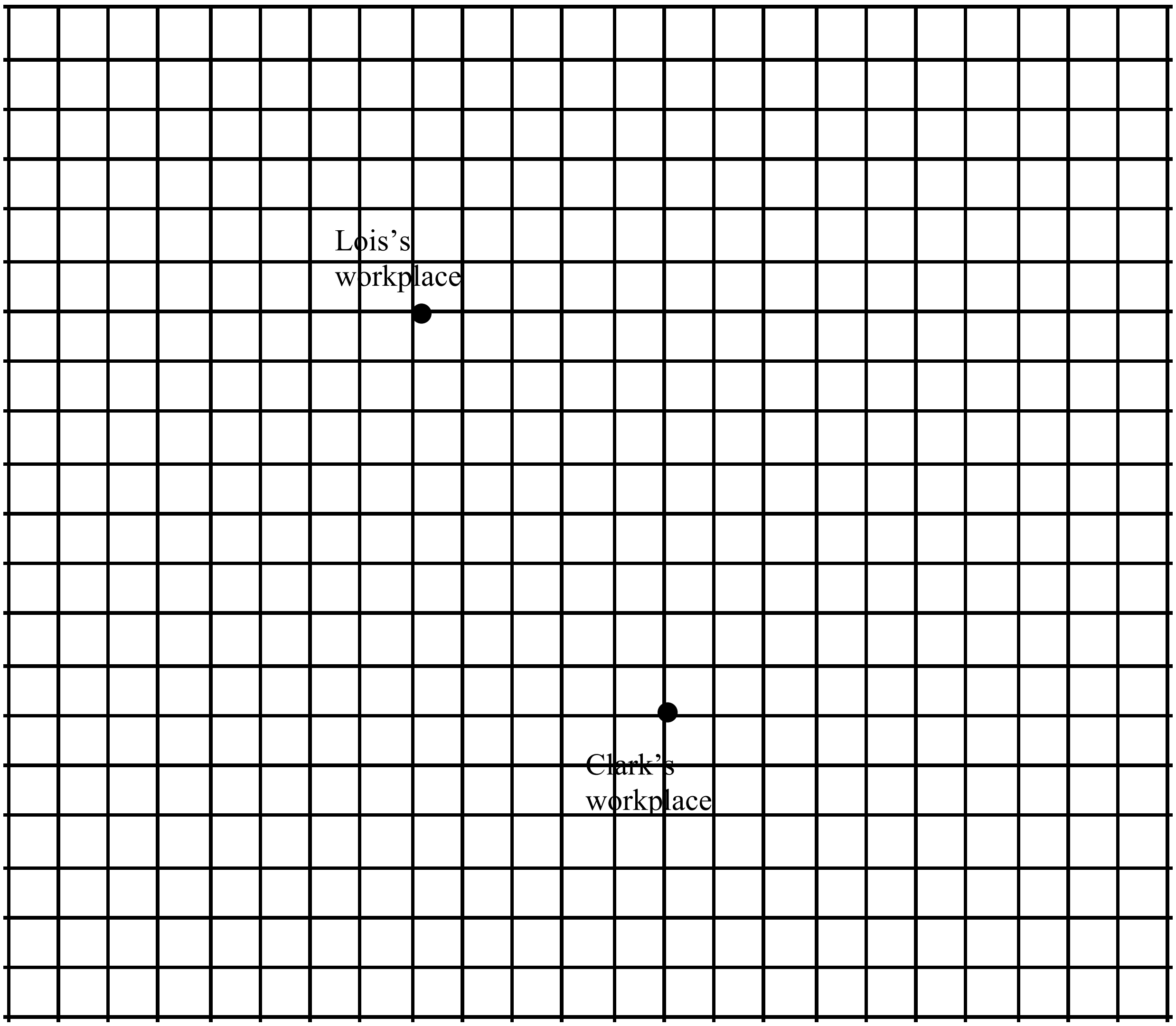

Figure 1

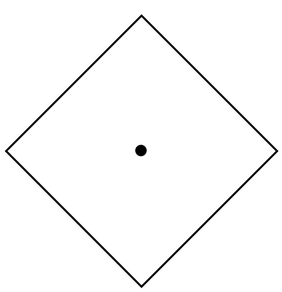

Figure 2

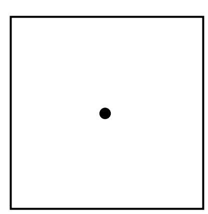

Figure 3

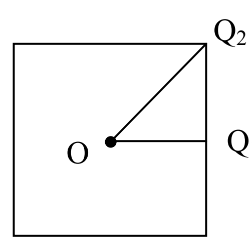

Figure 4

Appendix B

Jigsaw for Comparison Charts

1. *In your home group:*

2. *Private think time: Try to write down any similarities and differences between Euclidean and spherical geometries for your group's assigned term on your own–don't discuss with your group mates until everyone is ready.*

3. *Go around: Share your ideas in your home group, all ideas one person at a time.*

4. *Discuss: Come to a consensus about similarities and differences (try to include everyone's ideas.) Make sure everyone in your group understands the ideas discussed. You will be the expert group on your term.*

5. *Move to your new group, based on the rotation chart. Go around: Each person shares what their original group came up with for their assigned term. Listen carefully and ask any clarifying questions you may have. Record the information you learn in the appropriate row of your comparison chart*

6. *Return to your home group. Go around and share what you have learned from each of the experts in the other groups. Be sure to ask any needed clarifying questions. Fill in any of the remaining rows of the comparison chart.*

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