The Harmonic Oscillator
Ket & Wave Function notation

Read McIntyre 9.4

PH451/551

Reading Quiz

① Write in ket notation: $\int_{-\infty}^{\infty} \varphi_m^*(x)\varphi_n(x)\,dx = \delta_{mn}$

② Write in wave function notation: $c_n = \langle n|\psi\rangle$

③ Write the "closure" relation in ket notation (another way of writing the identity operator in terms of eigenkets)

Recap

① Ladder operators:

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

② Ground state wave function

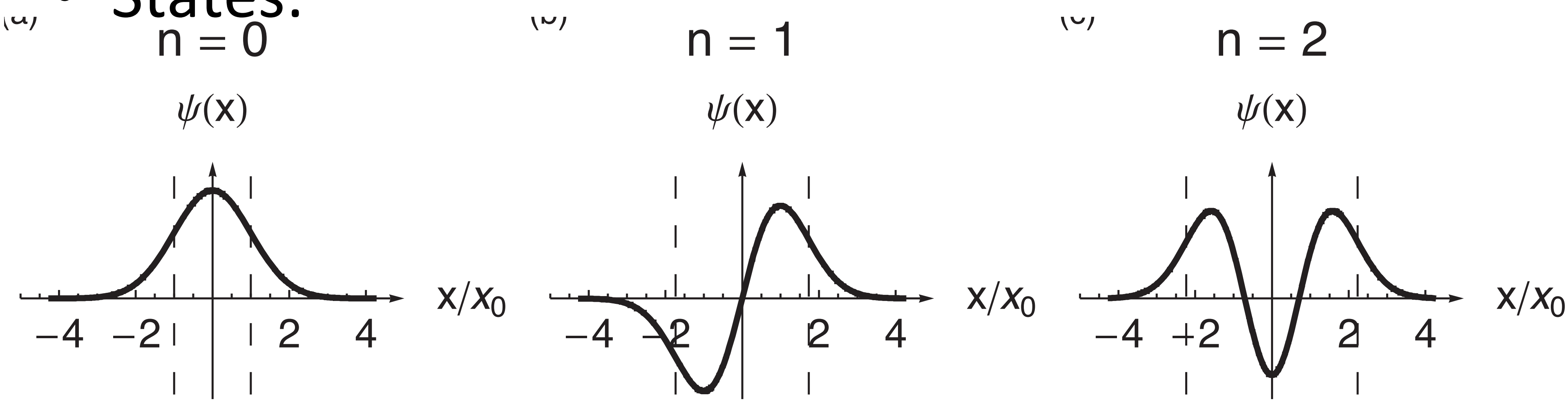
$$\sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{\hbar}{m\omega}\frac{d}{dx}\right)\overbrace{\left(\frac{m\omega}{2\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar}}^{\varphi_0(x)}=0$$

③ Generate other states

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n|0\rangle; \quad \varphi_n(x) = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n\varphi_0(x)$$

Recap

- States:



$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

$$\varphi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{m\omega}{2\hbar}}\, 2x e^{-m\omega x^2/2\hbar}$$

Small white board: Dimensions of prefactor?

Hermite Polynomials

- Dimensionless length: $\xi \equiv \left(\frac{m\omega}{\hbar}\right)^{1/2} x$

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} (1) e^{-\xi^2/2}$$

$$\varphi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2}} (2\xi) e^{-\xi^2/2}$$

$$\varphi_2(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^2 \times 2}} \left(4\xi^2 - 2\right) e^{-\xi^2/2}$$

$$\varphi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n \times n!}} H_n(\xi) e^{-\xi^2/2}$$

Classical limit

- Probability density:

$$P(x) = \psi^*(x)\psi(x)$$

- Go to PhET and look at functions

Kets & wave functions

- Wave function: $\varphi_n(x) \doteq |n\rangle$ $\qquad \varphi_n(x) = \langle x|n\rangle$
- General function:

$$|\psi\rangle = ?\left[|0\rangle + i|1\rangle + e^{i\delta}|3\rangle - |4\rangle\right]$$

Normalize
Probability to measure energy $\frac{5}{2}\hbar\omega$?
What is c_1? c_2?
What is expectation value of position, momentum, energy?

All of the above in wave function notation

Kets & wave functions

- Ket: $\left|\psi\right\rangle = ?\left[\left|0\right\rangle + i\left|1\right\rangle + e^{i\delta}\left|3\right\rangle - \left|4\right\rangle\right]$

$$c_n = \left\langle n \middle| \psi \right\rangle = \left\langle n \right|\left[\frac{\left|0\right\rangle + i\left|1\right\rangle + e^{i\delta}\left|3\right\rangle - \left|4\right\rangle}{\sqrt{1^2 + \left(-ii\right) + \left(e^{-i\delta}e^{i\delta}\right) + \left(-1\right)^2}}\right]$$

$$= \left\langle n \right|\left[\frac{\left|0\right\rangle + i\left|1\right\rangle + e^{i\delta}\left|3\right\rangle - \left|4\right\rangle}{2}\right]$$

$$\sum_n c_n{}^2 = 1 \qquad P_n = \left|c_n{}^2\right| \qquad \left\langle m \middle| n \right\rangle = \delta_{mn}$$

$$\left\langle \psi \right| \hat{H} \left| \psi \right\rangle = \left\langle \psi \right|\left\{ \tfrac{1}{2}E_0\left|0\right\rangle + i\,\tfrac{1}{2}E_1\left|1\right\rangle + e^{i\delta}\,\tfrac{1}{2}E_3\left|3\right\rangle - \tfrac{1}{2}E_4\left|4\right\rangle \right\}$$

$$= \tfrac{1}{4}\left[E_0 + E_1 + E_3 + E_4\right]$$

Kets & wave functions

- Wave function:

$$\psi(x) = \langle x|\psi\rangle = \tfrac{1}{2}\langle x|\left[|0\rangle + i|1\rangle + e^{i\delta}|3\rangle - |4\rangle\right]$$

$$c_n = \int_{space} \varphi_n^*(x)\psi_{norm}(x)\,dx$$

Expectation values position, momentum, energy?

All of the above in wave function notation