The Harmonic Oscillator Wave Functions

Read McIntyre 9.3

PH451/551

Reading Quiz

① Write the ground state HO wave function

(set *m* = ω = *hbar* =1)

$$\varphi_0(x) = ?$$

② Complete: $a^\dagger|n\rangle = ?$

③ Complete: $a|n\rangle = ?$

Recap

① Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

② Raising & lowering operators defined ("ladder operators")

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)$$

③ Eigenvalue equation

$$H\left|E\right\rangle = E\left|E\right\rangle$$

④ Lowering op on ground state gives zero

$$H\left|E_{ground}\right\rangle = \frac{\hbar\omega}{2}\left|E_{ground}\right\rangle$$

Recap

⑤ Ladder of eigenstates with energies separated by hbar*ω

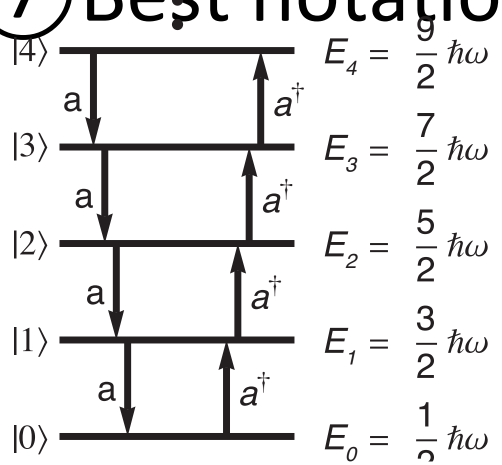
$$Ha^{\dagger}|E\rangle = (E + \hbar\omega)a^{\dagger}|E\rangle$$

$$Ha|E\rangle = (E - \hbar\omega)a|E\rangle$$

⑥ Energy spectrum

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega;\ n = 0,1,2,....$$

⑦ Best notation



$$H|n\rangle = E_n|n\rangle$$

$$\hbar\omega\left(a^{\dagger}a + \tfrac{1}{2}\right)|n\rangle = \left(n + \tfrac{1}{2}\right)\hbar\omega|n\rangle$$

$$(n = 0,1,2,3...)$$

Action of the ladder operators

$$Ha|E_n\rangle = (E_n - \hbar\omega)a|E_n\rangle$$

The state $a|n\rangle$ is an eigenstate of the HO Hamiltonian with an energy hbar*ω less than the energy of the eigenstate $|n\rangle$

$$a|n\rangle = c|n-1\rangle$$

c is some unknown constant. So a|n> and |n-1> are "the same" or "point in the same direction" …

$$\left|a|n\rangle\right|^2 = \left|c|n-1\rangle\right|^2$$

Show in detail that $c = \sqrt{n}$

Action of the ladder operators

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$a^{\dagger}|1\rangle = \sqrt{2}|2\rangle \Rightarrow |2\rangle = \frac{1}{\sqrt{2}}a^{\dagger}|1\rangle$$

Example:

$$a^{\dagger}|0\rangle = \sqrt{1}|1\rangle \Rightarrow |1\rangle = a^{\dagger}|0\rangle$$

$$|2\rangle = \frac{1}{\sqrt{2}}a^{\dagger}a^{\dagger}|0\rangle$$

What is the ground state?

- Show

$$a|0\rangle = 0$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)\varphi_0(x) = 0$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\varphi_0(x) = 0$$

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \quad \text{(Gaussian)}$$

Class exercise: generate the *n*=1 state.

- Do (without text)

$$a^\dagger|0\rangle = |1\rangle$$

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \quad \text{(Gaussian)}$$