The Harmonic Oscillator Spectrum

Read McIntyre 9.1-9.2

PH451/551

Reading Quiz

① Write the lowering operator *a* in terms of operators *x* and *p* (set $m = \omega = hbar = 1$)

② Write the raising operator a^+

③ Write the Hamiltonian in terms of raising and lowering operators (put the constants back in)

④ What is the energy spectrum?

⑤ What is the commutator of *a* with a^+?

Reading Quiz

① Write the lowering operator *a* in terms of operators *x* and *p* (set *m* = ω = *hbar* =1)

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

② Write the raising operator a^+ (set *m* = ω = *hbar* =1)

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)$$

③ Write the Hamiltonian in terms of raising and lowering operators (put the constants back in)

$$H = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

④ What is the energy spectrum?

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega$$

⑤ What is the commutator of *a* with a^+?

$$\left[a, a^\dagger\right] = 1$$

Energy spectrum

① There must be a lowest energy state. $\left| E_{lowest} \right\rangle$

② Therefore the lowering operator acts on the lowest state to give zero. $a\left| E_{lowest} \right\rangle = 0$

③ Therefore the lowest energy is $\frac{\hbar\omega}{2}$ (n=0)

$$\overbrace{\hbar\omega\left(a^{\dagger}a + \tfrac{1}{2}\right)}^{H}\left| E_{lowest} \right\rangle = \tfrac{1}{2}\hbar\omega\left| E_{lowest} \right\rangle$$

④ Energy eigenstates are connected by the raising/lowering (ladder) operators

$$Ha^{\dagger}\left| E_{lowest} \right\rangle = \left(1 + \tfrac{1}{2}\right)\hbar\omega a^{\dagger}\left| E_{lowest} \right\rangle$$

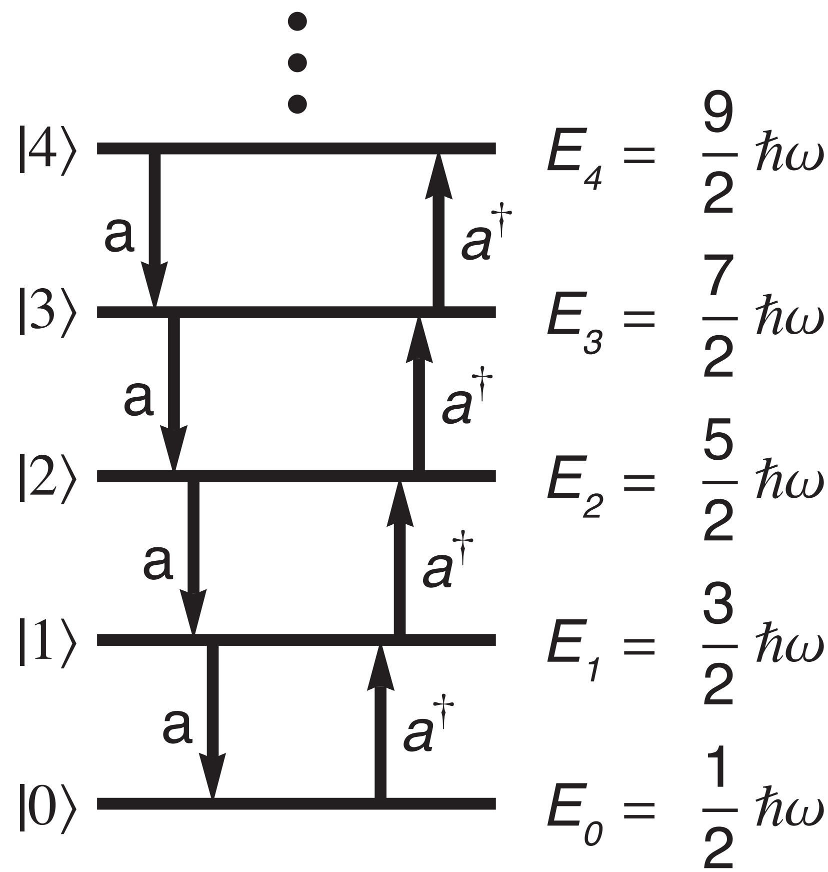
⑤ Therefore spectrum is $E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega$

Energy spectrum

⑥ Energy eigenstates are evenly spaced in energy

⑦ Therefore, there is only ONE energy difference! Spectroscopy shows absorption at a SINGLE frequency.

(period independent of amplitude – PH421!)



$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega$$

Group exercise
practice commutators

- Show $\left[H,a^{\dagger}\right]=\hbar\omega a^{\dagger}$

$$\left[H,a^{\dagger}\right]$$

$$=Ha^{\dagger}-a^{\dagger}H=\hbar\omega\left(a^{\dagger}a+\tfrac{1}{2}\right)a^{\dagger}-a^{\dagger}\hbar\omega\left(a^{\dagger}a+\tfrac{1}{2}\right)$$

$$=\hbar\omega\left(a^{\dagger}aa^{\dagger}+\cancel{\tfrac{1}{2}a^{\dagger}}-a^{\dagger}a^{\dagger}a-\cancel{\tfrac{1}{2}a^{\dagger}}\right)$$

$$=\hbar\omega\left(a^{\dagger}aa^{\dagger}-a^{\dagger}a^{\dagger}a\right)=\hbar\omega a^{\dagger}\left(aa^{\dagger}-a^{\dagger}a\right)=\hbar\omega a^{\dagger}\left[a,a^{\dagger}\right]$$

$$=\hbar\omega a^{\dagger}$$

Now what?

- With $\left[H, a^{\dagger}\right] = \hbar\omega a^{\dagger}$

$$Ha^{\dagger}|E\rangle = \left(\left[H, a^{\dagger}\right] + a^{\dagger}H\right)|E\rangle$$
$$= \left[H, a^{\dagger}\right]|E\rangle + a^{\dagger}H|E\rangle$$
$$= \hbar\omega a^{\dagger}|E\rangle + a^{\dagger}E|E\rangle$$
$$= \hbar\omega a^{\dagger}|E\rangle + Ea^{\dagger}|E\rangle$$
$$= \left(\hbar\omega + E\right)a^{\dagger}|E\rangle$$

Translate into spoken and written words

$$Ha^{\dagger}|E\rangle = \left(\hbar\omega + E\right)a^{\dagger}|E\rangle$$
$$Ha|E\rangle = \left(-\hbar\omega + E\right)a|E\rangle$$