Addition of angular momentum: two spin-1/2

Read McIntyre 11.4,5,6

PH451/551

Reading Quiz

F=total (spin) angular momentum

1. How do we label eigenstates of F^2, F_z?

S= electron spin a.m.; I = proton spin a.m.

2. How do we label eigenstates of S^2, $S_{z,}$ I^2, I_z

Reading Quiz

F=total (spin) angular momentum

1. How do we label eigenstates of F^2, F_z?

$$|F, m_F\rangle$$

$$F^2 = \hbar^2 F(F+1)|F, m_F\rangle; F_z = \hbar m_F |F, m_F\rangle$$

S= electron spin a.m.; I = proton spin a.m.

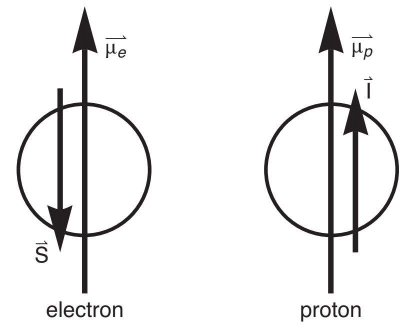
2. How do we label eigenstates of S^2, S_z, I^2, I_z

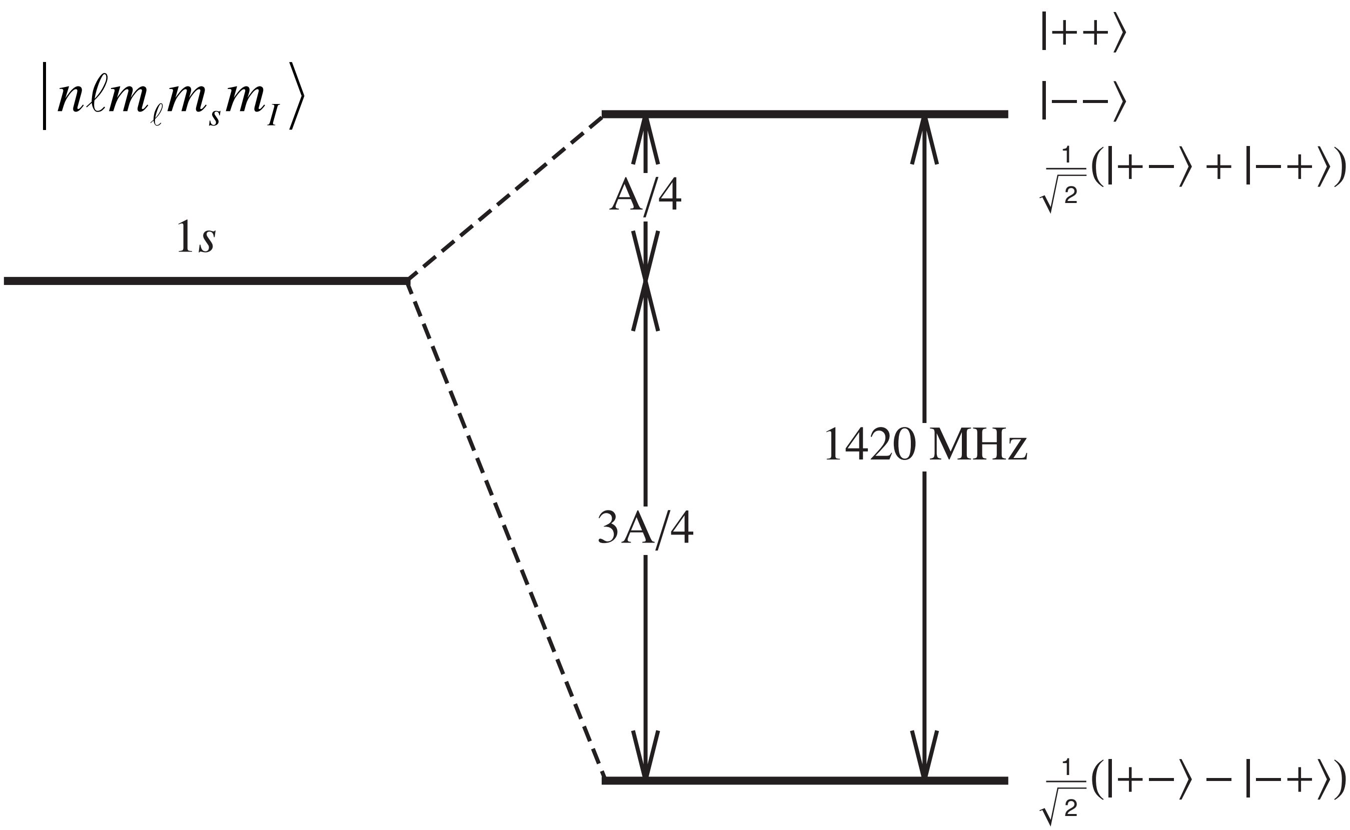
$$|S, m_S, I, m_I\rangle = |\tfrac{1}{2}, m_S, \tfrac{1}{2}, m_I\rangle = |m_S, m_I\rangle$$

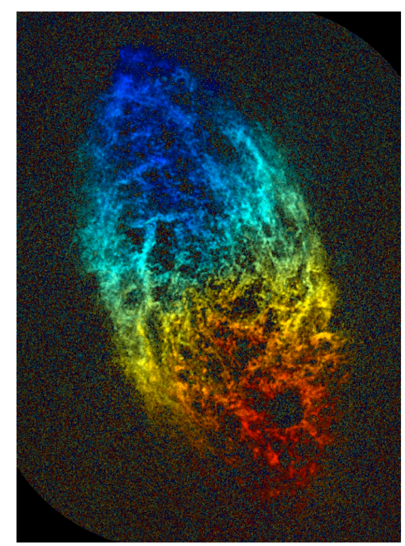
$$= |+,+\rangle, |-,+\rangle, |+,-\rangle, |-,-\rangle, |\tfrac{1}{2}, -\tfrac{1}{2}\rangle etc$$

The hyperfine interaction lifts the degeneracy of the H ground state

1. Hyperfine interaction: $H'_{hf} = \frac{\mu_0}{4\pi}\frac{g_e\mu_B g_p\mu_N}{\hbar^2}\frac{8\pi}{3}\mathbf{S}\bullet\mathbf{I}\,\delta(\mathbf{r}) = \frac{A}{\hbar^2}\mathbf{S}\bullet\mathbf{I}$

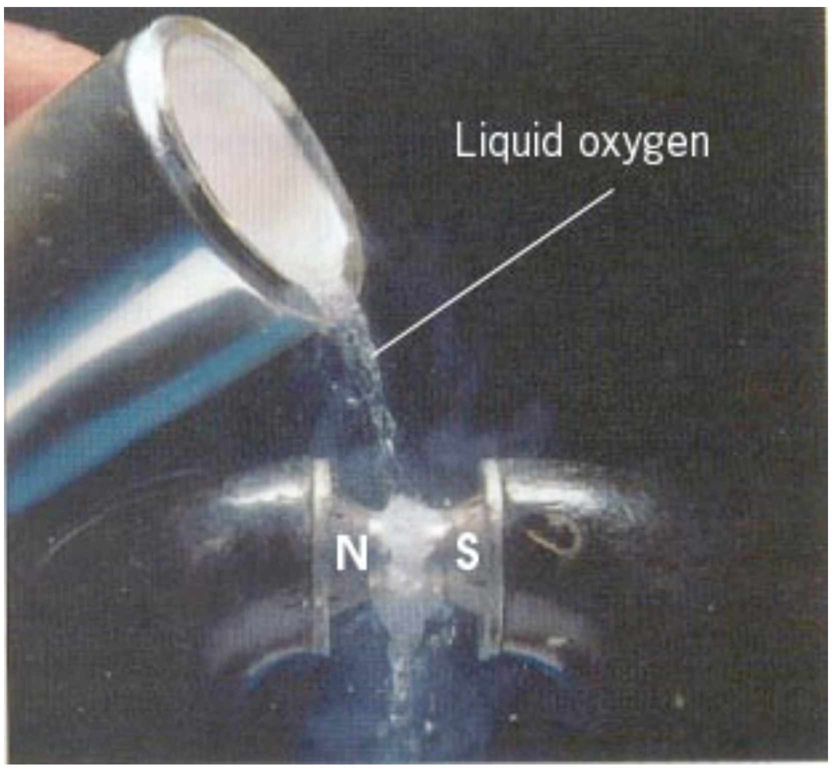




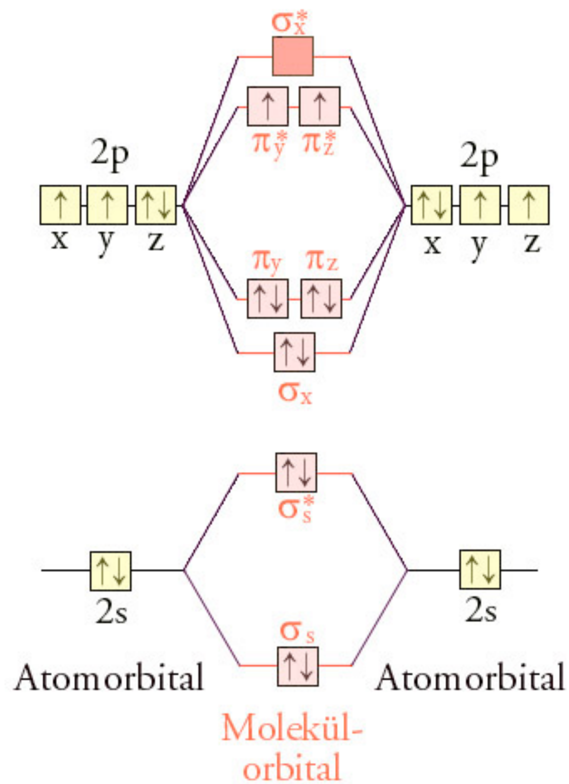


Aside: another singlet/triplet example

1. Oxygen stablity & magnetism:
http://en.wikipedia.org/wiki/Triplet_oxygen



http://coe.kean.edu/~afonarev/Physics/Magnetism/Magnetic%20Fields%20and%20Forces-eL.htm

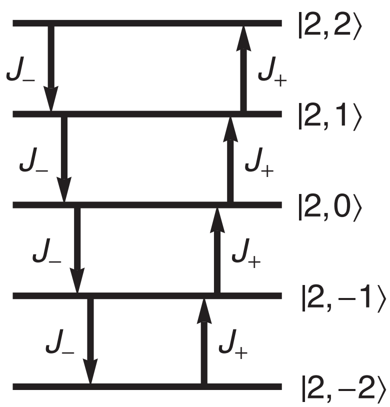


Two unpaired electrons => 4 degen. states. Degen. lifted by exchange interaction to give lower energy triplet (S=1, paramagnetic) and higher energy singlet (S=0, non magnetic).

Recap- system of 2 spin-1/2: e and p Uncoupled basis

1. Hyperfine interaction:

$$
\begin{array}{cc}
 & \begin{array}{cccc} |{+}{+}\rangle & |{+}{-}\rangle & |{-}{+}\rangle & |{-}{-}\rangle \end{array} \\
S\cdot I = S_+I_- + S_-I_+ + S_zI_z \doteq \dfrac{\hbar^2}{4} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
\begin{array}{c} \langle{+}{+}| \\ \langle{+}{-}| \\ \langle{-}{+}| \\ \langle{-}{-}| \end{array}
$$



$$J_+ = \left(J_x + iJ_y\right)$$

$$J_- = \left(J_x - iJ_y\right)$$

$$\left[J_+, J_-\right] = 2\hbar J_z$$

$$J_\pm\left|j, m_j\right\rangle = \hbar\left[j(j+1) - m_j(m_j \pm 1)\right]^{1/2}\left|j, m_j \pm 1\right\rangle$$

$$J_+\left|j, j\right\rangle = 0; \quad J_-\left|j, -j\right\rangle = 0$$

Recap- system of 2 spin-1/2: e and p
Uncoupled basis

1. Total angular momentum: $F = S + I$

$$S\cdot I = S_+ I_- + S_- I_+ + S_z I_z \doteq \begin{array}{c} \\ \frac{\hbar^2}{4} \end{array}\begin{array}{cc} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} \langle ++| \\ \langle +-| \\ \langle -+| \\ \langle --| \end{array} \end{array}$$

$E_1 = A/4, \quad |E_1\rangle = |++\rangle$

$E_2 = A/4, \quad |E_2\rangle = |--\rangle$

$E_3 = A/4, \quad |E_3\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle + |-+\rangle\right]$

$E_4 = -3A/4, \quad |E_4\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$

Another way to look at two spin ½:

1. Two magnetic dipoles exert torques on each other and change direction of a.m. of the other =>
   (a) z-comp. of a.m. of each particle is NOT conserved
   QM: S_z and I_z do not commute with H_{hf}
   We say, "m_S and m_I are bad quantum numbers"
   (b) but magnitude of a.m of each particle is conserved.
   QM: S^2 and I^2 commute with H_{hf}
   We say, "S and I are good quantum numbers"

2. To find other good quantum numbers for this problem and hence the "correct" or convenient basis, we must look for a conserved quantity. It will be total angular momentum:
   $$F = S + I$$

Another way to look at 2 spin ½:

1. Total angular momentum: $$F = S + I$$

$$F_z = S_z + I_z$$

$$\left|F, m_F\right\rangle$$

$$F^2 = \hbar^2 F(F+1)\left|F, m_F\right\rangle; F_z = \hbar m_F \left|F, m_F\right\rangle$$

2. How do these "coupled" basis states relate to the "uncoupled" states?
3. How do the quantum numbers *F*, m_F relate to quantum numbers *S*, m_S, *I*, m_I?

F_z in uncoupled basis:

1. Z-cmpt angular momentum: $$F_z = S_z + I_z$$

$$F_z \doteq \hbar \begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} |{+}{+}\rangle & |{+}{-}\rangle & |{-}{+}\rangle & |{-}{-}\rangle \end{array} \\ S_z \doteq \frac{\hbar}{2} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{array}{c} \langle{+}{+}| \\ \langle{+}{-}| \\ \langle{-}{+}| \\ \langle{-}{-}| \end{array} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} |{+}{+}\rangle & |{+}{-}\rangle & |{-}{+}\rangle & |{-}{-}\rangle \end{array} \\ I_z \doteq \frac{\hbar}{2} & \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{array}{c} \langle{+}{+}| \\ \langle{+}{-}| \\ \langle{-}{+}| \\ \langle{-}{-}| \end{array} \end{array}$$

2. Diagonal! Uncoupled states $|m_s m_I\rangle$ are eigenstates of F_z ! (but ambiguity about +-, -+ states)

$$F_z|m_S m_I\rangle = (S_z + I_z)|m_S m_I\rangle$$
$$= (m_S + m_I)\hbar|m_S m_I\rangle$$

$$F_z|FM_F\rangle = M_F\hbar|FM_F\rangle$$

$$M_F = m_S + m_I$$

F^2 in uncoupled basis:

1. Square of ang. Momentum: $F^2 = S^2 + I^2 + 2S \cdot I$

$$\mathbf{F}^2 \doteq \hbar^2 \begin{array}{c} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \end{array} \begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array}$$

$$2S \cdot I \doteq \frac{\hbar^2}{2} \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \langle ++| \\ \langle +-| \\ \langle -+| \\ \langle --| \end{array}$$

2. Not diagonal! Uncoupled states $|m_s m_I\rangle$ are not eigenstates of F^2 in general! But $|++\rangle$ and $|--\rangle$ are. $\mathbf{F}^2 |FM_F\rangle = F(F+1)\hbar^2 |FM_F\rangle$

$$\mathbf{F}^2 |++\rangle = 2\hbar^2 |++\rangle \Rightarrow F = 1$$

$$\mathbf{F}^2 |--\rangle = 2\hbar^2 |--\rangle \Rightarrow F = 1$$

coupled basis | uncoupled basis

$$|F=1, M_F=1\rangle = |++\rangle$$

$$|F=1, M_F=-1\rangle = |--\rangle$$

F^2 in uncoupled basis:

1. Square of ang. Momentum: $F^2 = S^2 + I^2 + 2S\cdot I$

$$\mathbf{F}^2 \doteq \hbar^2 \begin{array}{c} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & \boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}} & & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \end{array} \begin{array}{c} ++ \\ +- \\ -+ \\ -- \end{array}$$

$$2S\cdot I \doteq \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} \langle ++| \\ \langle +-| \\ \langle -+| \\ \langle --| \end{array}$$

2. Diagonalize subspace (HW)

coupled basis | uncoupled basis

$$|F=1, M_F=0\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle + |-+\rangle\right]$$

$$|F=0, M_F=0\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

Two ways to look a the same system

1. Square of ang. Momentum:

coupled basis | uncoupled basis

$$\left.\begin{aligned} |11\rangle &= |++\rangle \\ |10\rangle &= \frac{1}{\sqrt{2}}\big[|+-\rangle + |-+\rangle\big] \\ |1,-1\rangle &= |--\rangle \end{aligned}\right\} \textit{Triplet state} \qquad F = |S+I|$$

F=1, m_F = 1,0,-1

$$\left.|00\rangle = \frac{1}{\sqrt{2}}\big[|+-\rangle - |-+\rangle\big]\right\} \textit{Singlet state} \qquad F = |S-I|$$

F=0, m_F = 0

Clebsch-Gordan Coefficients

1.

$s=\frac{1}{2}$		F	1	1	1	0
	$I=\frac{1}{2}$	M_F	1	0	-1	0
m_s	m_I					
$\frac{1}{2}$	$\frac{1}{2}$		1	0	0	0
$\frac{1}{2}$	$-\frac{1}{2}$		0	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$
$-\frac{1}{2}$	$\frac{1}{2}$		0	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$-\frac{1}{2}$	$-\frac{1}{2}$		0	0	1	0

$$|11\rangle = |++\rangle$$

$$|10\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle + |-+\rangle\right]$$

$$|1,-1\rangle = |--\rangle$$

$$|00\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

$$|JM\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} C_{m_1 m_2 M}^{j_1 j_2 J} |j_1 j_2 m_1 m_2\rangle$$

Hyperfine interaction in coupled basis

1. HI

$$H'_{hf} \doteq \frac{A}{4}\begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} & \begin{array}{c} 11 \\ 10 \\ 1,-1 \\ 00 \end{array} \end{array}$$

2. Pity we didn't know about the coupled basis in the first place?!
3. But we learned how to add angular momentum (for the easiest case of just two spin states per particle)

Clebsch-Gordan Coefficients

1.

$s=\frac{1}{2}$	$I=\frac{1}{2}$	F	1	1	1	0
		M_F	1	0	-1	0
m_s	m_I					
$\frac{1}{2}$	$\frac{1}{2}$		1	0	0	0
$\frac{1}{2}$	$-\frac{1}{2}$		0	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$
$-\frac{1}{2}$	$\frac{1}{2}$		0	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$-\frac{1}{2}$	$-\frac{1}{2}$		0	0	1	0

$$|11\rangle = |++\rangle$$

$$|10\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle + |-+\rangle\right]$$

$$|1,-1\rangle = |--\rangle$$

$$|00\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

$$|JM\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} C_{m_1 m_2 M}^{j_1 j_2 J} |j_1 j_2 m_1 m_2\rangle$$

Clebsch-Gordan coefficients for j_1=j_2=1

$j_1=1$		j	2	2	2	2	2	1	1	1	0
	$j_2=1$	m	2	1	0	-1	-2	1	0	-1	0
m1	m2										
1	1		1	0	0	0	0	0	0	0	0
1	0		0	$\frac{1}{\sqrt{2}}$	0	0	0	$\frac{1}{\sqrt{2}}$	0	0	0
1	-1		0	0	$\frac{1}{\sqrt{6}}$	0	0	0	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{3}}$
0	1		0	$\frac{1}{\sqrt{2}}$	0	0	0	$-\frac{1}{\sqrt{2}}$	0	0	0
0	0		0	0	$\sqrt{\frac{2}{3}}$	0	0	0	0	0	$-\frac{1}{\sqrt{3}}$
0	-1		0	0	0	$\frac{1}{\sqrt{2}}$	0	0	0	$\frac{1}{\sqrt{2}}$	0
-1	1		0	0	$\frac{1}{\sqrt{6}}$	0	0	0	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{3}}$
-1	0		0	0	0	$\frac{1}{\sqrt{2}}$	0	0	0	$-\frac{1}{\sqrt{2}}$	0
-1	-1		0	0	0	0	1	0	0	0	0

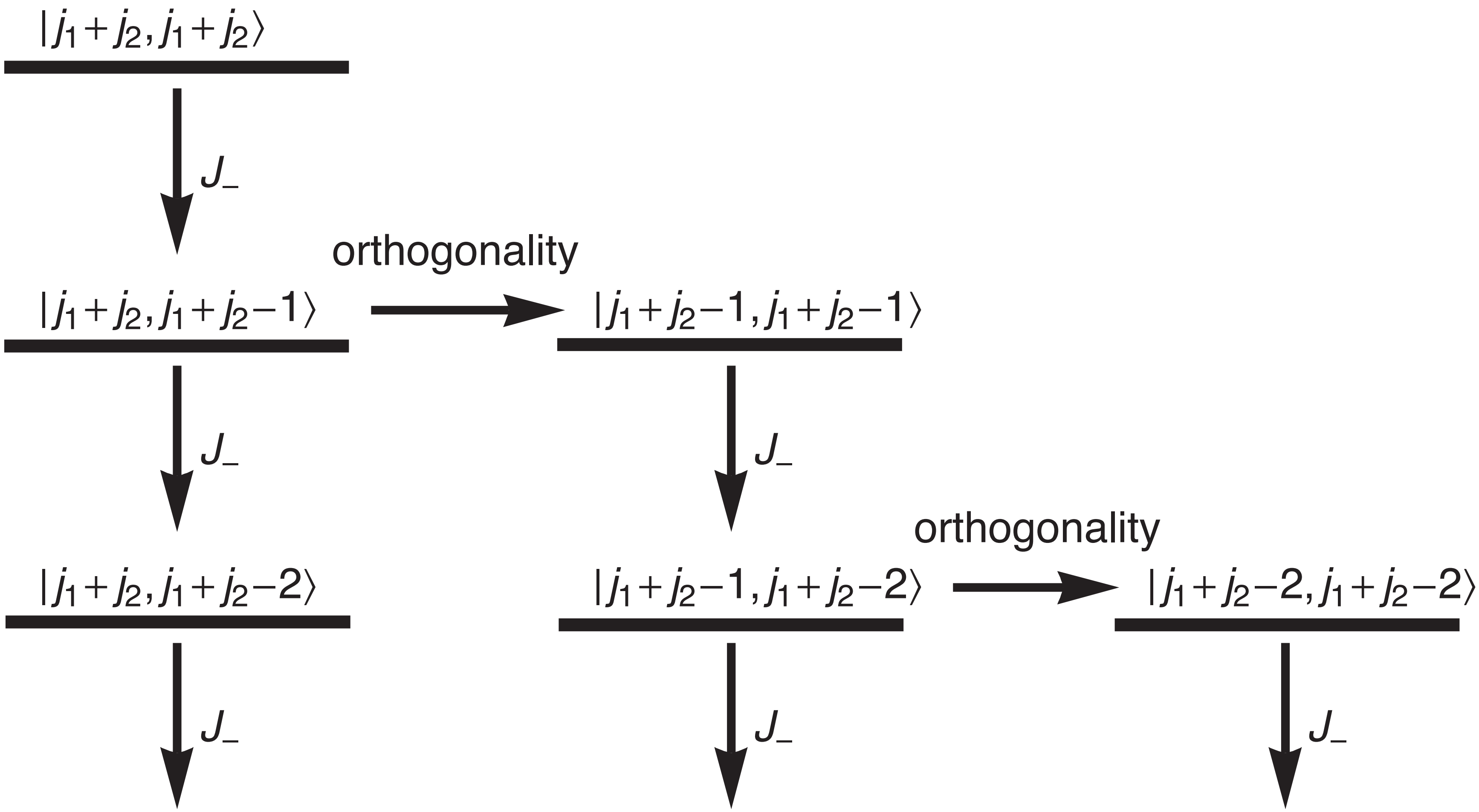
Clebsch-Gordan coefficients: "forward and backward"

$$\left|JM\right\rangle = \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} C^{j_1j_2J}_{m_1m_2M}\left|j_1j_2m_1m_2\right\rangle$$

$$\left|j_1j_2m_1m_2\right\rangle = \sum_{J=|j_1-j_2|}^{j_1+j_2}\left|JM\right\rangle\left\langle JM\right|j_1j_2m_1m_2\rangle$$

$$= \sum_{J=|j_1-j_2|}^{j_1+j_2} C^{j_1j_2J}_{m_1m_2M}\left|JM\right\rangle$$

Calculating Clebsch-Gordan coefficients



Commutators (unperturbed)

$$\left[H_0, S^2\right] = 0$$

$$\left[H_0, I^2\right] = 0$$

$$\left[H_0, L^2\right] = 0$$

$$\left[H_0, S_z\right] = 0$$

$$\left[H_0, I_z\right] = 0$$

$$\left[H_0, L_z\right] = 0$$

$$\left|n, \ell, m_\ell, s, m_s, I, m_I\right\rangle \overbrace{\rightarrow}^{often} \left|n, \ell, m_\ell, m_s, m_I\right\rangle$$

Commutators (perturbed)

$$H_{hf} = S \cdot I$$

$$\left[H_{hf}, S^2 \right] = 0 \qquad \left[H_{hf}, I^2 \right] = 0$$

$$\left[H_{hf}, F^2 \right] = 0$$

$$\left[H_{hf}, S_z \right] \neq 0 \qquad \left[H_{hf}, I_z \right] \neq 0$$

$$\left[H_{hf}, S_z + I_z \right] = 0$$

$$\left[H_{hf}, L^2 \right] = 0 \quad and \quad \left[H_{hf}, L_z \right] = 0 \quad (wait!)$$

$$\left| n, \ell, m_\ell, s, I, F, m_F \right\rangle$$

Commutators & conserved quantities

$$\frac{d}{dt}\langle\psi|A|\psi\rangle = \frac{\partial}{\partial t}\langle\psi|\{A|\psi\rangle\} + \langle\psi|\frac{\partial}{\partial t}A|\psi\rangle + \langle\psi|A\frac{\partial}{\partial t}|\psi\rangle$$

$$H|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$$

$$= \frac{1}{-i\hbar}\langle\psi|HA|\psi\rangle + \frac{1}{i\hbar}\langle\psi|AH|\psi\rangle + \langle\psi|\frac{\partial}{\partial t}A|\psi\rangle$$

$$\frac{d}{dt}\langle\psi|A|\psi\rangle = \frac{1}{i\hbar}\langle\psi|AH - HA|\psi\rangle + \langle\psi|\frac{\partial}{\partial t}A|\psi\rangle$$

$$\frac{d}{dt}\langle\psi|A|\psi\rangle = \frac{i}{\hbar}\langle\psi|[H,A]|\psi\rangle$$