Degen. perturbation theory applied to angular momentum system

Read McIntyre 11.4

PH451/551

Recap -degenerate perturbation theory

1. **Diagonalize the perturbation Hamiltonian in the degenerate subspace!** (and the first correction pops up on the diagonal)
2. => mix up the degenerate states to get the "right" basis that removes the infinities that would result if we blindly applied non-degen pert theory
3. Physics: the states with no component along the field remain unchanged. The ones with up and down components mix and change

Recap: Stark effect in H (*n* = 2 states)

1. Dipole energy: $H' = -\mathbf{d} \bullet \mathbf{E} = e\mathrm{E}r\cos\theta$

2. Perturbation:

$$H' \doteq \begin{array}{c} \begin{pmatrix} 0 & -3e\mathrm{E}a_0 & 0 & 0 \\ -3e\mathrm{E}a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} 200 \\ 210 \\ 211 \\ 21,-1 \end{array}$$

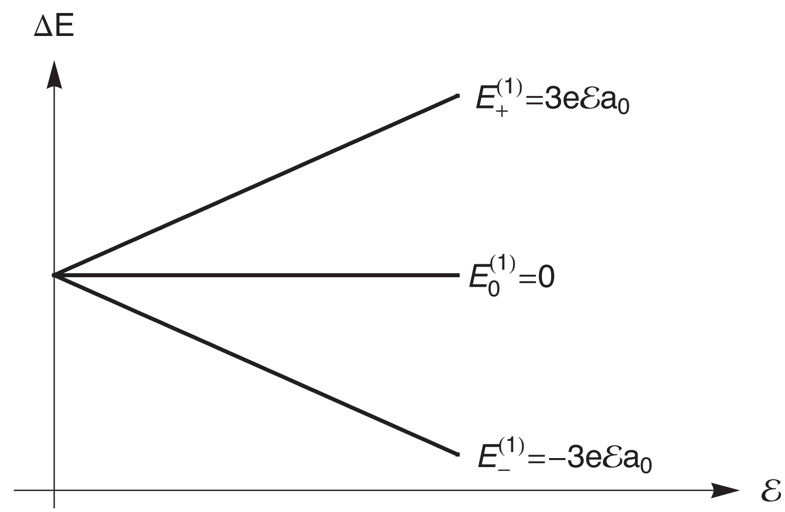
$$E = +3e\mathrm{E}a_0, -3e\mathrm{E}a_0, 0, 0$$

$$\left|\psi_+\right\rangle = \tfrac{1}{\sqrt{2}}\left[\left|200\right\rangle - \left|210\right\rangle\right]$$

$$\left|\psi_-\right\rangle = \tfrac{1}{\sqrt{2}}\left[\left|200\right\rangle + \left|210\right\rangle\right]$$

$$\left|\psi_3\right\rangle = \left|211\right\rangle$$

$$\left|\psi_4\right\rangle = \left|21,-1\right\rangle$$



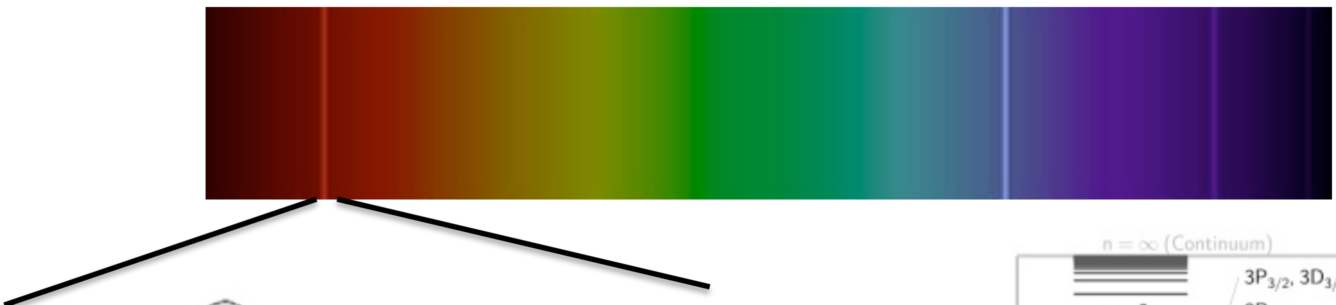
Reading Quiz

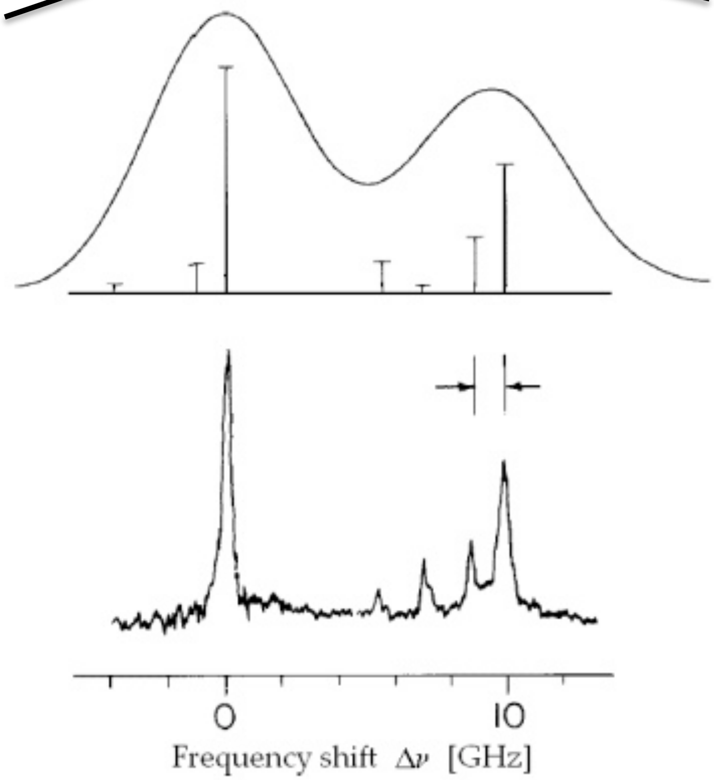
Hyperfine interaction?

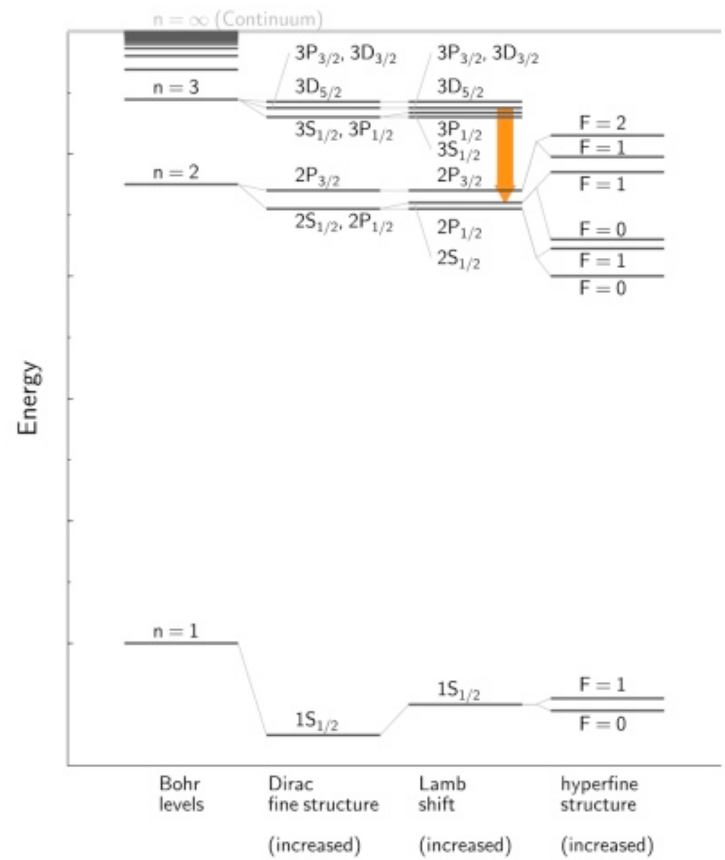
1. An electron in hydrogen makes a transition from the n=3 -> n=2 level. What is the energy, the wavelength and the frequency? Need numbers (and units)
2. What are "common" or "sensible" units?

Fine structure: beyond "1/n^2"

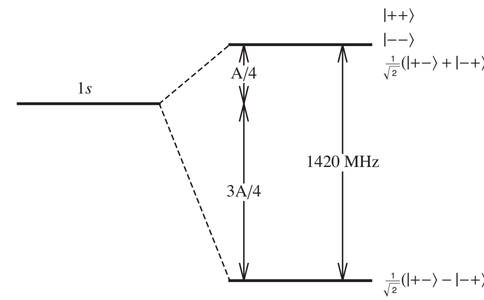
1. http://backreaction.blogspot.com/2007/12/hydrogen-spectrum-and-its-fine.html



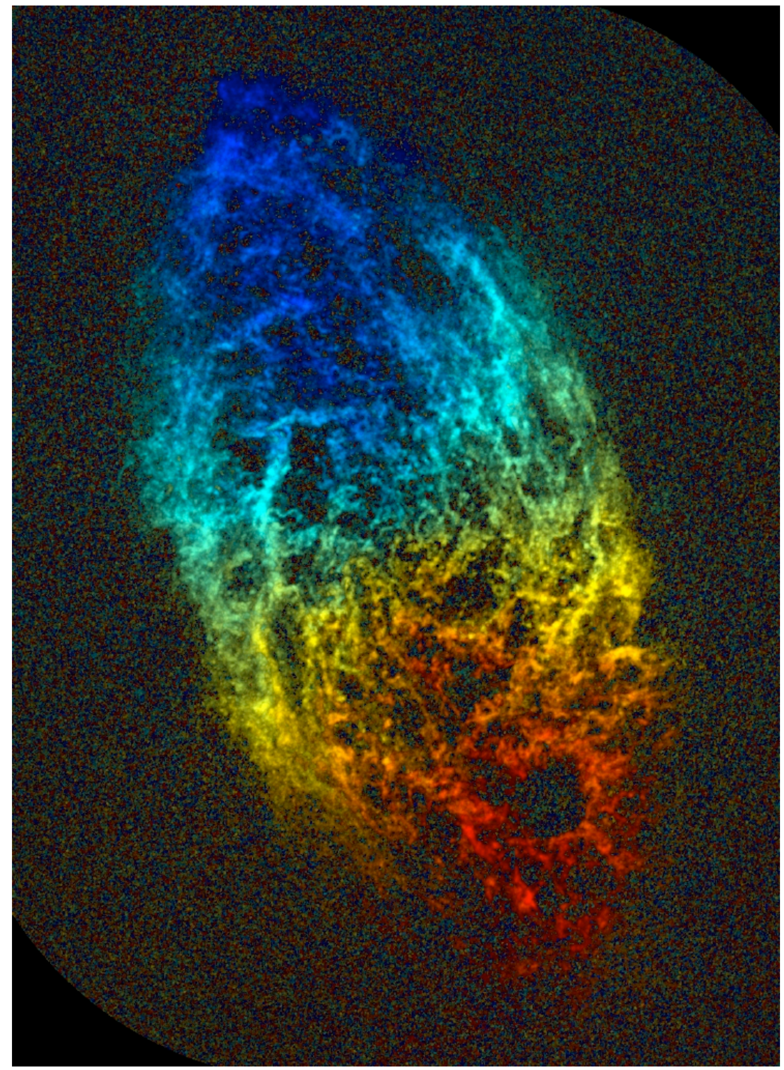




The 21 cm line … hyperfine

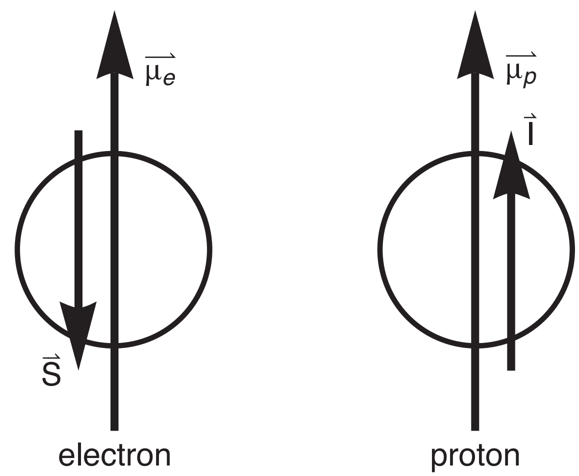


1. Forbidden transition
   s->s. Lifetime ≈11 million yr
2. Can see galaxies obscured in visible by dust
3. 1.42 GHz = 21 cm
   = ?eV =?K
4. *The HI radial velocity field of the nearby spiral galaxy M33 is shown here by colors corresponding to Doppler redshifts and blueshifts relative to the center of mass; brightness in this image is proportional to HI column density.*
   Image credit:
   http://images.nrao.edu/?id=51



Hyperfine interaction

1. The spins of the electron and proton introduce additional energy beyond the Coulomb interaction:



2. In symbols:

$$H'_{hf} = \frac{\mu_0}{4\pi}\frac{g_e\mu_B g_p\mu_N}{\hbar^2}\left[\frac{1}{r^3}\mathbf{I}\bullet\mathbf{L} - \frac{1}{r^3}\mathbf{S}\bullet\mathbf{I} + \frac{3}{r^5}(\mathbf{S}\bullet\mathbf{r})(\mathbf{I}\bullet\mathbf{r}) + \frac{8\pi}{3}\mathbf{S}\bullet\mathbf{I}\delta(\mathbf{r})\right]$$

$$H'_{hf} = \frac{\mu_0}{4\pi}\frac{g_e\mu_B g_p\mu_N}{\hbar^2}\frac{8\pi}{3}\mathbf{S}\bullet\mathbf{I}\delta(\mathbf{r}) = \frac{A}{\hbar^2}\mathbf{S}\bullet\mathbf{I}$$

Hyperfine interaction

1. Now we have something new – a system of TWO spins!

   - What are the spin operators?
   - Is there a total spin that is the sum?
   - What are the quantum numbers?
   - In other words, how do we add two spins, or more generally, two angular momenta?

A system of 2 spins – e and p

1. How do we write the states?

$$|e\ \ up\rangle|p\ \ up\rangle \equiv |+\rangle_e|+\rangle_p \equiv |++\rangle$$

What are the spin operators?

$$S_z \doteq \begin{array}{cc} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} & \\ \begin{pmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \end{pmatrix} & \begin{array}{c} \langle++| \\ \langle+-| \\ \langle-+| \\ \langle--| \end{array} \end{array}$$

$$I_z \doteq \begin{array}{cc} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} & \\ \begin{pmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ & \_\_ \end{pmatrix} & \begin{array}{c} \langle++| \\ \langle+-| \\ \langle-+| \\ \langle--| \end{array} \end{array}$$

A system of 2 spins – e and p

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$$S_z \doteq \begin{array}{cc} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} & \\ \begin{pmatrix} \hbar/2 & 0 & 0 & 0 \\ 0 & \hbar/2 & 0 & 0 \\ 0 & 0 & -\hbar/2 & 0 \\ 0 & 0 & 0 & -\hbar/2 \end{pmatrix} & \begin{array}{c} \langle++| \\ \langle+-| \\ \langle-+| \\ \langle--| \end{array} \end{array}$$

$$I_z \doteq \begin{array}{cc} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} & \\ \begin{pmatrix} \hbar/2 & 0 & 0 & 0 \\ 0 & -\hbar/2 & 0 & 0 \\ 0 & 0 & +\hbar/2 & 0 \\ 0 & 0 & 0 & -\hbar/2 \end{pmatrix} & \begin{array}{c} \langle++| \\ \langle+-| \\ \langle-+| \\ \langle--| \end{array} \end{array}$$

A system of 2 spins: e and p

1. Recall $$H'_{hf} = \frac{\mu_0}{4\pi}\frac{g_e\mu_B g_p\mu_N}{\hbar^2}\frac{8\pi}{3}\mathbf{S}\bullet\mathbf{I}\delta(\mathbf{r}) = \frac{A}{\hbar^2}\mathbf{S}\bullet\mathbf{I}$$

2. ? $$\mathbf{S}\bullet\mathbf{I} = S_x I_x + S_y I_y + S_z I_z$$

$$\mathbf{S}\bullet\mathbf{I} = \tfrac{1}{2}\left(S_+ I_- + S_- I_+\right) + S_z I_z$$

3. What do you think S+ and S- are? I+ and I-? What are their actions on the states?

Ladder operators in angular mom.

1. Recall

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)$$

$$\left[a, a^\dagger\right] = 1$$

$$J_+ = \left(J_x + iJ_y\right)$$

$$J_- = \left(J_x - iJ_y\right)$$

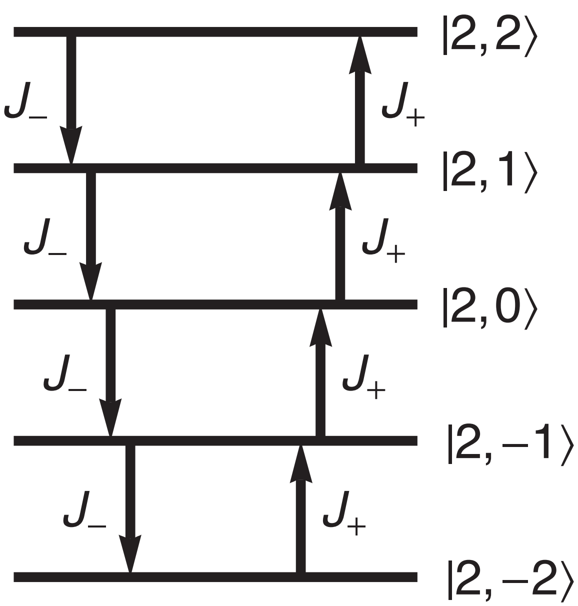
$$\left[J_+, J_-\right] = 2\hbar J_z$$

2. "J" is "S" or "I"

$$\mathbf{S} \bullet \mathbf{I} = \tfrac{1}{2}\left(S_+ I_- + S_- I_+\right) + S_z I_z$$

Ladder operators in angular mom.

1. Ladders within a particular j subspace!



$$J_+ = \left(J_x + iJ_y\right)$$

$$J_- = \left(J_x - iJ_y\right)$$

$$\left[J_+, J_-\right] = 2\hbar J_z$$

2. HW: $J_\pm \left| j, m_j \right\rangle = \hbar \left[j(j+1) - m_j(m_j \pm 1) \right]^{1/2} \left| j, m_j \pm 1 \right\rangle$

Matrices for hyperfine Hamiltonian?

1. S_z and I_z (remember J can stand for S, I, its sum, or any a.m.)

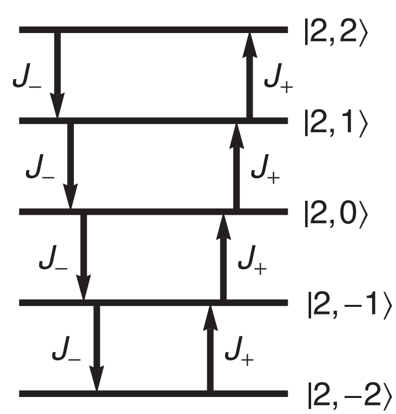
$$S_z \doteq \begin{array}{c} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} \\ \begin{pmatrix} \hbar/2 & 0 & 0 & 0 \\ 0 & \hbar/2 & 0 & 0 \\ 0 & 0 & -\hbar/2 & 0 \\ 0 & 0 & 0 & -\hbar/2 \end{pmatrix} \end{array} \begin{array}{c} \langle ++| \\ \langle +-| \\ \langle -+| \\ \langle --| \end{array}$$

$$I_z \doteq \begin{array}{c} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} \\ \begin{pmatrix} \hbar/2 & 0 & 0 & 0 \\ 0 & -\hbar/2 & 0 & 0 \\ 0 & 0 & +\hbar/2 & 0 \\ 0 & 0 & 0 & -\hbar/2 \end{pmatrix} \end{array} \begin{array}{c} \langle ++| \\ \langle +-| \\ \langle -+| \\ \langle --| \end{array}$$

$$\mathbf{S} \bullet \mathbf{I} = \tfrac{1}{2}\left(S_+ I_- + S_- I_+\right) + S_z I_z$$

$$J_\pm \left| j, m_j \right\rangle = \hbar \left[j(j+1) - m_j(m_j \pm 1) \right]^{1/2} \left| j, m_j \pm 1 \right\rangle$$

$$J_+ \left| j, j \right\rangle = 0; \quad J_- \left| j, -j \right\rangle = 0$$



Matrices for hyperfine Hamiltonian?

1. "extreme" states

$$S_z I_z|++\rangle = S_z \frac{\hbar}{2}|++\rangle = \frac{\hbar}{2} S_z|++\rangle = \frac{\hbar}{2}\frac{\hbar}{2}|++\rangle = \frac{\hbar^2}{4}|++\rangle$$

$$\tfrac{1}{2} S_+ I_-|++\rangle = \tfrac{1}{2} I_- S_+|++\rangle = 0$$

$$\tfrac{1}{2} S_- I_+|++\rangle = S_+ 0 = 0$$

$$S\cdot I|--\rangle = S_z I_z|--\rangle + \tfrac{1}{2} S_+ I_-|--\rangle + \tfrac{1}{2} S_- I_+|--\rangle = \frac{\hbar^2}{4}|--\rangle$$

$$S\cdot I \doteq \frac{\hbar^2}{4} \begin{array}{cc} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} & \\ \begin{pmatrix} \underline{1\quad} & \underline{\quad} & \underline{\quad} & \underline{0\quad} \\ \underline{0\quad} & \underline{\quad} & \underline{\quad} & \underline{0\quad} \\ \underline{0\quad} & \underline{\quad} & \underline{\quad} & \underline{0\quad} \\ \underline{0\quad} & \underline{\quad} & \underline{\quad} & \underline{1\quad} \end{pmatrix} & \begin{array}{c} \langle ++| \\ \langle +-| \\ \langle -+| \\ \langle --| \end{array} \end{array}$$

Matrices for hyperfine Hamiltonian?

1. "inner" states

$$
\begin{aligned}
S\cdot I\left|-+\right\rangle &= S_z I_z\left|-+\right\rangle + \tfrac{1}{2}S_+ I_-\left|-+\right\rangle + \tfrac{1}{2}S_- I_+\left|-+\right\rangle \\
&= -\frac{\hbar^2}{4}\left|-+\right\rangle + \tfrac{1}{2}S_+\hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\tfrac{1}{2}(-\tfrac{1}{2})\right]^{1/2}\left|--\right\rangle + 0 \\
&= -\frac{\hbar^2}{4}\left|-+\right\rangle + \tfrac{1}{2}\hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\tfrac{1}{2}(-\tfrac{1}{2})\right]^{1/2}\hbar\left[\tfrac{1}{2}\tfrac{3}{2}-\tfrac{1}{2}(-\tfrac{1}{2})\right]^{1/2}\left|+-\right\rangle \\
&= \frac{\hbar^2}{4}\left(-1\left|-+\right\rangle + 2\left|+-\right\rangle\right)
\end{aligned}
$$

$$
S\cdot I \doteq \frac{\hbar^2}{4}
\begin{array}{c}
\begin{array}{cccc} \left|++\right\rangle & \left|+-\right\rangle & \left|-+\right\rangle & \left|--\right\rangle \end{array} \\
\begin{pmatrix}
1 & \_\_ & 0 & 0 \\
0 & \_\_ & 2 & 0 \\
0 & \_\_ & -1 & 0 \\
0 & \_\_ & 0 & 1
\end{pmatrix}
\end{array}
\begin{array}{c} \left\langle ++\right| \\ \left\langle +-\right| \\ \left\langle -+\right| \\ \left\langle --\right| \end{array}
$$

$$
J_\pm\left|j,m_j\right\rangle = \hbar\left[j(j+1)-m_j(m_j\pm 1)\right]^{1/2}\left|j,m_j\pm 1\right\rangle
$$

Matrix for hyperfine Hamiltonian

1. With all the constants …

$$H'_{hf} = S \cdot I \doteq \underbrace{\frac{2\mu_0}{12} g_e \mu_B g_p \mu_N \left|\varphi_{1s}(0)\right|^2}_{\frac{A}{4}} \begin{array}{cc} \begin{array}{cccc} |+\,+\rangle & |+\,-\rangle & |-\,+\rangle & |-\,-\rangle \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} \langle +\,+| \\ \langle +\,-| \\ \langle -\,+| \\ \langle -\,-| \end{array} \end{array}$$

$$E_1 = A/4, \quad |E_1\rangle = |++\rangle$$

$$E_2 = A/4, \quad |E_2\rangle = |--\rangle$$

$$E_3 = A/4, \quad |E_3\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle + |-+\rangle\right]$$

$$E_4 = -3A/4, \quad |E_4\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$