ANGULAR MOMENTUM

Orbital angular momentum defined

$$\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} \doteq -i\hbar \hat{\mathbf{r}} \times \nabla$$

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \doteq -i\hbar\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right] \qquad \hat{L}_x \doteq i\hbar\left[\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right]$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \doteq -i\hbar\left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right] \qquad \hat{L}_y \doteq i\hbar\left[-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right]$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \doteq -i\hbar\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right] \qquad \hat{L}_z \doteq -i\hbar\frac{\partial}{\partial\phi}$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \qquad \hat{L}^2 \doteq -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Commutator relations for general angular momentum

$$\left[\hat{J}_x, \hat{J}_y\right] = i\hbar\hat{J}_z \qquad \left[\hat{J}_y, \hat{J}_z\right] = i\hbar\hat{J}_x \qquad \left[\hat{J}_z, \hat{J}_x\right] = i\hbar\hat{J}_y$$

$$\left[\hat{J}_x, \hat{J}^2\right] = \left[\hat{J}_y, \hat{J}^2\right] = \left[\hat{J}_z, \hat{J}^2\right] = 0$$

Ladder operators ...

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \qquad \hat{J}_+ = \hat{J}_-^\dagger \qquad \hat{L}_\pm \doteq \pm\hbar e^{\pm i\phi}\left[\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right]$$

... and their commutation relations

$$\left[\hat{J}_+, \hat{J}_-\right] = 2\hbar\hat{J}_z \qquad \left[\hat{J}_z, \hat{J}_\pm\right] = \pm\hbar\hat{J}_\pm \qquad \left[\hat{J}^2, \hat{J}_\pm\right] = 0$$

Other relationships

If $\hat{\vec{J}} \equiv \hat{\vec{L}} + \hat{\vec{S}}$, then $\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + 2\hat{\vec{L}}\cdot\hat{\vec{S}}$, and $\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + \hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+ + 2\hat{L}_z\hat{S}_z$

$$\hat{J}^2 = \hat{J}_\mp\hat{J}_\pm + \hat{J}_z^2 \pm \hbar\hat{J}_z$$

$$\hat{J}^2\left|jm_j\right\rangle = j(j+1)\hbar^2\left|jm_j\right\rangle \qquad \hat{J}_z\left|jm_j\right\rangle = m_j\hbar\left|jm_j\right\rangle \qquad m_j = -j, -j+1, \ldots\ldots, j-1, j$$

$$\hat{J}_\pm\left|jm_j\right\rangle = \hbar\left[j(j+1) - m_j(m_j \pm 1)\right]^{1/2}\left|j, m_j \pm 1\right\rangle$$

$$\hat{L}^2\left|\ell m_\ell\right\rangle = \ell(\ell+1)\hbar^2\left|\ell m_\ell\right\rangle \quad \ell = 0,1,2.... \qquad \hat{L}_z\left|\ell m_\ell\right\rangle = m_\ell\hbar\left|\ell m_\ell\right\rangle \qquad m_\ell = -\ell, -\ell+1, \ldots\ldots, \ell-1, \ell$$

$$\hat{S}^2\left|sm_s\right\rangle = s(s+1)\hbar^2\left|sm_s\right\rangle \qquad \hat{S}_z\left|sm_s\right\rangle = m_s\hbar\left|sm_s\right\rangle \qquad m_s = -s, -s+1, \ldots\ldots, s-1, s$$