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PH 451/551 Quantum Mechanics Capstone Winter 201x
TOTAL POINTS: xx Weniger 116, time

There are xx questions, for a total of xx points. They will surely present different levels of difficulty, so work quickly on the ones you find easy, leaving time for the ones you find harder. On average, budget about 1 point per minute.

Equations and integrals that may be useful:

$$\langle\psi|\hat{A}^{\dagger}|\phi\rangle = \langle\phi|\hat{A}|\psi\rangle^{*} \qquad \sum_n |n\rangle\langle n| = 1 \qquad \Delta Q \equiv \sqrt{\left\langle\left(Q-\langle Q\rangle\right)^2\right\rangle}$$

$$E_n = n^2\frac{\hbar^2\pi^2}{2mL^2} \qquad E_n = -\frac{1}{n^2}\frac{1}{2}\alpha^2 mc^2 \qquad E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega$$

$$\varphi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \qquad \alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \qquad a_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2}$$

$$V(x) = \tfrac{1}{2}m\omega^2x^2 \qquad a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p_x}{m\omega}\right) \qquad a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p_x}{m\omega}\right)$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega$$

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \varphi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{m\omega}{\hbar}}\,\tfrac{1}{\sqrt{2}}\,2x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\varphi_n(x) = \frac{1}{2^{n/2}}\frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega}{2\hbar}x^2}$$

$$E_n^{(1)} = \left\langle n^{(0)}\middle|H'\middle|n^{(0)}\right\rangle \qquad E_n^{(2)} = \sum_{m\neq n}\frac{\left|\left\langle n^{(0)}\middle|H'\middle|m^{(0)}\right\rangle\right|^2}{E_n^{(0)}-E_m^{(0)}}$$

$$\left|n^{(1)}\right\rangle = \sum_{m\neq n} c_{nm}^{(1)}\left|m^{(0)}\right\rangle \qquad c_{nm}^{(1)} = \frac{\left\langle m^{(0)}\middle|H'\middle|n^{(0)}\right\rangle}{E_n^{(0)}-E_m^{(0)}} \qquad H' = -\vec{d}\bullet\vec{\mathrm{E}}$$

$$\vec{\mu} = g\frac{q}{2m}\vec{S} \qquad \mu_B = \frac{e\hbar}{2m_e} \qquad \mu_N = \frac{e\hbar}{2m_p}$$

$$\vec{d} = q\vec{r}$$

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{S}^2 \doteq \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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$$S_x \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$S_z \doteq \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad S^2 \doteq 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$J^2 \left| j, m_j \right\rangle = \hbar^2 j(j+1) \left| j, m_j \right\rangle \qquad J_z \left| j, m_j \right\rangle = \hbar m_j \left| j, m_j \right\rangle$$

$$\left[J_x, J_y \right] = i\hbar J_z \qquad J_\pm = J_x \pm iJ_y$$

$$J_\pm \left| j, m_j \right\rangle = \hbar \left[j(j+1) - m_j (m_j \pm 1) \right]^{\frac{1}{2}} \left| j, m_j \pm 1 \right\rangle$$

$$|JM\rangle = \sum_{m_1 m_2} | j_1 j_2 m_1 m_2 \rangle \langle j_1 j_2 m_1 m_2 | JM \rangle$$

$$\vec{J}_1 + \vec{J}_2 = J_1{}^2 + 2\vec{J}_1 \cdot \vec{J}_2 + J_2{}^2$$

$$H'_{hf} = \frac{A}{\hbar^2} \mathbf{S \cdot I}$$

$$H'_{rel} = -\frac{p^4}{8m^3c^2} \qquad H'_{SO} = \frac{e^2}{8\pi\varepsilon_0 m^2 c^2 r^3} \mathbf{L \cdot S} \qquad H'_Z = -\boldsymbol{\mu} \bullet \mathbf{B} = \frac{\mu_B B}{\hbar}\left(g_\ell L_z + g_e S_z \right)$$

$$E_{fs}^{(1)} = -\frac{1}{2}\alpha^4 mc^2 \frac{1}{n^3}\left[\frac{1}{j+\frac{1}{2}} - \frac{3}{4n} \right] \qquad E_{SO} = \frac{1}{4}\alpha^4 mc^2 \frac{j(j+1) - \ell(\ell+1) - \frac{3}{4}}{n^3 \ell\left(\ell+\frac{1}{2}\right)(\ell+1)}$$

$$c_f(t) = \frac{1}{i\hbar}\int_0^t \langle f | H'(t') | i \rangle e^{i\frac{E_f - E_i}{\hbar}t'} dt' \qquad R_{i\to f} = \frac{2\pi}{\hbar^2}\left| V_{fi} \right|^2 \delta(\omega_{fi} - \omega)$$

$$H|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle \qquad H|\varphi_n\rangle = E_n|\varphi_n\rangle \qquad |\psi\rangle_{t=0} = \sum_n c_n |\varphi_n\rangle$$

$$\int_0^\infty e^{-a^2x^2} dx = \frac{1}{2a}\sqrt{\pi} \qquad \int_0^\infty x e^{-x^2} dx = \frac{1}{2} \qquad \int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$$

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = 1 - 2\sin^2\theta \qquad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\int \sin mx \sin nx \, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, \ \left(m^2 \neq n^2\right)$$

$$\int \cos mx \cos nx \, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}, \ \left(m^2 \neq n^2\right)$$

$$\int \sin mx \cos nx \, dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}, \ \left(m^2 \neq n^2\right)$$

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$s=\frac{1}{2}$		F	1	1	1	0
	$I=\frac{1}{2}$	M_F	1	0	-1	0
m_s	m_I					
$\frac{1}{2}$	$\frac{1}{2}$		1	0	0	0
$\frac{1}{2}$	$-\frac{1}{2}$		0	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$
$-\frac{1}{2}$	$\frac{1}{2}$		0	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$-\frac{1}{2}$	$-\frac{1}{2}$		0	0	1	0

$j_1=1$		j	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	$j_2=\frac{1}{2}$	m	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
m1	m2							
1	$\frac{1}{2}$		1	0	0	0	0	0
1	$-\frac{1}{2}$		0	$\frac{1}{\sqrt{3}}$	0	0	$\sqrt{\frac{2}{3}}$	0
0	$\frac{1}{2}$		0	$\sqrt{\frac{2}{3}}$	0	0	$-\frac{1}{\sqrt{3}}$	0
0	$-\frac{1}{2}$		0	0	$\sqrt{\frac{2}{3}}$	0	0	$\frac{1}{\sqrt{3}}$
-1	$\frac{1}{2}$		0	0	$\frac{1}{\sqrt{3}}$	0	0	$-\sqrt{\frac{2}{3}}$
-1	$-\frac{1}{2}$		0	0	0	1	0	0

$j_1=\frac{3}{2}$		j	2	2	2	2	2	1	1	1
	$j_2=\frac{1}{2}$	m	2	1	0	-1	-2	1	0	-1
m1	m2									
$\frac{3}{2}$	$\frac{1}{2}$		1	0	0	0	0	0	0	0
$\frac{3}{2}$	$-\frac{1}{2}$		0	$\frac{1}{2}$	0	0	0	$\frac{\sqrt{3}}{2}$	0	0
$\frac{1}{2}$	$\frac{1}{2}$		0	$\frac{\sqrt{3}}{2}$	0	0	0	$-\frac{1}{2}$	0	0
$\frac{1}{2}$	$-\frac{1}{2}$		0	0	$\frac{1}{\sqrt{2}}$	0	0	0	$\frac{1}{\sqrt{2}}$	0
$-\frac{1}{2}$	$\frac{1}{2}$		0	0	$\frac{1}{\sqrt{2}}$	0	0	0	$-\frac{1}{\sqrt{2}}$	0
$-\frac{1}{2}$	$-\frac{1}{2}$		0	0	0	$\frac{\sqrt{3}}{2}$	0	0	0	$\frac{1}{2}$
$-\frac{3}{2}$	$\frac{1}{2}$		0	0	0	$\frac{1}{2}$	0	0	0	$-\frac{\sqrt{3}}{2}$
$-\frac{3}{2}$	$-\frac{1}{2}$		0	0	0	0	1	0	0	0

$j_1=1$		j	2	2	2	2	2	1	1	1	0
	$j_2=1$	m	2	1	0	-1	-2	1	0	-1	0
m1	m2										
1	1		1	0	0	0	0	0	0	0	0
1	0		0	$\frac{1}{\sqrt{2}}$	0	0	0	$\frac{1}{\sqrt{2}}$	0	0	0
1	-1		0	0	$\frac{1}{\sqrt{6}}$	0	0	0	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{3}}$
0	1		0	$\frac{1}{\sqrt{2}}$	0	0	0	$-\frac{1}{\sqrt{2}}$	0	0	0
0	0		0	0	$\sqrt{\frac{2}{3}}$	0	0	0	0	0	$-\frac{1}{\sqrt{3}}$
0	-1		0	0	0	$\frac{1}{\sqrt{2}}$	0	0	0	$\frac{1}{\sqrt{2}}$	0
-1	1		0	0	$\frac{1}{\sqrt{6}}$	0	0	0	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{3}}$
-1	0		0	0	0	$\frac{1}{\sqrt{2}}$	0	0	0	$-\frac{1}{\sqrt{2}}$	0
-1	-1		0	0	0	0	1	0	0	0	0

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1. [20 points] Perturbed HO

A particle is bound in the harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$. A static perturbation $H' = \gamma x^2$ is applied to the system.

(a) Calculate the first-order corrections to (all) the energy eigenvalues using the perturbation formalism.

(b) What is the exact energy? Explain why your answer is consistent with (a) above.

2. [20 points] Degenerate perturbation theory

A particle in a two-dimensional harmonic oscillator potential $U(x,y) = \frac{1}{2}m\omega_0^2\left(x^2 + y^2\right)$ has energy eigenstates that are characterized by TWO quantum numbers n_x and n_y, each of which can have values 0, 1, 2, 3, … ∞ .

The energy eigenstates are products of the 1-d energy eigenstates:

$$\left|n_x, n_y\right\rangle \doteq \varphi_{n_x}(x)\varphi_{n_y}(y)$$

with corresponding energy eigenvalues

$$E_{n_x,n_y} = \left(n_x + \tfrac{1}{2}\right)\hbar\omega_0 + \left(n_y + \tfrac{1}{2}\right)\hbar\omega_0 = \left(n_x + n_y + 1\right)\hbar\omega_0$$

(a) (5 points) What is the energy of the first excited state (the next state above the lowest-energy state)? Why is this energy level degenerate, and what is the degeneracy of the level?

(b) (15 points) Calculate the new energy eigenvalues and eigenstates (to first order) of these degenerate states that comprise the first excited state when a small perturbation to the Hamiltonian of the form $H'(x) = U_0\sqrt{\dfrac{\hbar}{m\omega}}\ \delta(x)$ is applied where $\delta(x)$ is the Dirac delta function. Proceed as follows: Construct the matrix that represents the perturbation H' in the degenerate subspace and use it to determine the first-order changes, if any, to the <u>energy eigenvalues</u> and the <u>eigenstates</u>.

3. [30 points] Fine structure

In the simplest treatment of the hydrogen atom, the electron and proton experience only the Coulomb interaction, and the energy eigenstates are designated $\left|n\ \ell\ m_\ell\ s\ m_s\right\rangle$.

(a) What are the allowed values of these quantum numbers?

(b) If the system is in the "2p" state, what are the particular values of the quantum numbers and what is the (unperturbed) energy of the 2p state? What is its degeneracy?

(c) In class we used degenerate perturbation theory to show that adding interactions to the Hamiltonian introduced "fine structure" to the H-atom spectrum and that the fine-structure corrections are $E_{fs}^{(1)} = -\dfrac{1}{2}\alpha^4 mc^2 \dfrac{1}{n^3}\left[\dfrac{1}{j+\frac{1}{2}} - \dfrac{3}{4n}\right]$.

Use this expression to show how the unperturbed 2p levels are changed: draw a diagram that shows the new level(s), correctly labeled, and find the size of the correction(s) in eV.

(d) What wavelength resolution is necessary to resolve the fine structure in the 1s <–> 2p transition? Give a fractional value rather than an absolute wavelength.

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4. [20 points] Two spin systems

Two particles "A" and "B" form the quantum system of interest. Only the spin angular momentum is relevant in this problem. A is spin-1/2 particle and B is a spin-1 particle.

(a) Write down all possible kets that describe the state of system in the
  (i) coupled, and
  (ii) uncoupled bases. (State what the numbers in the kets represent in both cases).

(b) Use the Clebsch-Gordan coefficients to write the coupled states as linear combinations of the uncoupled states.

(c) If the system is in the state where the total angular momentum is given by $j = 3/2$, and the z-component of the total angular momentum is $m_j = -1/2$, what is (and explain why) the probability that a measurement of the z-component of the angular momentum of B yields:
  (i) $\hbar$,
  (ii) $0\hbar$,
  (iii) $-\hbar$?

5. [20 points] Identical particles

Two indistinguishable, non-interacting, uncharged spin-1/2 fermions are confined to a region of length L in one-dimension. In your answers, explain your reasoning.

(a) Find the energy eigenvalue(s) of the ground state.
(b) Find the energy eigenstate(s) of the ground state.
(c) Specify and discuss the degeneracy of the ground state.
(d) Find the energy eigenvalue(s) of the first excited state.
(e) Find the energy eigenstate(s) of the first excited state.
(f) Specify and discuss the degeneracy of the first excited state.

END OF EXAM