L1 – Center of mass, reduced mass, angular momentum

Read course notes sections 2-6

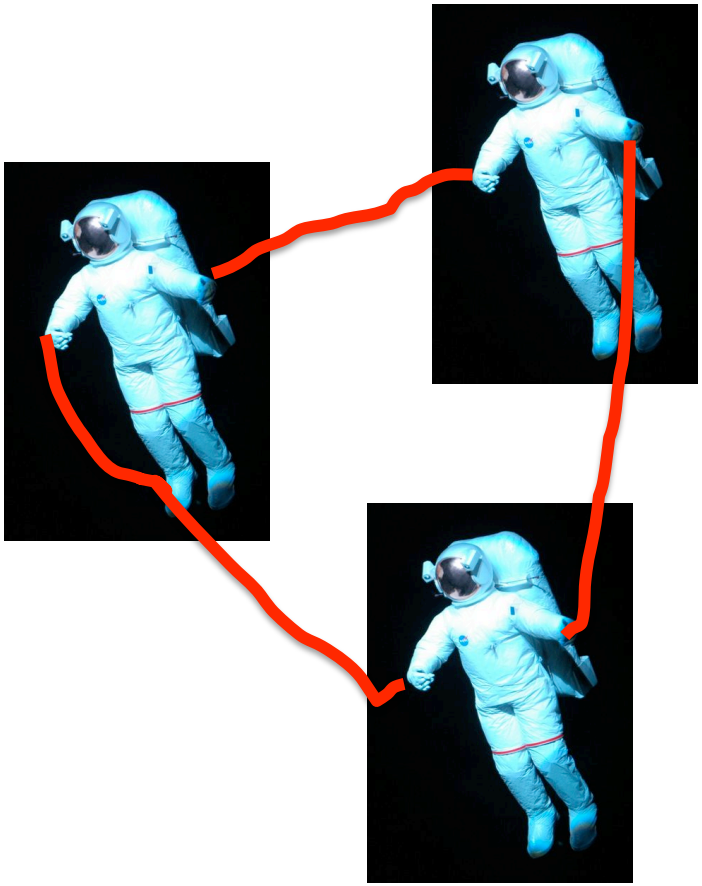
Taylor 3.3-3.5

Newton's Law for 3 bodies (or *n*)

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = \vec{F}_1 + 0 + \vec{f}_{12} + \vec{f}_{13}$$

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = \vec{F}_2 + \vec{f}_{21} + 0 + \vec{f}_{23}$$

$$m_3 \frac{d^2\vec{r}_3}{dt^2} = \vec{F}_3 + \vec{f}_{31} + \vec{f}_{32} + 0$$



What are $m_i\,?$ $\vec{r}_i\,?$ $\vec{F}_i\,?$ $\vec{f}_{ij}\,?$

Center of mass & its motion

- Position of center of mass is the (mass) weighted average of individual mass positions

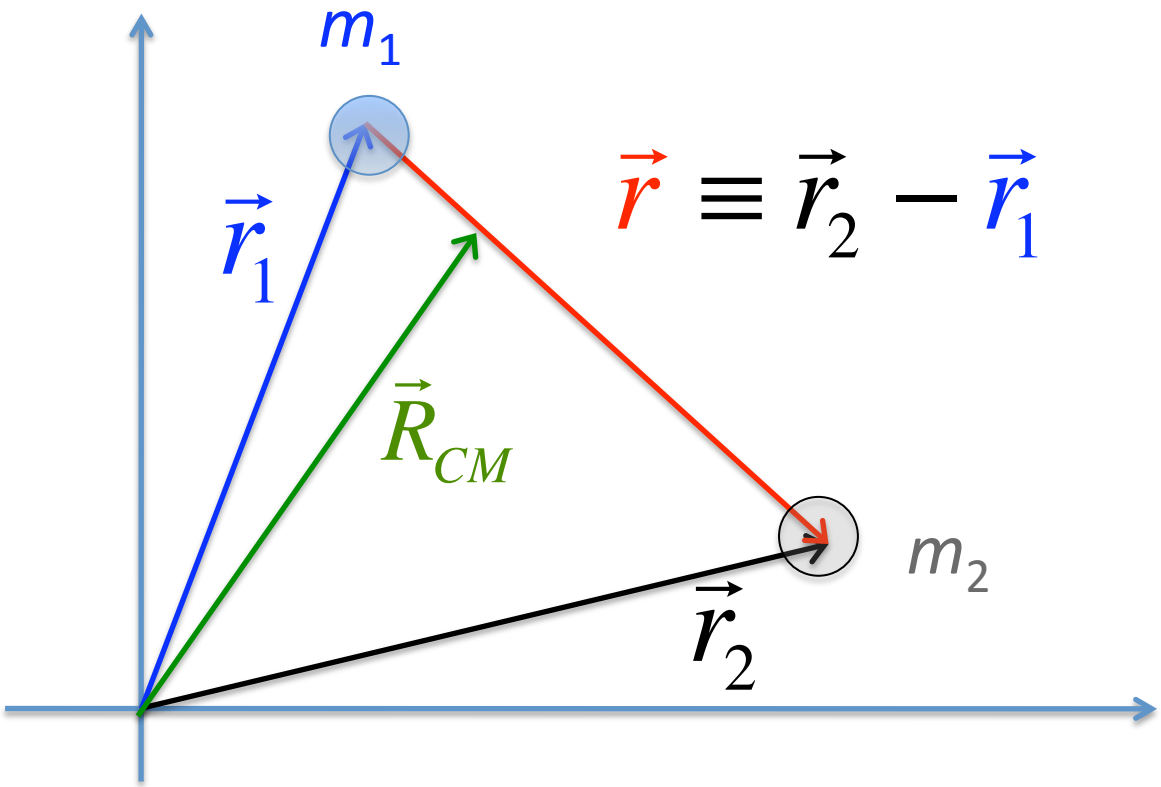
$$\vec{R}_{CM} \equiv \frac{m_1}{M}\vec{r}_1 + \frac{m_1}{M}\vec{r}_2 + \frac{m_1}{M}\vec{r}_3 + \ldots = \sum_n \frac{m_n}{M}\vec{r}_n$$

$$\sum_n m_n \frac{d^2\vec{r}_n}{dt^2} = \sum_n \vec{F}_n$$

$$M\frac{d^2\vec{R}}{dt^2} = \sum_n \vec{F}_n \qquad\qquad M\frac{d^2\vec{R}}{dt^2} = 0 \Rightarrow ?$$

2-body problem is "solvable"
CoM

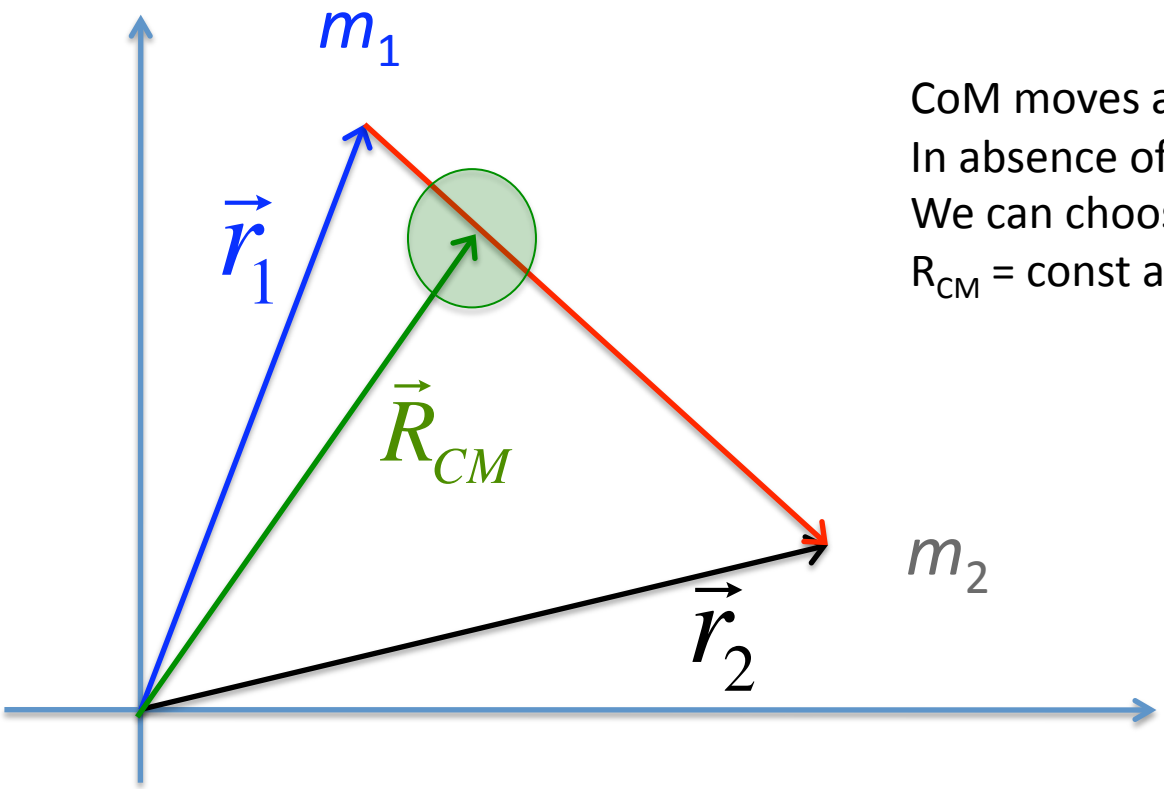
$$R_{CM} \equiv \frac{m_2}{M}\vec{r}_2 + \frac{m_1}{M}\vec{r}_1$$



$$\vec{r} \equiv \vec{r}_2 - \vec{r}_1$$

2-body problem is "solvable"
CoM

$$R_{CM} \equiv \frac{m_2}{M}\vec{r}_2 + \frac{m_1}{M}\vec{r}_1$$



CoM moves as if all the mass is located there.
In absence of ext forces, V_{CM} = constant
We can choose const vel = 0
R_{CM} = const and we can choose = 0.

2-body problem is "solvable" relative coordinates

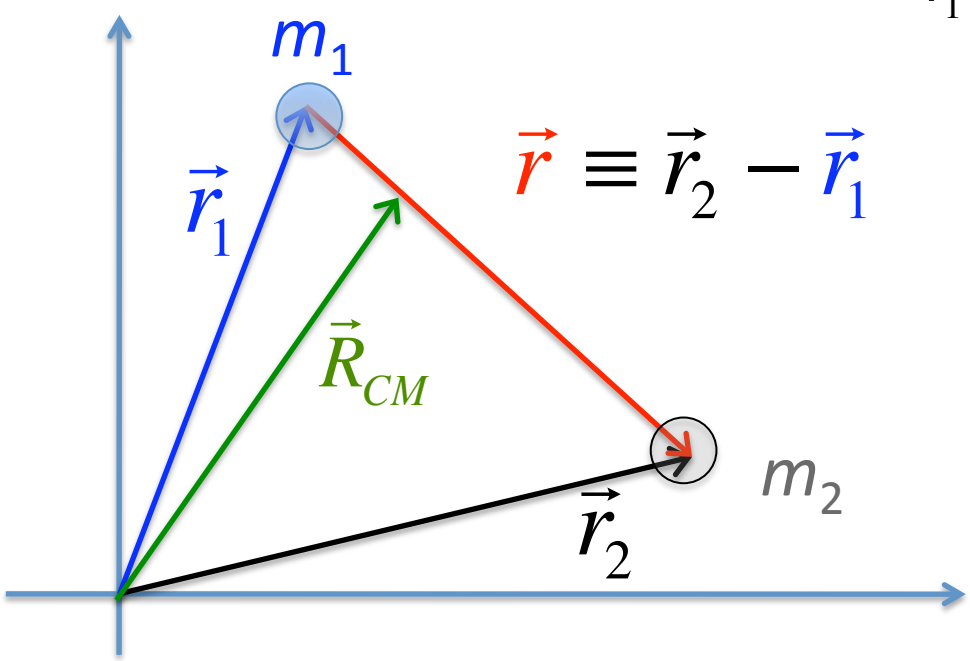
REDUCED MASS

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

$$R_{CM} \equiv \frac{m_2}{M}\vec{r}_2 + \frac{m_1}{M}\vec{r}_1 \qquad \overbrace{= 0}^{\text{can choose if Fext=0}}$$

$$\vec{r}_2 = \frac{m_1}{m_1 + m_2}\vec{r}\,; \qquad m_2\vec{r}_2 = \mu\vec{r}$$

$$\vec{r}_1 = -\frac{m_2}{m_1 + m_2}\vec{r}\,; \qquad m_1\vec{r}_1 = -\mu\vec{r}$$



$$\vec{r} \equiv \vec{r}_2 - \vec{r}_1$$

Check limits, directions for sense; Students to derive mathematically at home.

2-body problem is "solvable" relative coordinates

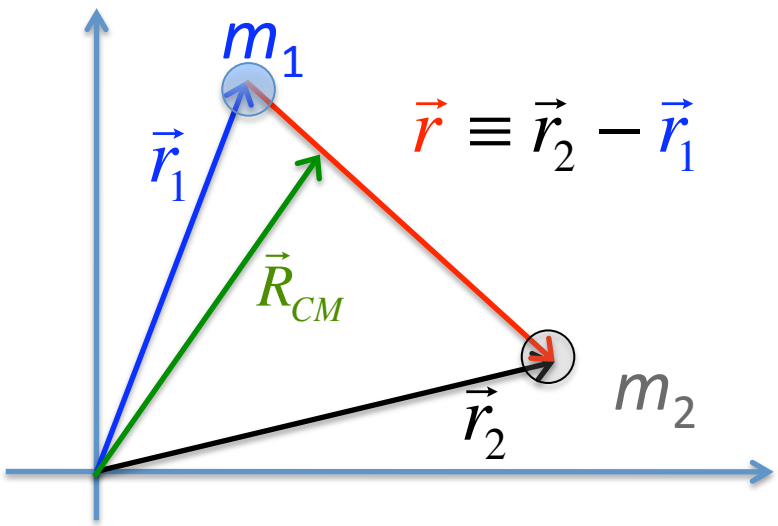
$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

$$m_2 \vec{r}_2 = \mu \vec{r}\,; \quad m_1 \vec{r}_1 = -\mu \vec{r}$$

$$m_1 \frac{d^2 \vec{r}_1}{dt^2} = \vec{f}_{12}$$

$$m_2 \frac{d^2 \vec{r}_2}{dt^2} = \vec{f}_{21}$$

$$\mu \frac{d^2 \vec{r}}{dt^2} = \vec{f}_{21} = f(r)\hat{r} \text{ for central force}$$



Relative motion obeys a single-particle equation. Fictitious particle, mass μ moving in a central force field. Vector r is measures RELATIVE displacement; we have to "undo" to get actual motion of each of particles 1 and 2. Often m1>>m2.

Central force

- Define central force

$$\vec{f}_{21} = -\vec{f}_{12} = f(r)\hat{r}; \quad \text{gravity: } f(r) = -\frac{Gm_1m_2}{r^2}$$

- f_{12} reads "the force on 1 caused by 2"
- Depends on magnitude of separation and *not* orientation
- Points towards origin in a 1-particle system
- Derivable from potential (conservative)

$$\vec{f}_{12} = -\nabla U(r); \quad \text{gravity: } U(r) = -\frac{Gm_1m_2}{r}$$

- Work done is path independent

Angular momentum

- Define (must specify an origin!)

$$\vec{L} \equiv \vec{r} \times \vec{p}; \quad \vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{L}_{TOT} = \sum_i \vec{r}_i \times \vec{p}_i$$

$$\vec{L}_{TOT} = \sum_i \vec{r}_i \times \vec{p}_i = \vec{r}_1 \times m_1 \frac{d\vec{r}_1}{dt} + \vec{r}_2 \times m_2 \frac{d\vec{r}_2}{dt}$$

$$= \vec{r}_1 \times \left(-\mu \frac{d\vec{r}}{dt} \right) + \vec{r}_2 \times \mu \frac{d\vec{r}}{dt}$$

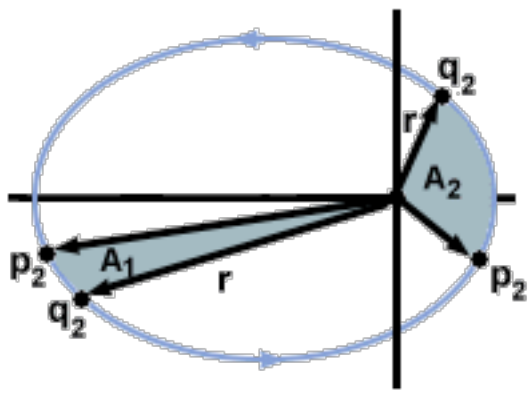
$$= \left(\vec{r}_1 - \vec{r}_2 \right) \times \mu \frac{d\vec{r}}{dt}$$

$$\vec{L}_{TOT} = \vec{r} \times \mu \vec{v}; \qquad \frac{d\vec{L}_{TOT}}{dt} = 0$$

Show the last statement is true for central force.
Angular momentum is conserved if no external TORQUE.

Conservation of AM & Kepler's 2nd law

- Equal areas equal times (cons. angular mom.)
- Later (once we've explored polar coordinates).



Summary

- Define R_{CM}; total mass M
- Define r, relative coord; $\mu = m_1 m_2 / M$
- No ext forces => CoM constant momentum
- No external torque => AM conserved
- AM conserved -> what is value? (see later)
- AM conserved -> Orbit in a plane (hwk)