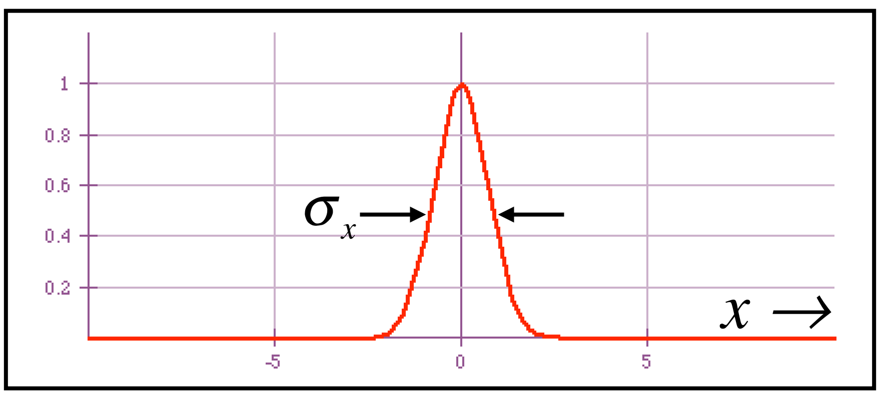
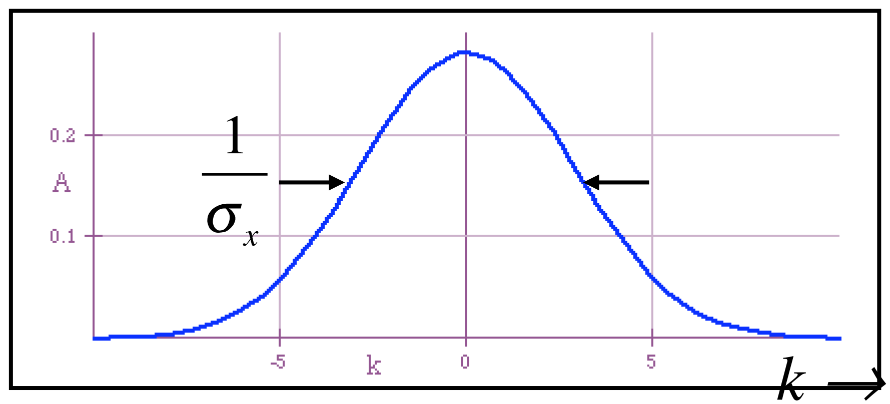
WAVE PACKETS & SUPERPOSITION

Reading:
Main 9.
PH421 Fourier notes





Non dispersive wave equation

$$\frac{\partial^2}{\partial x^2}\psi(x,t)=\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(x,t)$$

So far, we know that this equation results from
- application of Newton's law to a taut rope
- application of the Maxwell equations to a dielectric medium

(What do the quantities represent in each case?)

So far we've discussed single-frequency waves, but we did an experiment with a pulse … so let's look more formally at *superposition*. You must also review Fourier discussion from PH421.

Example: Non dispersive wave equation

$$\frac{\partial^2}{\partial x^2}\psi(x,t)=\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(x,t)$$

$$\psi(x,t)=A\cos kx\cos\omega t+B\cos kx\sin\omega t$$
$$+C\sin kx\cos\omega t+D\sin kx\sin\omega t$$

$$\psi(x,t)=F\cos\left(kx-\omega t+\phi\right)+G\cos\left(kx+\omega t+\theta\right)$$

$$\psi(x,t)=He^{i(kx-\omega t)}+H^{*}e^{-i(kx-\omega t)}+Je^{i(kx+\omega t)}+J^{*}e^{-i(kx+\omega t)}$$

$$\psi(x,t)=\mathrm{Re}\left[Le^{i(kx-\omega t)}\right]+\mathrm{Re}\left[Me^{i(kx+\omega t)}\right]$$

With $v = \omega/k$, but we are at liberty (so far…) to pick ANY ω!
A, B, C, D; and F, G, ϕ, θ arbitrary real constants;
H, J, L, M arbitrary complex constants.

Example: Non dispersive wave equation

$$\frac{\partial^2}{\partial x^2}\psi(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(x,t)$$

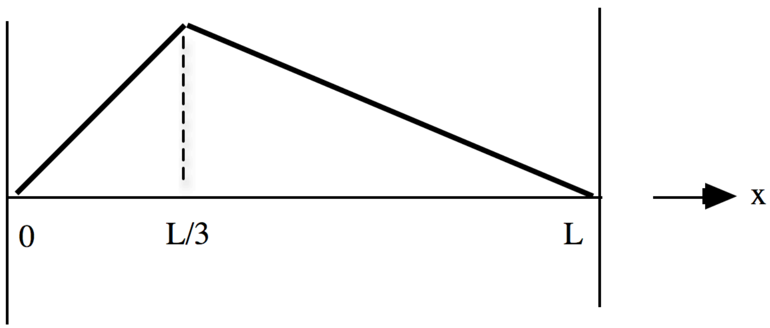
A general solution is the superposition of solutions of all possible frequencies (or wavelengths). So any shape is possible!

$$\psi(x,t) = \sum_{\omega} \begin{matrix} A_\omega \cos\frac{\omega}{v}x\cos\omega t + B_\omega \cos\frac{\omega}{v}x\sin\omega t \\ +C_\omega \sin\frac{\omega}{v}x\cos\omega t + D_\omega \sin\frac{\omega}{v}x\sin\omega t \end{matrix}$$

$$\psi(x,t) = \int\left\{\mathrm{Re}\left[L(k)e^{i(kx-vkt)}\right] + \mathrm{Re}\left[M(k)e^{i(kx+vkt)}\right]dk\right\}$$

Example 1:

Wave propagating in rope (phase vel. v) with fixed boundaries at $x = 0, L$. (Finish for homework)



$$\psi(x,0) = \text{given; what is it?}$$

$$\left.\frac{\partial \psi(x,t)}{\partial t}\right|_{t=0} = \text{given; what is it?}$$

Which superposition replicates the initial shape of the wave at $t = 0$? How can we choose coefficients A_ω, B_ω, C_ω, D_ω to replicate the initial shape & movement of the wave at $t = 0$?

$$\psi(x,t) = \sum_\omega \begin{array}{l} A_\omega \cos\frac{\omega}{v}x\cos\omega t + B_\omega \cos\frac{\omega}{v}x\sin\omega t \\ +C_\omega \sin\frac{\omega}{v}x\cos\omega t + D_\omega \sin\frac{\omega}{v}x\sin\omega t \end{array}$$

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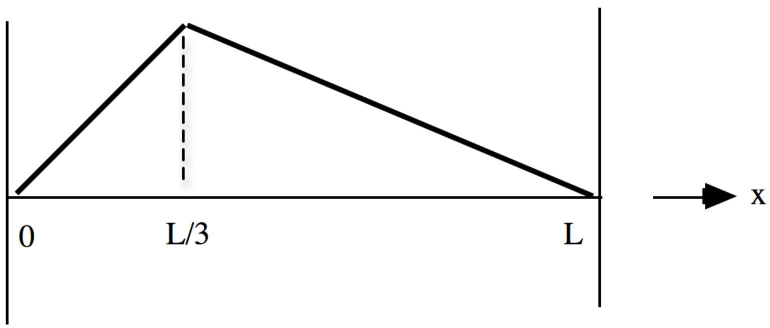

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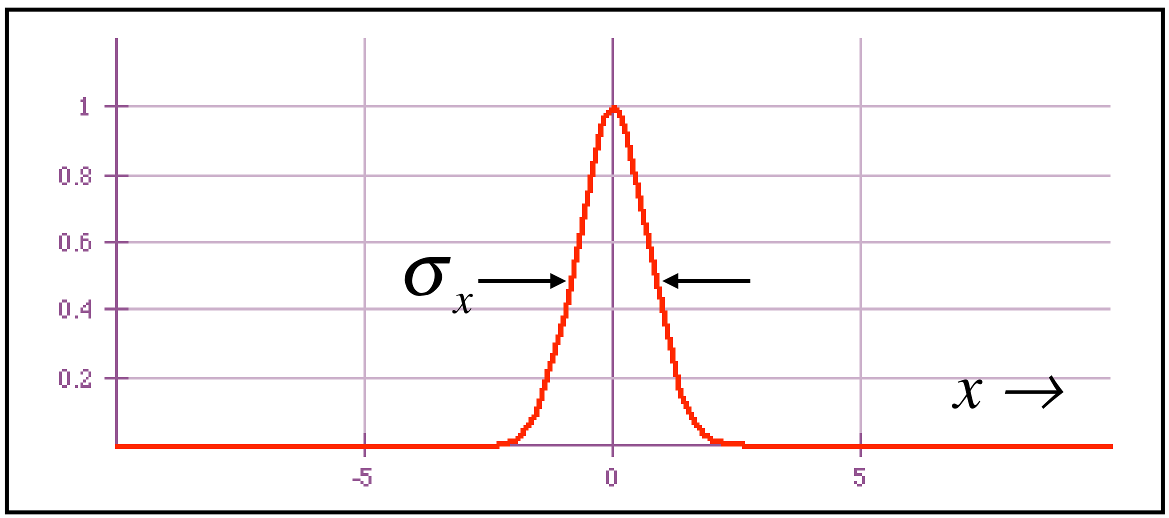
Example 2:
Class activity - build a Gaussian wave packet

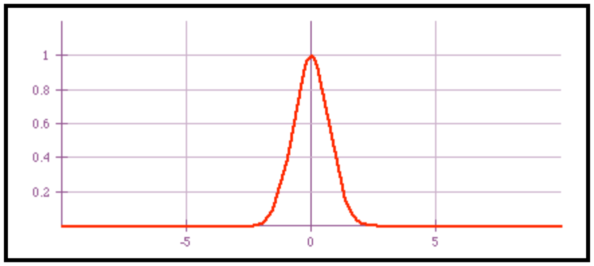
This shape can propagate in a rope, right? Therefore it must be a solution to the wave equation for the rope. It's obviously not a single-frequency harmonic. Then which superposition is it? How is this problem different from the fixed-end problem we recently discussed?

$$\psi(x,t) = A_0 e^{-\frac{(x-vt)^2}{2\sigma_x^2}}$$

$$\psi(x,0) = A_0 e^{-\frac{x^2}{2\sigma_x^2}}$$



Gaussian wave packet



$$\psi(x,t) = A_0 e^{-\frac{(x-vt)^2}{2\sigma_x^2}}$$

Which superposition replicates the initial shape of the wave at $t = 0$?

$$\psi(x,0) = A_0 e^{-\frac{x^2}{2\sigma_x^2}}$$

$$\psi(x,0) = \int_{-\infty}^{\infty} dk\, L(k) e^{ikx}$$

How can we choose coefficients $L(k)$ to replicate the initial shape of the wave at $t = 0$?
(note we dropped the "Re" - we'll put it back at the end)

Build a Gaussian wave packet

$$\psi(x,0) = \int_{-\infty}^{\infty} dk\, L(k) e^{ikx}$$

$$A_0 e^{-\frac{x^2}{2\sigma_x^2}} = \int_{-\infty}^{\infty} dk\, L(k) e^{ikx}$$

Plug in for $\psi(x,0)$

$$L(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, A_0 e^{-\frac{x^2}{2\sigma_x^2}} e^{-ikx}$$

Recognize a FT, and invert it (this is a big step, we have to review PH421 and connect to previous example)

$$\int_{-\infty}^{\infty} dy\, e^{vy} e^{-uy^2} = \sqrt{\frac{\pi}{u}}\, e^{\frac{v^2}{4u}}$$

← Here's a handy integral from a table

Now evaluate *L(k)*

Build a Gaussian wave packet

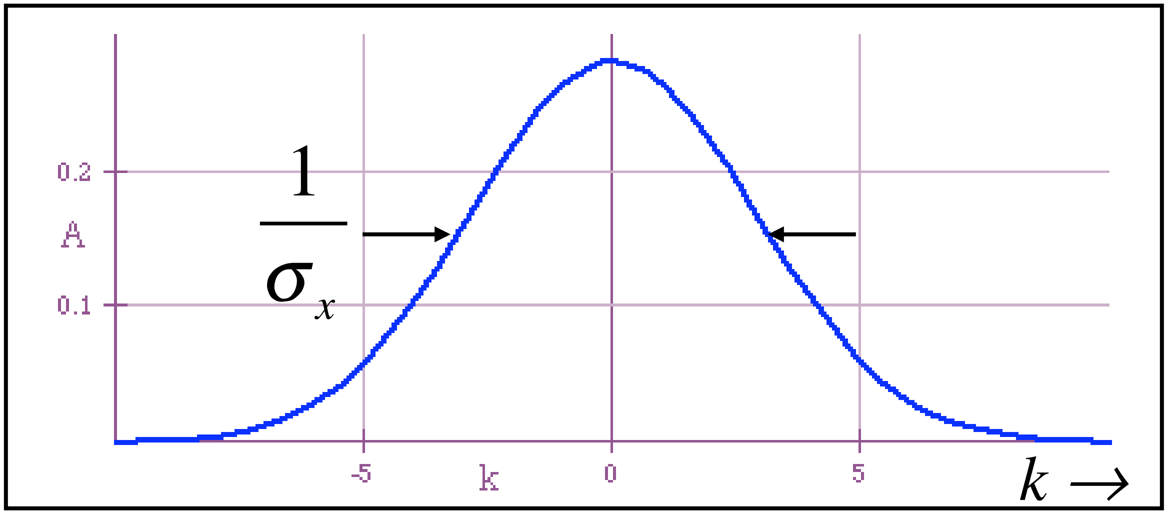
$$L(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, A_0 e^{-\frac{x^2}{2\sigma_x^2}} e^{-ikx}$$

$$\int_{-\infty}^{\infty} dy\, e^{vy} e^{-uy^2} = \sqrt{\frac{\pi}{u}}\, e^{\frac{v^2}{4u}}$$

$$v \to ik; u \to \frac{1}{2\sigma_x^2}$$

$$L(k) = \frac{1}{2\pi}\sqrt{\frac{\pi}{\frac{1}{2\sigma_x^2}}}\, e^{\frac{(ik)^2}{4\left(\frac{1}{2\sigma_x^2}\right)}}$$

$$L(k) = \sqrt{\frac{\sigma_x^2}{2\pi}}\, e^{-\frac{\sigma_x^2 k^2}{2}}$$



Build a Gaussian wave packet

$$\psi(x,0)=\int_{-\infty}^{\infty} dk\, L(k)e^{ikx}$$

$$\psi(x,0)=\int_{-\infty}^{\infty} dk\sqrt{\frac{\sigma_x^2}{2\pi}}\,e^{-\frac{\sigma_x^2k^2}{2}}\,e^{ikx}$$

Now put back the time dependence … remember each *k*-component evolves with its own velocity … which just happens to be the same for this non-dispersive case

$$\psi(x,t)=\int_{-\infty}^{\infty} dk\sqrt{\frac{\sigma_x^2}{2\pi}}\,e^{-\frac{\sigma_x^2k^2}{2}}\,e^{i(kx-kvt)}$$

$$\psi(x,t)=A_0e^{-\frac{(x-vt)^2}{2\sigma_x^2}}$$

Build a Gaussian wave packet - summary

$$\psi(x,t) = A_0 e^{-\frac{(x-vt)^2}{2\sigma_x^2}}$$

Initial pulse - shape described by Gaussian spatial function

$$\psi(x,t) = \int_{-\infty}^{\infty} dk \sqrt{\frac{\sigma_x^2}{2\pi}}\, e^{-\frac{\sigma_x^2 k^2}{2}} e^{i(kx-kvt)}$$

Written in terms of sinusoids of different wavelengths (and freqs)

There is a general feature of a packet of any shape: To make a narrow pulse in space, we need a wide distribution of *k*-values; to make a wide pulse in space, a narrow range of *k* values is necessary. The Gaussian spatial profile is special – it happens that the "strength" of each *k*-contribution is also a Gaussian distribution in *k*-space.

Build a Gaussian wave packet - summary

$$\psi(x,t) = A_0 e^{-\frac{(x-vt)^2}{2\sigma_x^2}}$$

Initial pulse - shape described by Gaussian spatial function

$$\psi(x,t) = \int_{-\infty}^{\infty} dk \sqrt{\frac{\sigma_x^2}{2\pi}}\, e^{-\frac{\sigma_x^2 k^2}{2}} e^{i(kx-kvt)}$$

Written as a “sum” of “sinusoids” of different wavelengths (and freqs)

- Important to be able to deconstuct an arbitrary waveform into its constituent single-frequency or single-wavevector components. E.g. R and T values depend on k. Can find out how pulse propagates.
- Especially important when the relationship between k and ω is not so simple - *i.e.* the different wavelength components travel with different velocities (“dispersion”) as we will find in the QM discussion

Build a Gaussian wave packet - check

$$\psi(x,t)=\int_{-\infty}^{\infty} dk\sqrt{\frac{\sigma_x^2}{2\pi}}\,e^{-\frac{\sigma_x^2k^2}{2}}e^{i(kx-kvt)}\qquad \int_{-\infty}^{\infty} dy\,e^{vy}e^{-uy^2}=\sqrt{\frac{\pi}{u}}\,e^{\frac{v^2}{4u}}$$

$$\psi(x,t)=\sqrt{\frac{\pi}{\frac{\sigma_x^2}{2}}}\sqrt{\frac{\sigma_x^2}{2\pi}}\,e^{\frac{-(x-vt)^2}{4\frac{\sigma_x^2}{2}}}$$

$$\psi(x,t)=A_0e^{-\frac{(x-vt)^2}{2\sigma_x^2}}$$

Group velocity

With what velocity does a recognizable feature of a wave packet travel? This velocity is called the group velocity - the velocity of the group of waves.

For our rope example, it's rather trivial. Every wave of different wavelength and hence different frequency that goes into the packet has the same phase velocity. Thus all the components of the packet move at the same speed and the shape remains the same.

In this case, the group velocity, $d\omega/dk$, is the same as the phase velocity ω/k. But this is no longer true when components of different wavelengths travel at different frequencies. Coming soon to a theatre near you ……..

WAVE PACKETS & SUPERPOSITION - REVIEW

- *Other shapes than sinusoids can propagate in systems*
- *Other shapes are superpositions of sinusoids*
- *Fourier integrals/series for unconstrained/constrained conditions*
- *Fourier transform of a narrow/wide Gaussian is a wide/ narrow Gaussian*
- *Group velocity, phase velocity*
- *Mathematical representations of the above*