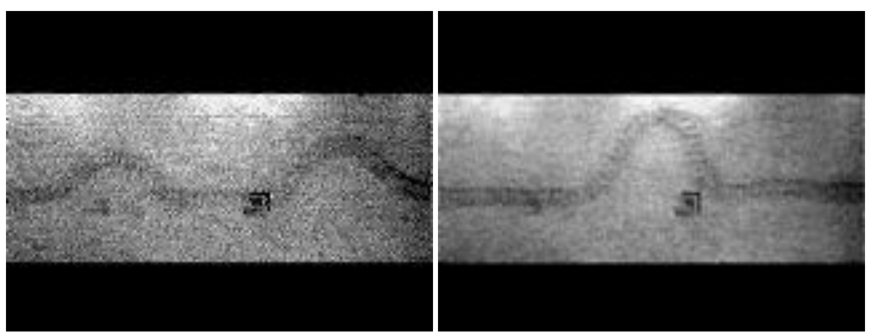
REFLECTION AND TRANSMISSION

Reading:
Main 9.2
GEM 9.1.3



$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi(x,t)}{\partial t^2}$$

What happens when a wave encounters a medium where it propagates with a different velocity?

http://www.kettering.edu/~drussell/Demos/reflect/reflect.html

- Animations courtesy of Dr. Dan Russell, Kettering University

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi(x,t)}{\partial t^2}$$

$$v = \frac{\omega}{k}$$

$$v_1 = \frac{\omega}{k_1} \qquad v_2 = \frac{\omega}{k_2}$$

Medium changes ($i.e$ v changes) at $x = 0$

v is one constant for $x < 0$ and another constant for $x > 0$.

Need piecewise function for ψ

Traveling wave solutions, with wave incident from the left

Frequency must be same on both sides (why?), therefore k changes (and λ)

$$\psi_{inc}(x,t) = \mathrm{Re}\left[Ae^{i(-\omega t + k_1 x)} \right]$$

$$\psi_{ref}(x,t) = \mathrm{Re}\left[Be^{i(-\omega t - k_1 x)} \right]$$

$$\psi_{trans}(x,t) = \mathrm{Re}\left[Ce^{i(-\omega t + k_2 x)} \right]$$

$$\psi_{Left}(x,t) = \psi_{inc}(x,t) + \psi_{ref}(x,t) \qquad \psi_{Right}(x,t) = \psi_{trans}(x,t)$$

$$\psi_{left}(x,t) = \mathrm{Re}\left[Ae^{i(-\omega t + k_1 x)} \right] + \mathrm{Re}\left[Be^{i(-\omega t - k_1 x)} \right]$$

$$\psi_{right}(x,t) = \mathrm{Re}\left[Ce^{i(-\omega t + k_2 x)} \right]$$

$$\psi_{Left}(0,t) = \psi_{Right}(0,t)$$ ← Rope must be continuous

$$\boxed{A + B = C}$$

$$\left.\frac{\partial \psi_{Left}}{\partial x}\right|_{x=0,t} = \left.\frac{\partial \psi_{Right}}{\partial x}\right|_{x=0,t}$$

Transverse force on vanishingly small rope element (massless) must be zero

$$\boxed{iAk_1 - iBk_1 = iCk_2}$$

$$\psi_{Left}(0,t) = \psi_{Right}(0,t)$$ $\leftarrow$ Rope must be continuous

$$\mathrm{Re}\left[Ae^{i(-\omega t)}\right] + \mathrm{Re}\left[Be^{i(-\omega t)}\right] = \mathrm{Re}\left[Ce^{i(-\omega t)}\right]$$

$$A\cos(\omega t) + B\cos(\omega t) = C\cos(\omega t)$$

$$A + B = C$$

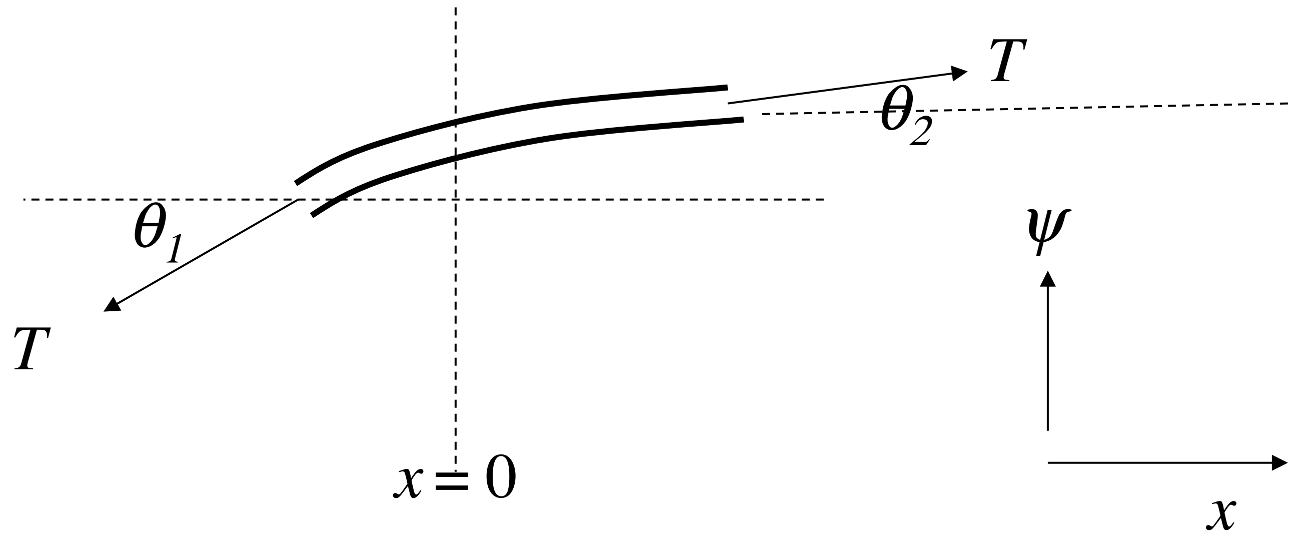
$$\left.\frac{\partial \psi_{Left}}{\partial x}\right|_{x=0,t} = \left.\frac{\partial \psi_{Right}}{\partial x}\right|_{x=0,t}$$ $\leftarrow$ Transverse force on massless rope element is zero

$$\mathrm{Re}\left[ik_1 Ae^{i(-\omega t)}\right] + \mathrm{Re}\left[-ik_1 Be^{i(-\omega t)}\right] = \mathrm{Re}\left[ik_2 Ce^{i(-\omega t)}\right]$$

$$k_1 A\sin(\omega t) - k_1 B\sin(\omega t) = k_2 C\sin(\omega t)$$

$$k_1 A - k_1 B = k_2 C$$

Force exerted by left side of rope on right side is $-T\left.\frac{\partial\psi_{Left}}{\partial x}\right|_{x=x_0}$



Force exerted by right side of rope on left side is $+T\left.\frac{\partial\psi_{Right}}{\partial x}\right|_{x=x_0}$

$$\boxed{iAk_1 - iBk_1 = iCk_2}$$

Transverse force on vanishingly small rope element (massless) must be zero

$$\boxed{A + B = C \qquad Ak_1 - Bk_1 = Ck_2}$$

Solve simultaneously

$$\boxed{R_\psi \equiv \frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}} \quad \boxed{T_\psi \equiv \frac{C}{A} = \frac{2k_1}{k_1 + k_2}}$$

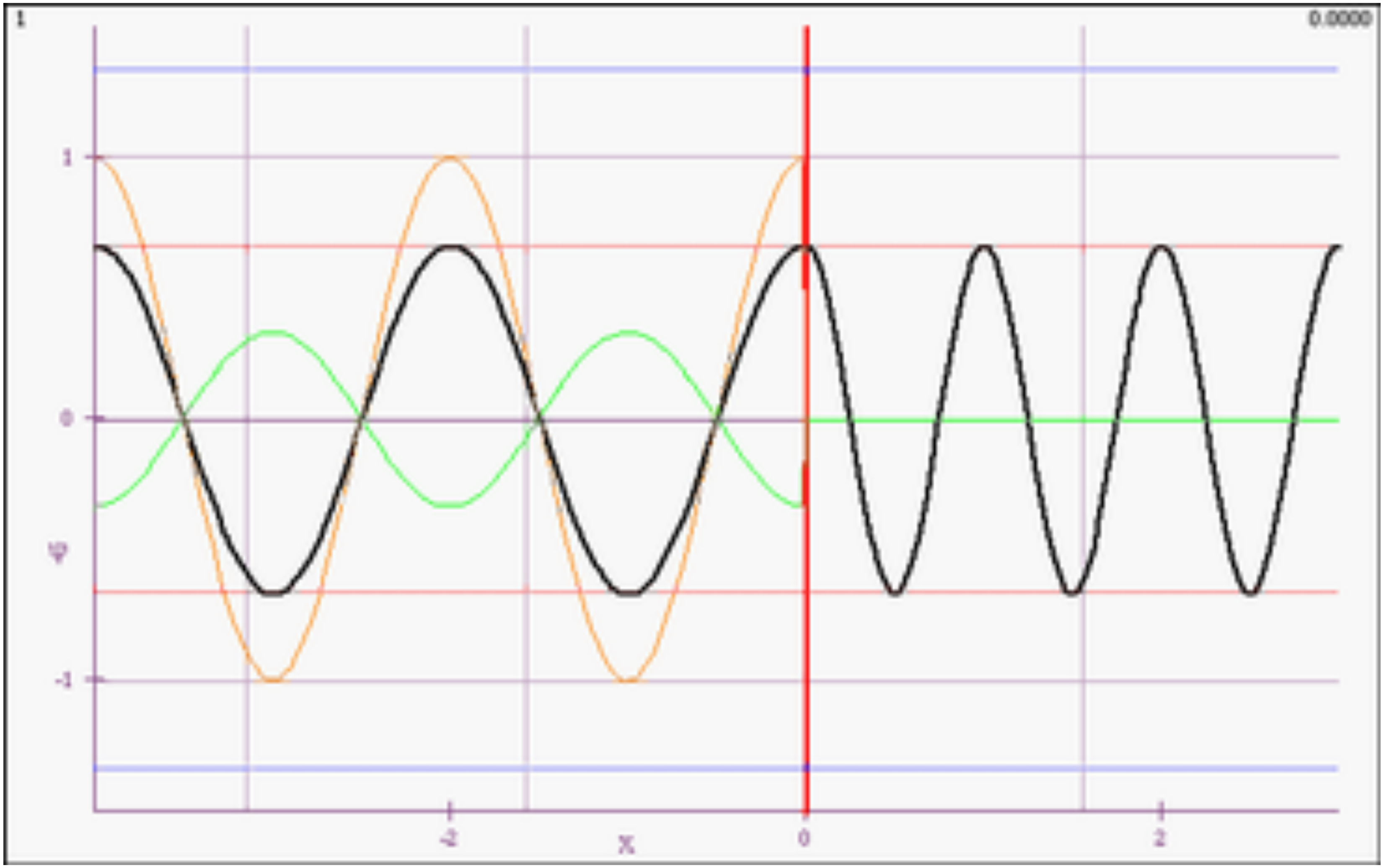
Displacement reflection and transmission coeffs!

$$\psi_{Left}(x,t) = e^{i(-\omega t + k_1 x)} + \frac{k_1 - k_2}{k_1 + k_2} e^{i(-\omega t - k_1 x)}$$

$$\psi_{Right}(x,t) = \frac{2k_1}{k_1 + k_2} e^{i(-\omega t + k_2 x)}$$

Boundary

$v_1 > v_2$ $v_2 < v_1$



Incident $(\rightarrow)$ + *reflected* $(\leftarrow)$ *Transmitted* $(\rightarrow)$

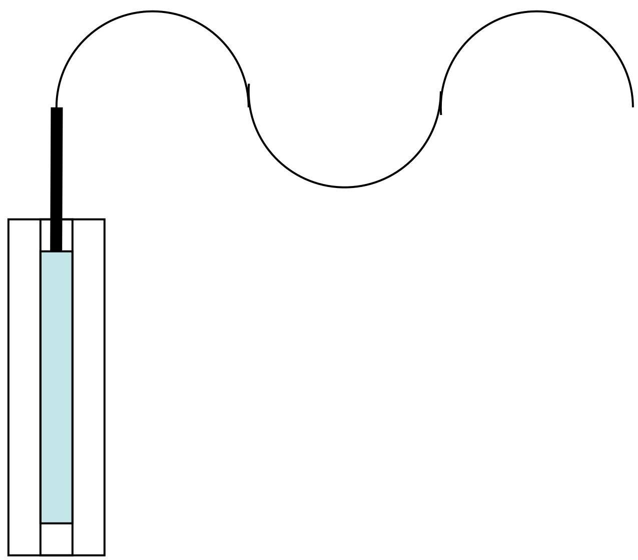
Impedance, Z

$$F_{appl} \equiv Z\frac{\partial\psi}{\partial t}$$

Defines Z as the ratio of the applied force to the resulting (material) velocity

For rope system: piston applies force at $x = 0$ producing wave in direction of +ve x
(τ = tension)

$$-\tau\frac{\partial\psi}{\partial x} = Z\frac{\partial\psi}{\partial t}$$



Impedance, Z

is *defined* as the ratio of the applied force to the resulting (material) velocity

$$-\tau \frac{\partial \psi}{\partial x} \equiv Z \frac{\partial \psi}{\partial t}$$

$$\frac{\partial \psi(x - vt)}{\partial t} = -v \frac{\partial \psi(x - vt)}{\partial x}$$

Traveling wave eqn

$$\Rightarrow \quad -\tau = -Zv$$

$$\boxed{Z = \frac{\tau}{v} = \frac{\tau k}{\omega}}$$

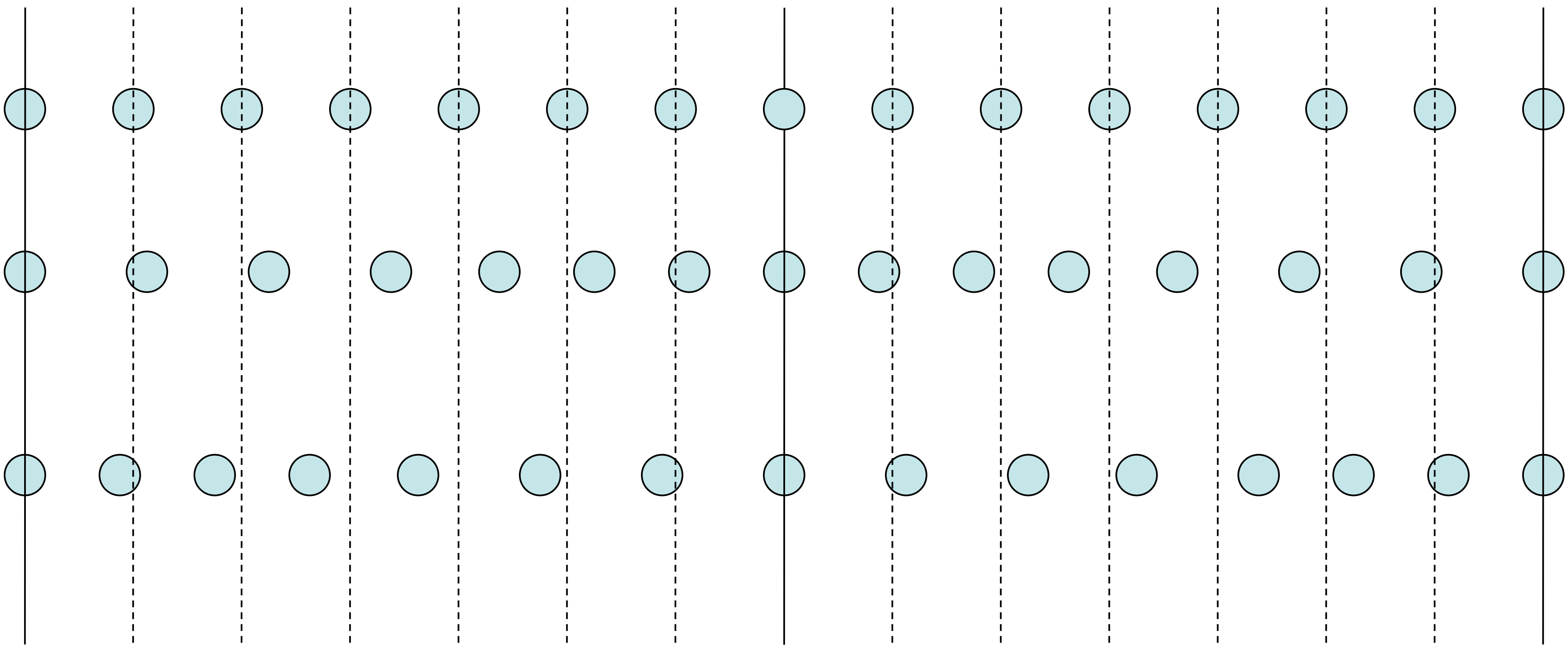
$$\boxed{Z_{rope} = \frac{\tau}{v} = \sqrt{\tau \mu}}$$

Impedance proportional to k *i*f τ is constant.

$$\psi_{Left}(x,t) = \mathrm{Re}\left[e^{i(-\omega t + k_1 x)}\right] + \mathrm{Re}\left[\frac{Z_1 - Z_2}{Z_1 + Z_2} e^{i(-\omega t - k_1 x)}\right]$$

$$\psi_{Right}(x,t) = \mathrm{Re}\left[\frac{2Z_1}{Z_1 + Z_2} e^{i(-\omega t + k_2 x)}\right]$$

Pressure/force and displacement



Exercise: Draw $\psi(x,0)$; $p(x,0)$

$$F(x,t) = -\tau \frac{\partial \psi(x,t)}{\partial x}$$

Also obeys the 1-D wave equation $\Rightarrow$ it has reflection and transmission coefficients, too!

$$F_{Left}(x,t) = \boxed{-\tau i k_1 A} e^{i(-\omega t + k_1 x)} \boxed{+ \tau i k_1 B} e^{i(-\omega t - k_1 x)}$$

$$F_{Right}(x,t) = \boxed{-\tau i k_2 C} e^{i(-\omega t + k_2 x)}$$

$$\boxed{R_F \equiv \frac{i k_1 B}{-i k_1 A} = \frac{k_2 - k_1}{k_1 + k_2}}$$

$$\boxed{T_F \equiv \frac{-i k_2 C}{-i k_1 A} = \frac{2 k_2}{k_1 + k_2}}$$

Force reflection and transmission coeffs!

$$\psi_{Left}(x,t) = \mathrm{Re}\left[e^{i(-\omega t + k_1 x)}\right] + \mathrm{Re}\left[\frac{Z_1 - Z_2}{Z_1 + Z_2} e^{i(-\omega t - k_1 x)}\right]$$

$$\psi_{Right}(x,t) = \mathrm{Re}\left[\frac{2Z_1}{Z_1 + Z_2} e^{i(-\omega t + k_2 x)}\right]$$

$$F_{Left}(x,t) = \mathrm{Re}\left[e^{i(-\omega t + k_1 x)}\right] + \mathrm{Re}\left[\frac{Z_2 - Z_1}{Z_1 + Z_2} e^{i(-\omega t - k_1 x)}\right]$$

$$F_{Right}(x,t) = \mathrm{Re}\left[\frac{2Z_2}{Z_1 + Z_2} e^{i(-\omega t + k_2 x)}\right]$$

Electric circuits:
force like voltage; $\partial\psi/\partial t$ like current

This is where the “impedance” idea is more familiar. A driving voltage produces a current in a circuit; the proportionality constant is the impedance. The various circuit elements produce currents that are in phase with, ahead of, or behind the driving voltage.

There are analogies between the electric and mechanical systems. See Main Ch 10 for a comprehensive listing.

	$Z_1 = Z_2$	$Z_2 = 0$	$Z_2 = \infty$
R			
T			
R_F			
T_F			

REFLECTION AND TRANSMISSION - REVIEW

- *Continuity conditions (relationship of positions, forces etc at boundary)*
- *Reflection and transmission coefficients for ψ,*
- *Reflection and transmission coefficients for $d\psi/dx$,*
- *Free & fixed boundaries,*
- *Phase change at boundary*
- *impedance (mechanical and electrical)*
- *Mathematical representations of the above*