THE NON-DISPERSIVE WAVE EQUATION

Reading:
Main 9.1.1
GEM 9.1.1
Taylor 16.1, 16.2, 16.3
(Thornton 13.4, 13.6, 13.7)

$$\boxed{\frac{\partial^2}{\partial x^2}\psi(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(x,t)}$$

Example: Non dispersive wave equation

(a second order linear partial differential equation)

$$\boxed{\frac{\partial^2}{\partial x^2}\psi(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(x,t)}$$

Demonstrate that both standing and traveling waves satisfy this equation (HW)

PROVIDED $\omega/k = v$.

Thus we recognize that v represents the wave velocity. (What does "velocity" mean for a standing wave?)

Example: Non dispersive wave equation

$$\boxed{\frac{\partial^2}{\partial x^2}\psi(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi(x,t)}$$

Separation of variables: $\psi(x,t) = X(x)T(t)$

Then

$$\begin{aligned}\psi(x,t) = {} & A\cos kx\cos\omega t \\ & + B\cos kx\sin\omega t \\ & + C\sin kx\cos\omega t \\ & + D\sin kx\sin\omega t\end{aligned}$$

With $v = \omega/k$, A, B, C, D arbitrary constants

Separation of variables:

Assume solution

$$\psi(x,t) = X(x)T(t)$$

Plug in and find

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{v^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2}$$

Argue both sides must equal constant and set to a constant. For the moment, we'll say the constant must be negative, and we'll call it $-k^2$ and this is the same k as in the wavevector.

Separation of variables:

Then
$$\frac{d^2X(x)}{dx^2} = -k^2X(x)$$

$$X(x) = B'_p \cos kx + B'_q \sin kx$$

And
$$\frac{d^2T(t)}{dt^2} = -v^2k^2T(t) = -\omega^2T(t)$$

$$T(t) = B_p \cos \omega t + B_q \sin \omega t$$

$$\psi(x,t) = X(x)T(t)$$

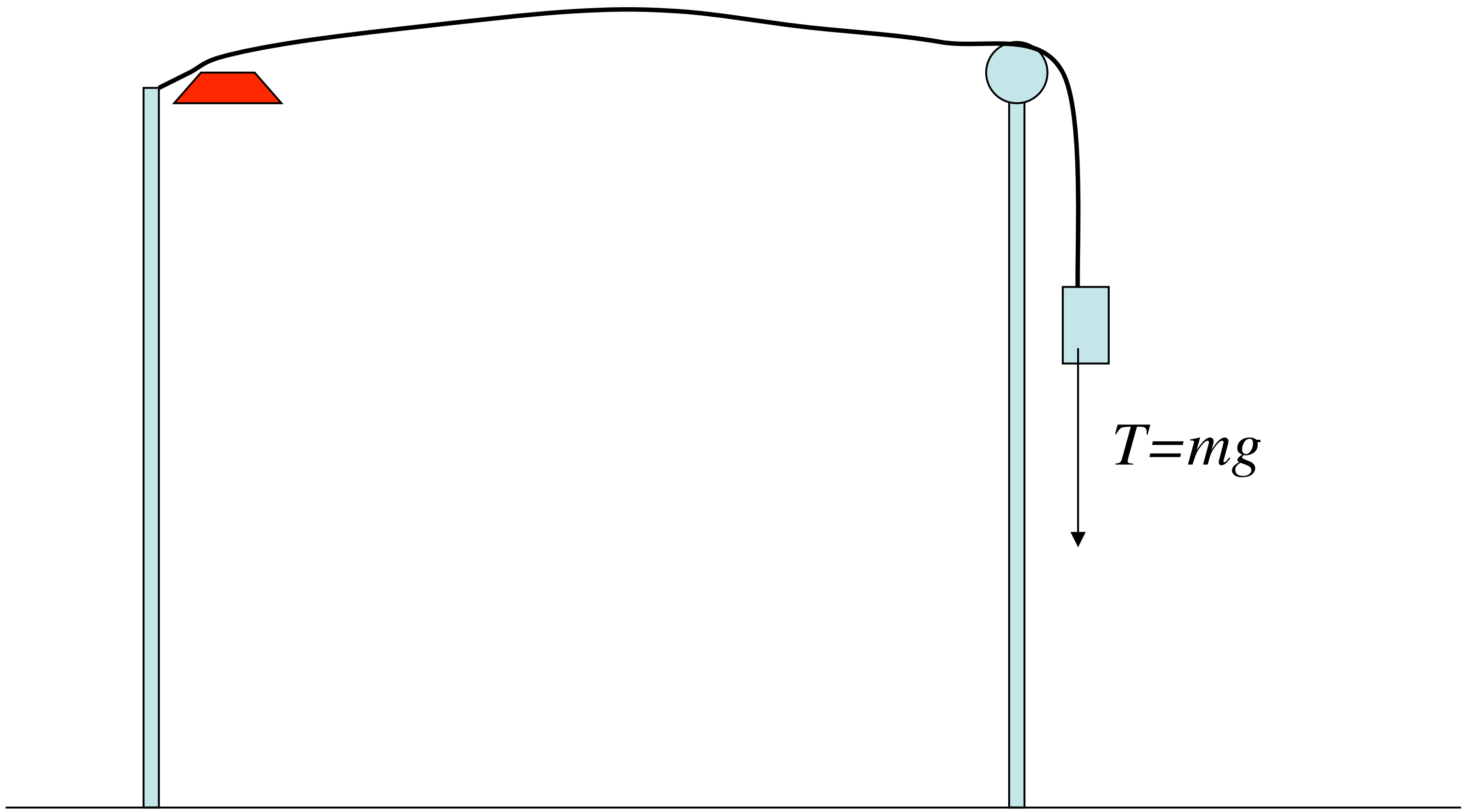
Example: Non dispersive wave equation

Specifying initial conditions determines the (so far arbitrary) coefficients:

$$\psi(x,0); \quad \left.\frac{\partial\psi(x,t)}{\partial t}\right|_{t=0}$$

In-class worksheet had different examples of initial conditions

	A	*B*	*C*	*D*
1				
2				
3				
4				

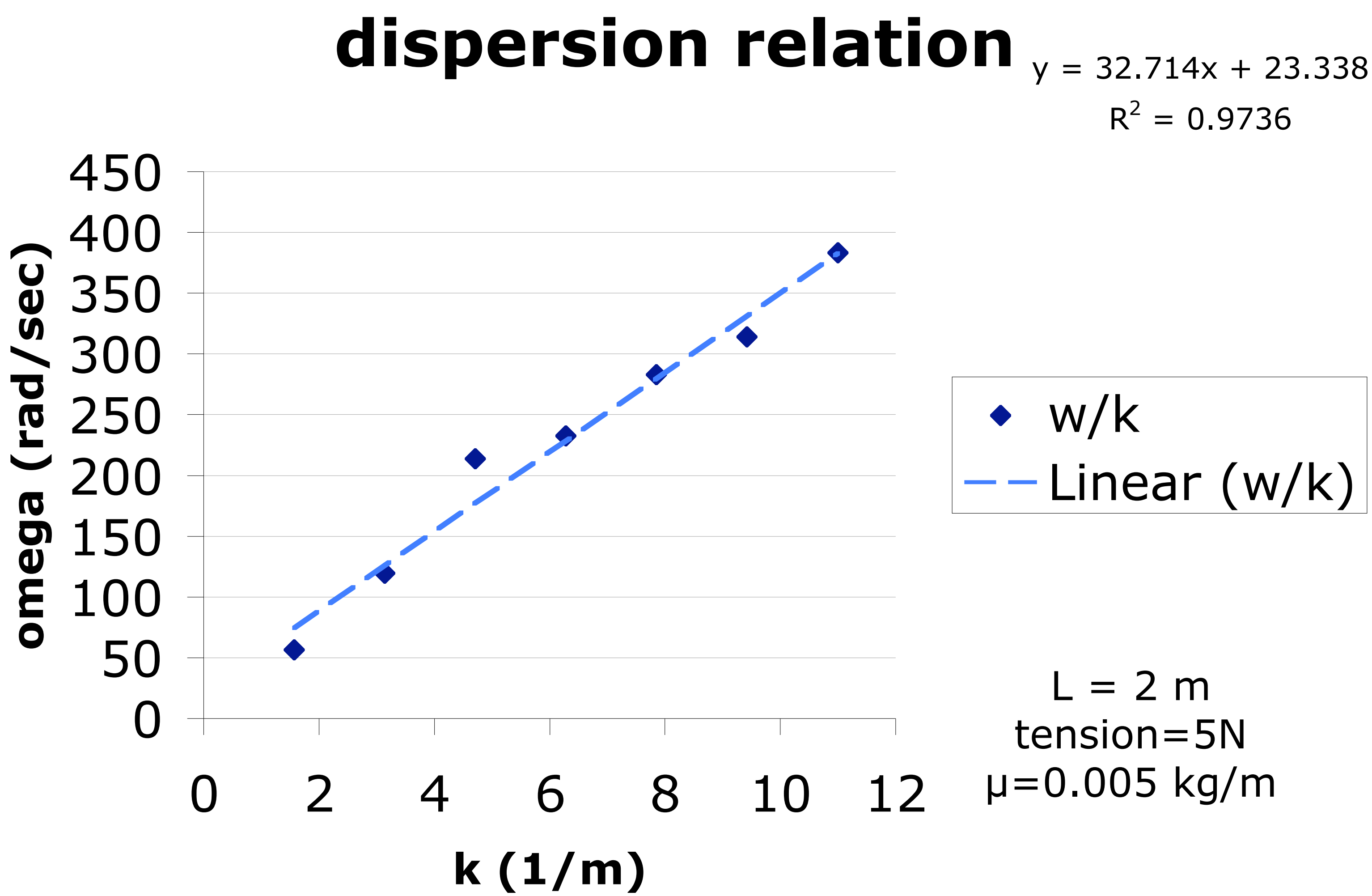


Generate standing waves in this system and measure values of ω (or f) and k (or λ).
Plot ω *vs.* k - this is the DISPERSION RELATION.
Determine phase velocity for system.

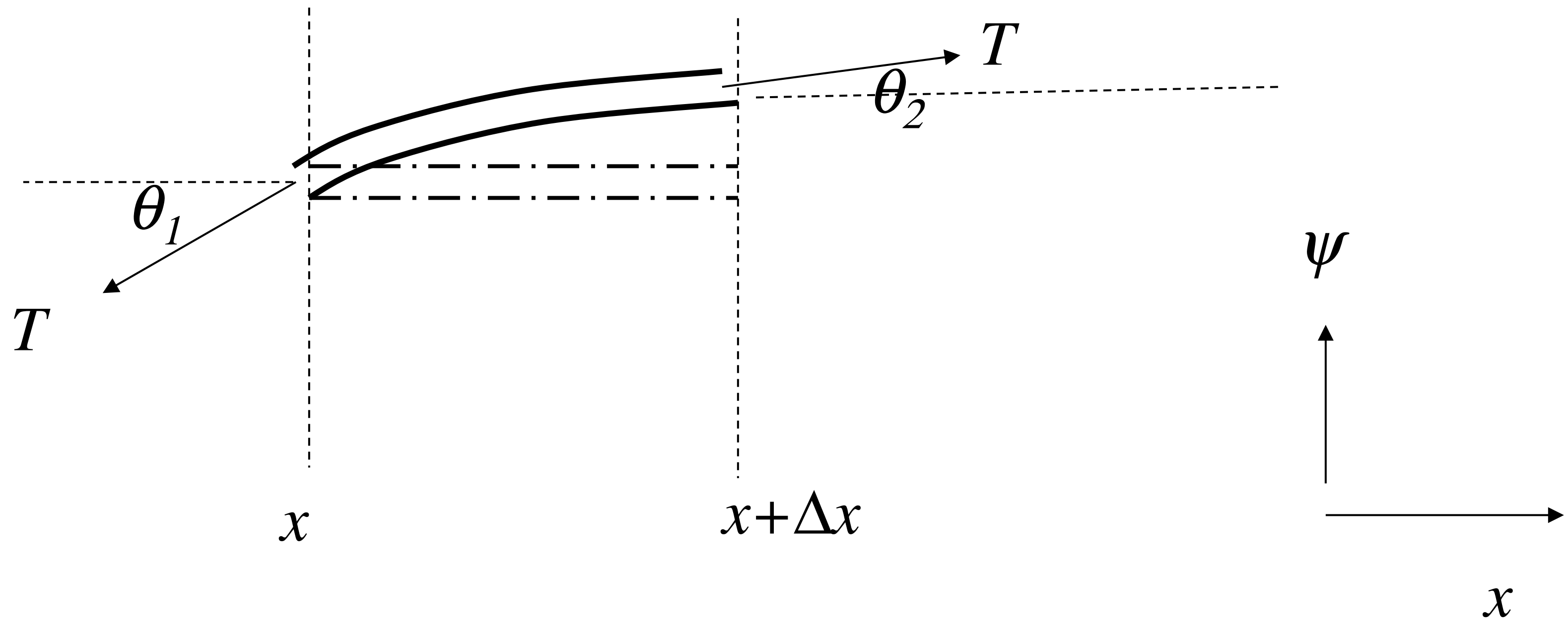
The dispersion relation is an important concept for a system in which waves propagate. It tells us the velocity at which waves of particular frequency propagate (phase velocity) and also the **group velocity**, or the velocity of a **wave packet** (superposition of waves). The group velocity is the one we associate with transfer of information (more in our QM discussion).

Non dispersive: waves of different frequency have the **same** velocity (e.g. electromagnetic waves in vacuum)

Dispersive: waves of different frequency have **different** velocity (e.g. electromagnetic waves in medium; water waves; QM)

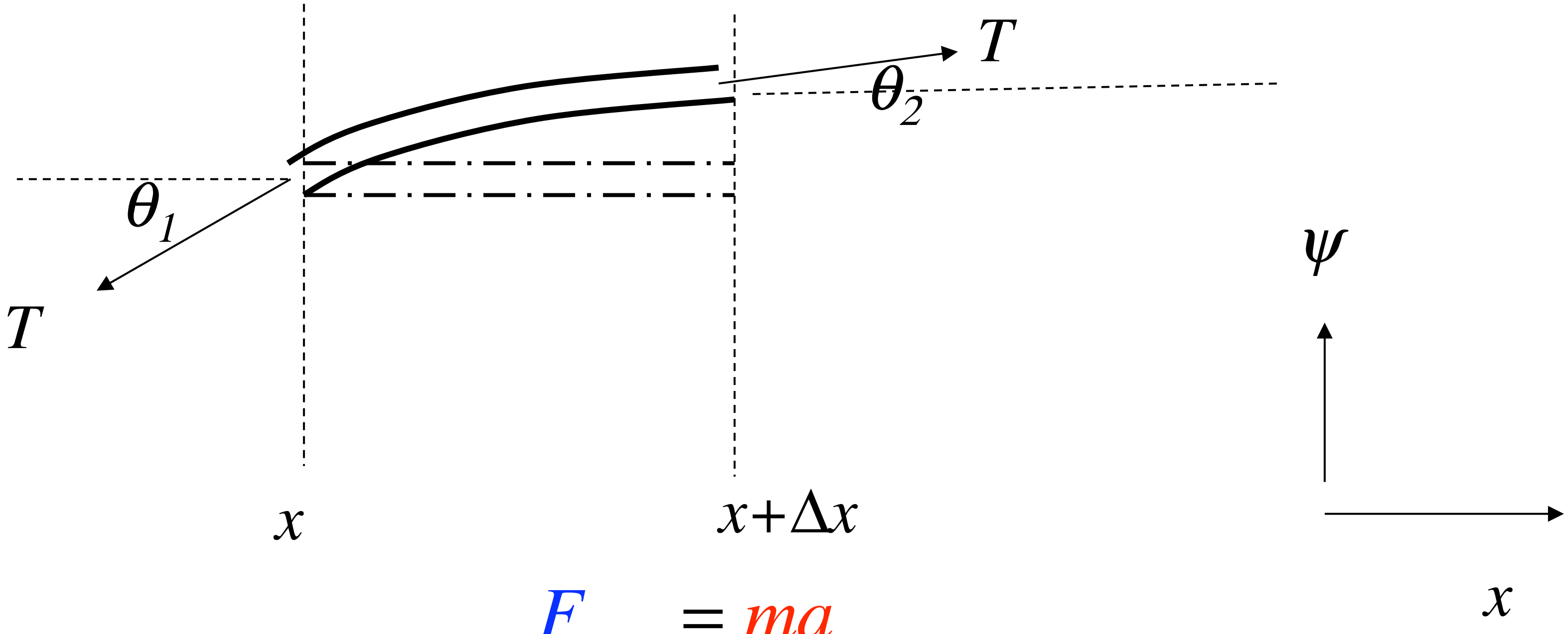

dispersion relation
y = 32.714x + 23.338
R2 = 0.9736
omega (rad/sec)
450
400
350
300
250
200
150
100
50
0
0
2
4
6
8
10
12
k (1/m)
w/k
Linear (w/k)
L = 2 m
tension=5N
μ=0.005 kg/m

Waves in a rope: Application of Newton's law results in the non dispersive wave equation! (for small displacements from equilibrium; $\cos\theta \approx 1$)



Given system parameters:

T = tension in rope; μ = mass per unit length of rope



$$F_{perp} = ma_{perp}$$

$$F_{perp} = T\sin\theta_2 - T\sin\theta_1$$
$$= T\left(\tan\theta_2\cos\theta_2 - \tan\theta_1\cos\theta_1\right)$$

$$ma_{perp} = \mu\Delta x\frac{\partial^2\psi}{\partial t^2}$$

$$F_{perp} = T\left(\left.\frac{\partial\psi}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial\psi}{\partial x}\right|_{x}\right) = T\frac{\partial^2\psi}{\partial x^2}\Delta x$$

$$\frac{\partial^2\psi}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2\psi}{\partial x^2}$$

Non disp wave eqn with $v^2 = T/\mu$

THE NON-DISPERSIVE WAVE EQUATION -REVIEW

- *(non-dispersive) wave equation*
- *Separation of variables*
- *initial conditions,*
- *dispersion, dispersion relation*
- *superposition, traveling wave <-> standing wave,*
- *(non-dispersive) wave equation*
- *Mathematical representations of the above*