PH424 - 1-DIMENSIONAL WAVES — 2011 FINAL EXAM

21 February, 2011, 7pm WGR 304 — 100 points, 110 minutes

- Work with symbols first, and put in numbers (with units) as a last step.
- Always explain your reasoning to demonstrate insight. Yes/no answers, statements without explanation, and answers without working shown, are not distinguishable from guesses.
- Answer all questions. The best strategy is to attempt every problem.
- Sketches should be large, clear, and neatly drawn, with labels and indicators highlighting the points to which you wish to draw attention. Sketches that are untidy, smudged or otherwise of poor quality will be disregarded.

$$\frac{\partial^2\psi(x,t)}{\partial t^2}+\Gamma\frac{\partial\psi(x,t)}{\partial t}=v^2\frac{\partial^2\psi(x,t)}{\partial x^2}$$

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=\hat{H}\psi(x,t)\quad\text{where}\quad\psi(x,t)=\sum_n c_n\varphi_n(x)e^{-iE_nt/\hbar}\quad\text{and}\quad\hat{H}\varphi(x)=E_n\varphi_n(x)$$

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t}=\hat{H}|\psi(t)\rangle\quad\text{where}\quad|\psi(t)\rangle=\sum_n c_n|\varphi_n\rangle e^{-iE_nt/\hbar}\quad\text{and}\quad\hat{H}|\varphi_n\rangle=E_n|\varphi_n\rangle$$

$$\langle Q\rangle=\langle\Phi|\hat{Q}|\Phi\rangle=\int_{-\infty}^{\infty}\Phi^*(x)\hat{Q}\Phi(x)dx$$

$$j=\frac{\hbar}{2im}\left[\psi^*\nabla\psi-\psi\nabla\psi^*\right]$$

$$\hat{p}=-i\hbar\frac{\partial}{\partial x}$$

$$Z=\frac{F_{appl}}{v_{trans}}\qquad w(x,t)=\frac{Z}{2v}\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2+\frac{Zv}{2}\left(\frac{\partial\psi(x,t)}{\partial x}\right)^2$$

Uniform string:

$$v=\sqrt{\frac{\tau}{\mu}}\qquad Z=\sqrt{\tau\mu}\qquad Z=\sqrt{\tau\mu}\left[1-i\sqrt{\frac{\Gamma}{2\omega}}\right]\text{(with attenuation)}$$

Cable electromagnetic waves:

$$v=\sqrt{\frac{1}{L_0C_0}}\qquad Z=\sqrt{\frac{L_0}{C_0}}\qquad Z=\sqrt{\frac{L_0}{C_0}}\left[1-i\frac{R_0}{2\omega L_0}\right]\text{(with attenuation)}$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(k)e^{ikx}dk\Leftrightarrow g(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-ikx}dx$$

Constants:

$$h=6.6\times10^{-34}Js=4.1\times10^{-15}eVs$$

$$\hbar=1.1\times10^{-34}Js=6.5\times10^{-16}eVs$$

$$m_{electron}=9.1\times10^{-31}kg=0.5MeV/c^2$$

$$m_{proton}=1.7\times10^{-27}kg=1.0GeV/c^2$$

$c = 3.0 \times 10^{8}\, m/s = 3.0 \times 10^{10}\, cm/s$

$hc = 1240\ eV\ nm$

Integrals

$$\int x^2 \sin^2(ax)\,dx = \frac{x^3}{6} - \left[\frac{x^2}{4a} - \frac{1}{8a^3}\right]\sin(2ax) - \frac{x\cos(2ax)}{4a^2}$$

$$\int x \sin^2(ax)\,dx = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

Trig identities

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos^2 A + \sin^2 A = 1$

$\cos^2 A - \sin^2 A = \cos 2A$

Rope/cable analogies (from Main 10.3)

Rope		Cable	
Displacement	ψ	Charge	ψ
Transverse velocity	$\frac{\partial \psi}{\partial t}$	Current	$\frac{\partial \psi}{\partial t}$
Slope of rope	$\frac{\partial \psi}{\partial x}$	Charge density	$\frac{\partial \psi}{\partial x}$
Mass/length	μ	Inductance/length	L_0
Tension	τ	$(\text{Capacitance/length})^{-1}$	$1/C_0$
Resistance/length	R_0	Resistance/length	R_0
PE density	$\frac{1}{2}\tau\left(\frac{\partial \psi}{\partial x}\right)^2$	electric energy density	$\frac{1}{2C_0}\left(\frac{\partial \psi}{\partial x}\right)^2$
KE density	$\frac{1}{2}\mu\left(\frac{\partial \psi}{\partial t}\right)^2$	Magnetic energy density	$\frac{1}{2}L_0\left(\frac{\partial \psi}{\partial t}\right)^2$
Impedance (no damping)	$\sqrt{\tau\mu}$	Impedance (no damping)	$\sqrt{\frac{L_0}{C_0}}$
Velocity	$\sqrt{\frac{\tau}{\mu}}$	Propagation velocity	$\sqrt{\frac{1}{L_0 C_0}}$

[1] [20 points]

A quantum state can be represented (at $t=0$) by $\left|\Phi\right\rangle_{t=0} = \frac{-i}{\sqrt{5}}\left|\varphi_1\right\rangle + \frac{2}{\sqrt{5}}\left|\varphi_3\right\rangle$ where the $\left|\varphi_n\right\rangle$ are the normalized eigenstates of the Hamiltonian. E_n are the corresponding energy eigenvalues.

Are the statements below true or false? If true, demonstrate that they are true. If false, give the correct value or expression for the quantity on the left side.

(a) $\left\langle\varphi_3\middle|\Phi\right\rangle_{t=0} = \frac{4}{5}$

(b) $\left\langle E\right\rangle = -\frac{1}{5}E_1 + \frac{2}{5}E_3$

(c) The probability that a measurement of the energy yields E_1 is $\wp\left(E_1\right) = \frac{i^2}{\sqrt{5}}$

(d) $\left|\Phi(t)\right\rangle = \left|\Phi\right\rangle_{t=0} \exp^{-\frac{i\langle E_1 - E_3\rangle t}{\hbar}}$

[2]. [10 points]

You generate a voltage pulse in a coaxial cable of impedance 75 Ω. It doesn't quite reach the oscilloscope, so you find another coaxial cable to use as an extension. That cable has an impedance of 120 Ω. Assuming no loss in either cable, what size voltage pulse is measured at the oscilloscope?

[3]. [20 points]

(a) A transverse perfectly sinusoidal traveling wave propagates to the right in a stretched string with tension τ and mass per unit length μ. The string is perfectly terminated by a piston-like device as shown below. Suddenly, the tension in the string changes to 80% of its value, but the piston's motion remains unchanged. Why is there now a reflected wave?

(b) Under the same conditions as in (a), by how much does the wavelength of the wave change?

(c) A transverse pulse propagates in a stretched string with tension τ and mass per unit length μ. It joins seamlessly to another rope with the same tension τ, but with mass per unit length $\mu/4$. Is the pulse in the transmitted string upright or inverted with respect to the incident pulse, and why? Is the transmitted pulse broader, narrower, or the same width as the incident pulse, and why?

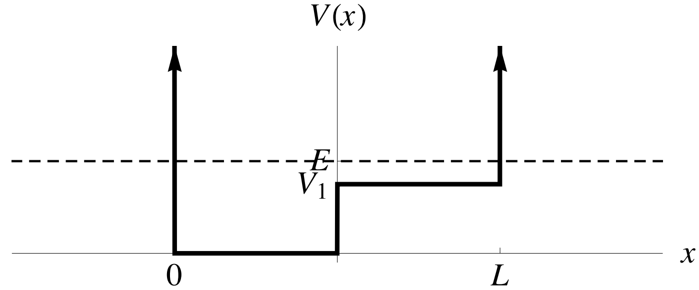
[4] [25 points]

For the asymmetric potential energy given below (an infinite well for 0<x<L but with a potential step at $x = a$)

(a) Give the form of the eigenfunctions that solve the energy eigenvalue equation $\hat{H}\varphi(x) = E\varphi(x)$, for $E > V_1$, for all x. Include any arbitrary constants that must be introduced.

(b) State the appropriate conditions that allow you to solve for the arbitrary constants and E. Apply these conditions to write down the relationships between the constants (you do not need to solve the equations, but you must note if any are zero).

(c) Sketch the qualitative form for a solution $\varphi_n(x)$ for the value of E shown, assuming that there are several solutions with lower energy (*i.e.* that this is one of the more energetic states). Note important features if they are not obvious from your plot.



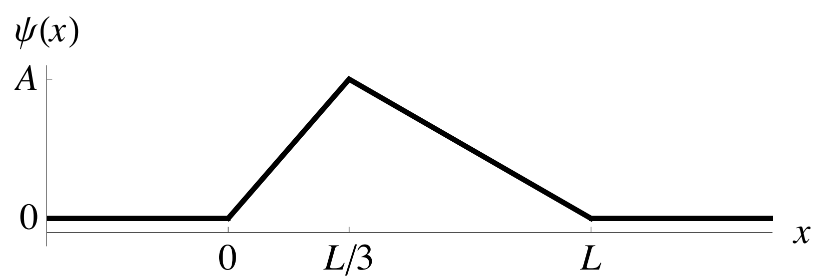
[5] **[25 points]**

An electron's potential energy is described by the "infinite square well potential energy", namely

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < L \\ \infty & x > L \end{cases}$$ where L is known and is the width of the potential well.

(a) Quote without proof the energy eigenvalues E_n and the energy eigenstates $\varphi_n(x)$ for this system.

Now, suppose the electron is prepared in a superposition of energy eigenstates such that the spatial representation of the wave function $\psi(x)$ is as shown below:



(b) What is the probability to measure the position of the electron between 0 and $L/3$? Express your answer in terms of L, and the as-yet undetermined A.

(c) If the energy of the electron were measured, what is the probability of measuring the energy E_3? Explain carefully how you arrive at your answer, writing down the necessary integrals, but you do not need to evaluate them. You must, however, demonstrate that you have the information needed to evaluate them.

(d) If the energy of the electron were measured as E_2, what would the wave function be after the measurement?

(e) Your classmate asserts that $A = \sqrt{3/L}$. Does this give a properly normalized wave function?