THE DRIVEN, DAMPED HARMONIC OSCILLATOR

Reading:
Main 5.1, 6.1
Taylor 5.5, 5.6

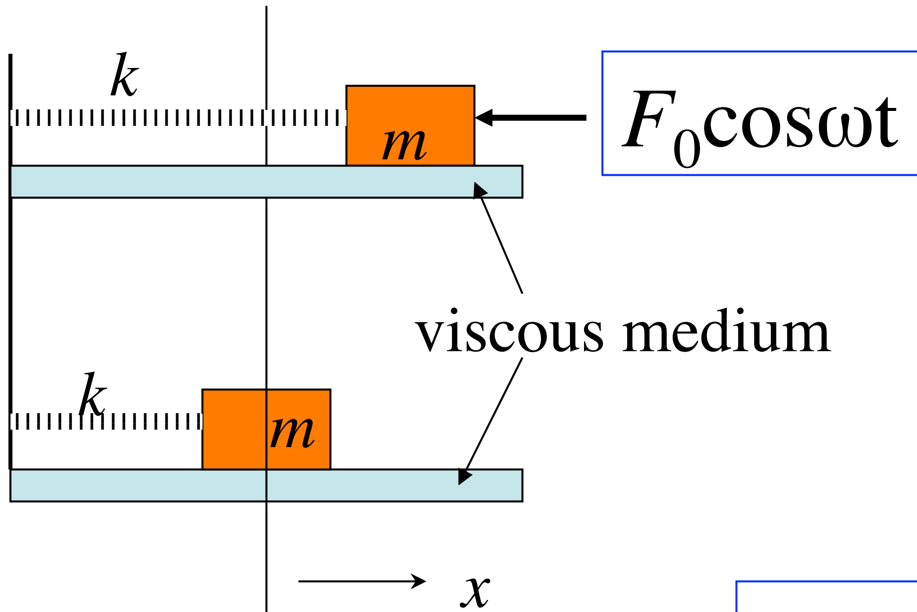
Natural motion of damped, driven harmonic oscillator

$$Force = m\ddot{x}$$

$$restoring + resistive + driving\ \ force = m\ddot{x}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\omega_1 = \omega_0\sqrt{1 - \frac{\beta^2}{\omega_0^2}}$$



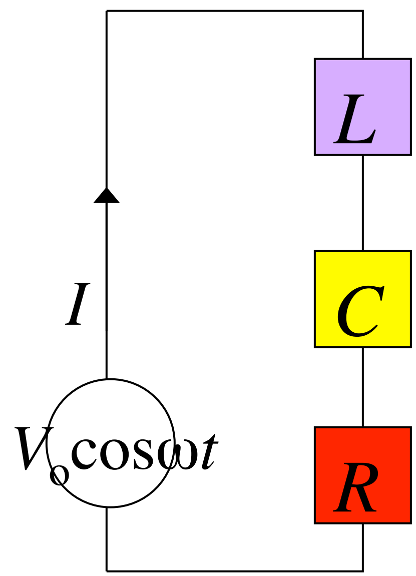
$$-kx - b\dot{x} + F_0 \cos(\omega t) = m\ddot{x}$$

$$m\ddot{x} + \omega_0^2 x + 2\beta\dot{x} + = F_0 \cos(\omega t)$$

Note ω and ω_0 are not the same thing!
ω is driving frequency
ω_0 is natural frequency

Natural motion of damped, driven harmonic oscillator

Apply Kirkoff's laws



$$V_0 \cos(\omega t) - L\ddot{q} - \frac{q}{C} - R\dot{q} = 0$$

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = \frac{V_0}{L} e^{i\omega t}$$

http://www.sciencejoywagon.com/physicszone/lesson/otherpub/wfendt/accircuit.htm

What is the response of the system? x(t), q(t), or in general, ψ(t)?

Qualitative questions first:

•What is the basic form of the system response after long times?

Sinusoidal. $\psi(t) = \psi_{max}\cos(\omega t+\phi)$ (after times longer than $1/\beta$)

•Is the frequency of the system response the same, smaller, or larger than the driving frequency?

The same - it must be!

•How does the magnitude of the response depend on the driving frequency?

It is large close to the natural frequency ω_0, and small at lower and at higher frequencies (this is called resonance)

•How does the phase of the response depend on the driving frequency?

We'll have to see. It changes, certainly. But it depends on what you mean by "response". Is it displacement/charge? Velocity/current?

$$V_{ext} = \mathrm{Re}\left[V_0 e^{i\omega t}\right]$$ V_0 real, constant, and known

$$q(t) = \mathrm{Re}\left[q_0 e^{i\omega t}\right]$$ Let's assume this form for *q(t)*

But now q_0 is complex: $$q_0 = \left|q_0\right| e^{i\phi_q}$$

This solution makes sure *q(t)* is oscillatory (and at the same frequency as F_{ext}), but may not be in phase with the driving force.

Task #1: Substitute this assumed form into the equation of motion, and find the values of $|q_0|$ and ϕ_q in terms of the known quantities. Note that these constants depend on driving frequency ω (but not on *t* - that's why they're "constants"). How does the shape vary with ω?

$$V_{ext} = \mathrm{Re}\left[V_0 e^{i\omega t}\right]$$

Assume V_0 real, and constant

$$q(t) = \mathrm{Re}\left[q_0 e^{i\omega t}\right]$$

$$I(t) = ?$$

Task #2: In the lab, you'll actually measure I (current) or dq/dt. So let's look at that: Having found $q(t)$, find $I(t)$ and think about how the shape of the amplitude and phase of I change with frequency.

$$V_{ext} = \mathrm{Re}\left[V_0 e^{i\omega t}\right]$$

Assume V_0 real, and constant

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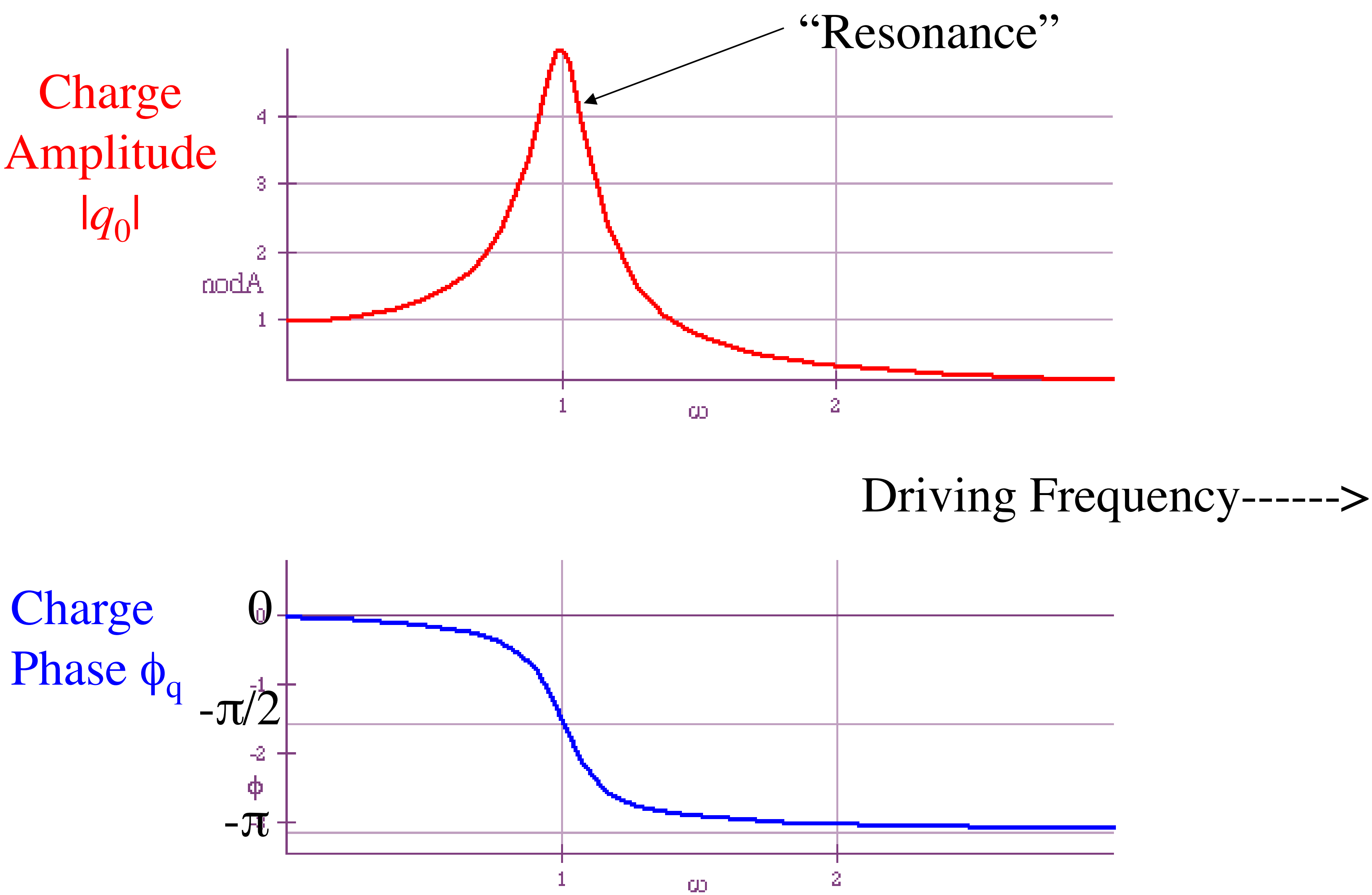
Task #1: Substitute this assumed form into the equation of motion, and find the values of $|q_0|$ and ϕ in terms of the known quantities. Note that these constants depend on ω (but not on t - that's why they're "constants"). How does the shape vary with ω?

$$|q_0| = \frac{V_0/L}{\left[\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2\right]^{1/2}}$$

$$\tan\phi = \frac{-2\beta\omega}{\omega_0^2 - \omega^2}$$


Charge
Amplitude
$|q_0|$
“Resonance”
4
3
2
modA
1
1
ω
2
Driving Frequency------>
Charge
Phase ϕ_q
0
-1
$-\pi/2$
-2
φ
$-\pi$
1
ω
2

$$V_{ext} = \mathrm{Re}\left[V_0 e^{i\omega t}\right] \qquad q(t) = \mathrm{Re}\left[q_0 e^{i\omega t}\right]$$

$$I(t) = \dot{q}(t) = \mathrm{Re}\left[\omega \left|q_0\right| e^{i\left(\phi_q + \pi/2\right)} e^{i\omega t}\right]$$

$$\left|I_0\right| = \frac{\omega V_0 / L}{\left[\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2\right]^{1/2}}$$

$$\phi_I = \frac{\pi}{2} + \arctan\frac{-2\beta\omega}{\omega_0^2 - \omega^2}$$

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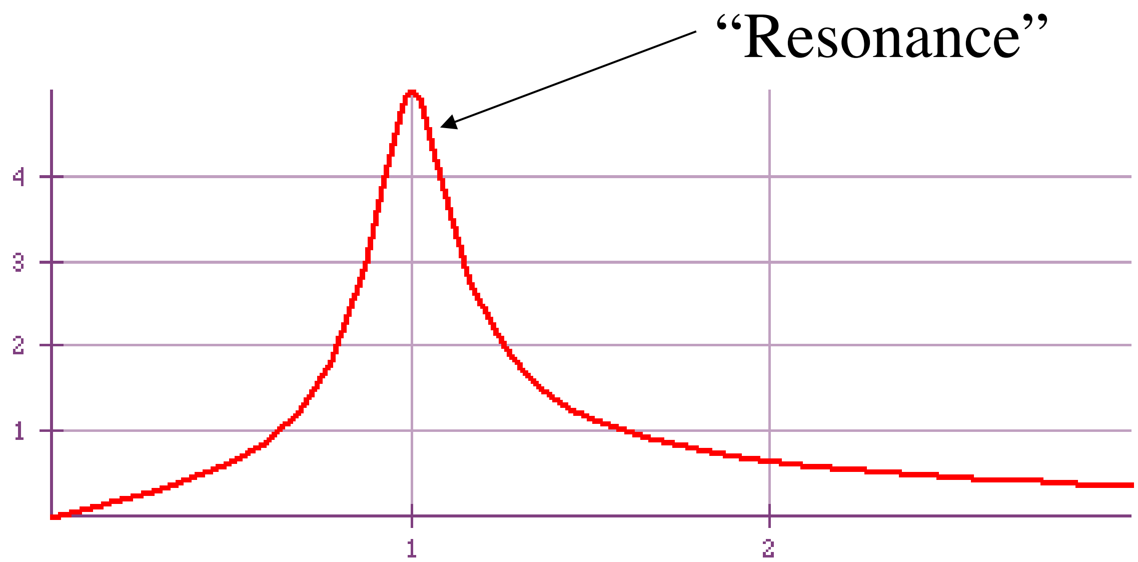
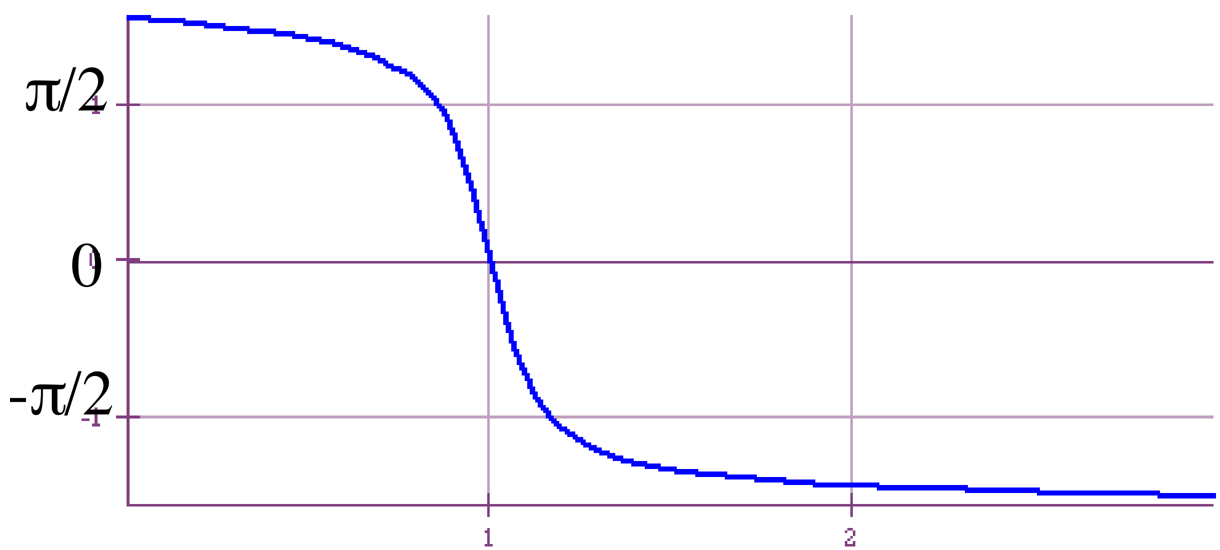
Current
Amplitude
$|I_0|$

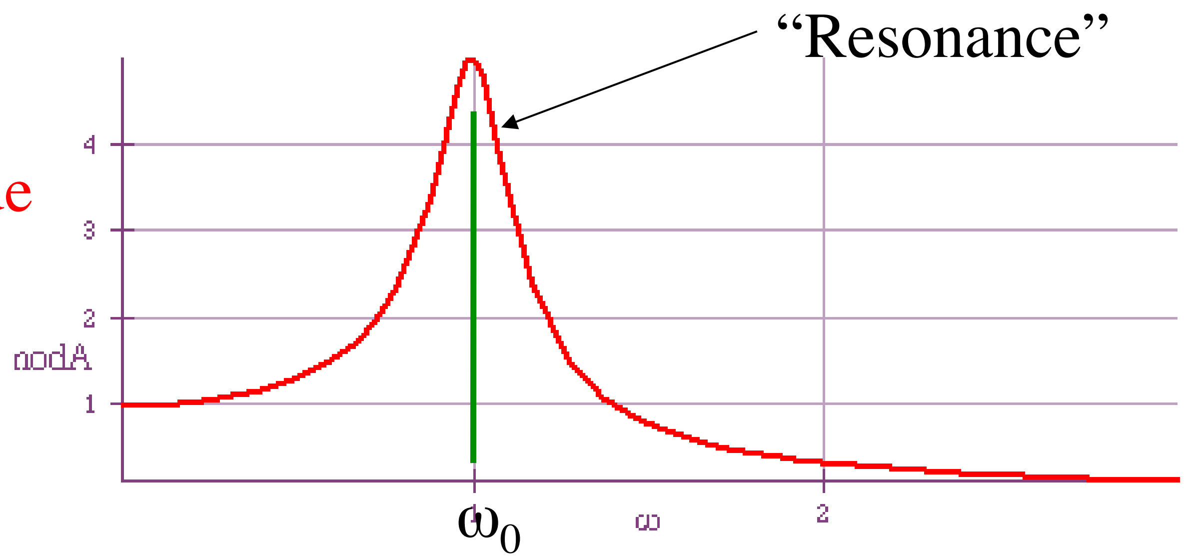


Driving Frequency------>

Current
Phase

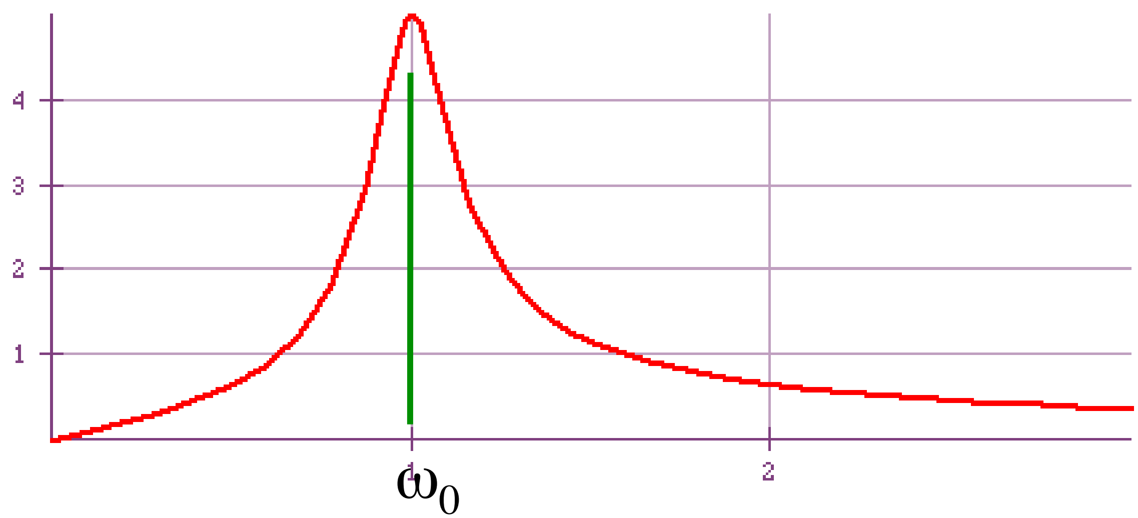


Charge Amplitude
$|q_0|$

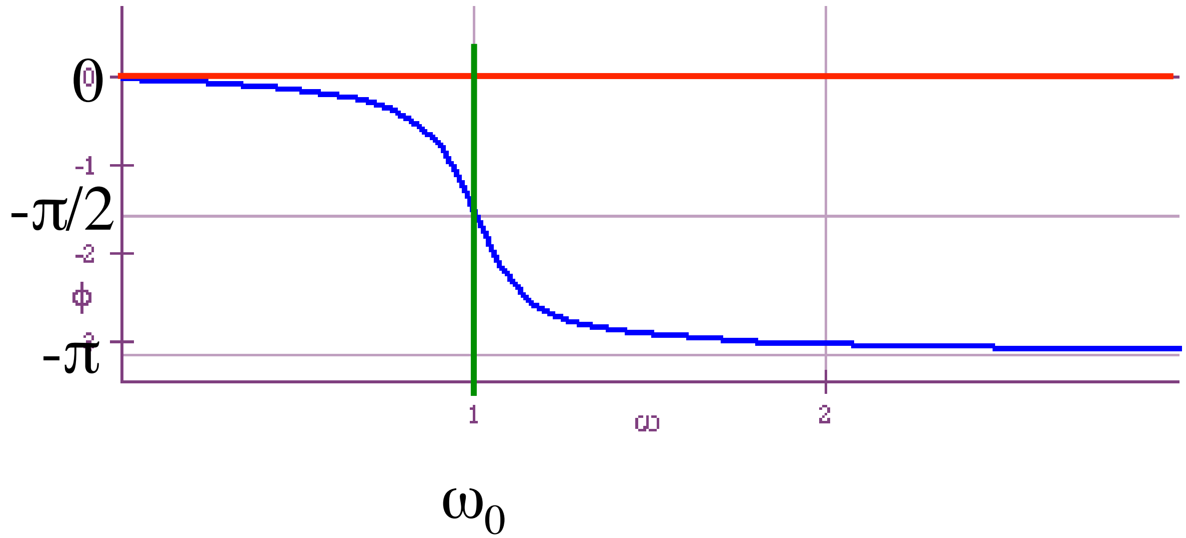


Driving Frequency------>

Current Amplitude
$|I_0|$



Charge Phase ϕ_q



Driving Frequency------>

Current Phase